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High-beta toroidal equilibria with flow in reduced fluid models

Atsushi ITO and Noriyoshi NAKAJIMA

National Institute for Fusion Science,322-6 Oroshi-cho, Toki 509-5292, Japan

A reduced set of equations for high-beta tokamak equilibria with flow comparable to the poloidal sound velocity are solved analytically for the single-fluid MHD case and numerically for the case of two-fluid model with ion Finite Larmor radius (FLR). The analytical solution for single-fluid MHD equilibria shows that the shift of the magnetic axis from the geometric axis is enhanced by a slightly super-poloidal-sonic flow and it produces a forbidden region of equilibrium by the poloidal-sonic flow. Numerical analysis shows that there are regular solutions for the two-fluid model with FLR that are singular for the single-fluid and Hall MHD models, and that the solutions depend on the sign of the $E \times B$ flow compared to that of the ion diamagnetic flow.

Keywords: two-fluid model, finite Larmor radius, equilibria with flow, magnetohydrodynamics

1 Introduction

In magnetically confined plasmas, equilibrium flows may suppress instability and turbulent transport to give rise to improved confinement modes where high-β is achieved. In such equilibria, the scale lengths characteristic of microscopic effects not included in single-fluid magnetohydrodynamics (MHD) cannot be neglected. Small scale effects on flowing equilibria due to the Hall current have been studied with two-fluid or Hall MHD models [1]. However, these models are consistent with kinetic theory only for cold ions. A consistent treatment of hot ions in a two-fluid framework must include the ion gyroviscosity and other finite Larmor radius (FLR) effects. In the fluid formalism of collisionless magnetized plasmas, these effects are incorporated by means of asymptotic expansions in terms of the small parameter $\delta \sim \rho_i/a$, where ρ_i is the ion Larmor radius and a is the macroscopic scale length. With a slow dynamics ordering, $v \sim \delta v_{th}$ where v and v_{th} are the flow and thermal velocities respectively, the ion FLR terms [2] are much simplified in the reduced models for large-aspect-ratio, high-β tokamaks [3] after relating δ to the inverse aspect ratio expansion parameter $\varepsilon \equiv a/R_0 \ll 1$, where R_0 is the characteristic scale length of the major radius.

We have derived reduced sets of two-fluid equations for axisymmetric equilibria with flow in the orders of the poloidal sound velocities [4]. The poloidal-sonic flow can be described by the reduced model with the relation $\delta \sim \epsilon$. This reduced set can describe the three models: single-fluid (ideal) MHD, Hall MHD, two-fluid model with ion FLR. For the single fluid case, we can find analytical solutions and study the effects of poloidal-sonic flow on the equilibrium profiles (Sec. 3). For the case of two-fluid model with ion FLR, we have found numerically regular solutions that depend on the sign of the $E \times B$ flow compared to that of the ion diamagnetic flow (Sec. 4).

2 Basic equations

The equations for two-fluid equilibria with hot ions are

$$\nabla \cdot (n\mathbf{v}) = 0, \tag{1}$$

$$\nabla \times \mathbf{E} = 0, \tag{2}$$

$$m_i n\mathbf{v} \cdot \nabla\mathbf{v} = \mathbf{j} \times \mathbf{B} - \nabla(p_i + p_e) - \lambda_i \nabla \cdot \Pi_i^{gv}, \tag{3}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{\lambda_H}{ne}(\mathbf{j} \times \mathbf{B} - \nabla p_e), \tag{4}$$

$$\mu_0 \mathbf{j} = \nabla \times \mathbf{B} \tag{5}$$

$$\mathbf{v} \cdot \nabla p_i + \gamma p_i \nabla \cdot \mathbf{v} + \lambda_i \left(\frac{2}{5}\gamma \nabla \cdot \mathbf{q}_i\right) = 0, \tag{6}$$

$$(\mathbf{v} - \lambda_H \mathbf{j}/ne) \cdot \nabla p_e + \gamma p_e \nabla \cdot (\mathbf{v} - \lambda_H \mathbf{j}/ne) + \lambda_e \left(\frac{2}{5}\gamma \nabla \cdot \mathbf{q}_e\right) = 0, \tag{7}$$

where m_i is the ion mass, n is the density, $\mathbf{v}$ is the ion flow velocity, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, $\mathbf{j}$ is the current density, p_i and p_e are the ion and electron pressures, Π_i^{gv} is the ion gyroviscous tensor, $\mathbf{q}_i$ and $\mathbf{q}_e$ are the ion and electron heat fluxes respectively, and $\gamma = 5/3$. The diagonal components of the pressure tensors are assumed to be isotropic. The electron mass m_e is neglected because $m_e \ll m_i$. The electron gyroviscosity is also neglected since $\rho_e \ll \rho_i$. We have introduced the artificial indices λ_i, λ_e and λ_H that label the two-fluid, non-ideal terms: $(\lambda_i, \lambda_e, \lambda_H) = (0, 0, 0)$ for single-fluid (ideal) MHD, $(0, 0, 1)$ for two-fluid MHD with adiabatic electron pressure but zero ion Larmor radius (Hall MHD) and $(1, 1, 1)$ for two-fluids with finite ion Larmor radius. Here we shall consider the corresponding toroidal axisymmetric equilibria, where, in cylindrical coordinates (R, φ, Z), the magnetic field $\mathbf{B}$ can be written as

$$\mathbf{B} = \nabla\psi(R, Z) \times \nabla\varphi + I(R, Z)\nabla\varphi \tag{8}$$

The asymptotic expansion is defined in terms of the inverse aspect ratio $\varepsilon \equiv a/R_0 \ll 1$ where a and R_0 are

author's e-mail: ito.atsushi2@nifs.ac.jp

the characteristic scale length of the minor and major radii respectively. The following high-β tokamak orderings for compressible reduced MHD are applied,

$$B_p \sim \varepsilon B_0, \quad p_i \sim p_e \sim \varepsilon\left(B_0^2/\mu_0\right), \quad |\nabla| \sim 1/a.$$

The variables are expanded as $f = f_1 + f_2 + f_3 + \ldots$ We assume slow dynamics ordering,

$$v \sim \delta v_{thi}, \quad m_i n v^2 \sim \|\Pi_i^{gv}\| \sim \delta^2 p_{i,e},$$

$$q_i \sim v p_{i,e} \sim \delta v_{thi} p_{i,e}.$$

The energy of flows in the order of the poloidal sound speed $v \sim C_{sp} \equiv (B_p/B_0)(\gamma p/nm_i)^{1/2}$ is the third order of the magnetic energy,

$$m_i n v^2 \sim \varepsilon^2 p \sim \varepsilon^3\left(B_0^2/\mu_0\right).$$

The equation for ψ_1 is identical to the reduced GS equation for single-fluid, static equilibria,

$$\Delta_2\psi_1 = -\mu_0 R_0^2\left[\left(\frac{2x}{R_0}\right)p_1' + g_*'\right] - \left(\frac{I_1^2}{2}\right)', \tag{9}$$

where $\Delta_2 \equiv (\partial^2/\partial R^2 + \partial^2/\partial Z^2)$, $p_1 \equiv p_{i1} + p_{e1}$ and

$$p_{i2} + p_{e2} + \frac{B_0}{\mu_0 R_0} I_2 \equiv g_*(\psi_1). \tag{10}$$

The following quantities are shown to be arbitrary functions of ψ_1,

$$\begin{aligned} n_0 &= n_0(\psi_1), \quad p_{i1} = p_{i1}(\psi_1), \\ p_{e1} &= p_{e1}(\psi_1), \quad I_1 = I_1(\psi_1). \end{aligned} \tag{11}$$

The effects of flow, two-fluid and ion FLR appears in the equation for ψ_2 [5, 4].

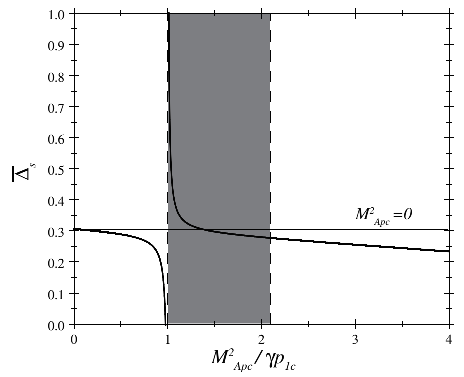

Fig. 1 The Shafranov shift as a function of poloidal Mach number. The shaded region is beyond the equilibrium beta limit.

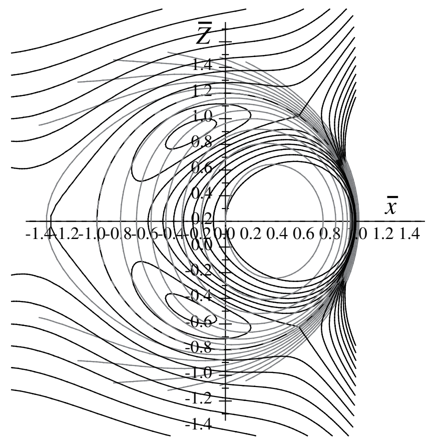

Fig. 2 The magnetic surfaces for $M_{Apc}^2 = 0$ (gray) and for $M_{Apc}^2 = 1.05\gamma p_{1c}$ (black).

3 Single-fluid equilibria

Single-fluid MHD equilibria are given by setting $(\lambda_i, \lambda_e, \lambda_H) = (0, 0, 0)$. For linear profiles of the lowest-order quantities, we have derived analytical equilibria of high-beta tokamaks with poloidal-sonic flow and the expressions for the shift of the magnetic axis, the shift of the pressure maximum and the equilibrium beta limit [6]. We assume linear profiles for the following free functions,

$$p_1 = \varepsilon\left(B_0^2/\mu_0\right) p_{1c}(\psi_1/\psi_c), \tag{12}$$

$$g_* + \frac{I_1^2}{2\mu_0 R_0^2} = \varepsilon^2\left(B_0^2/\mu_0\right) g_c\left(\frac{\psi_1}{\psi_c}\right), \tag{13}$$

$$M_{Ap}^2 = \varepsilon M_{Apc}^2(\psi_1/\psi_c), \tag{14}$$

$$M_{Ap}(\psi_1) \equiv -\left[\mu_0 m_i n_0(\psi_1)\right]^{1/2} R_0 \Phi_1'(\psi_1)/B_0 \tag{15}$$

is the leading order of the poloidal Alfvén Mach number, $\beta_1 \equiv \gamma p_1/(B_0^2/\mu_0)$. The fixed boundary conditions for ψ_1 and ψ_2 are given by assuming circular cross section as

$$\psi_1(a, \theta) = \psi_2(a, \theta) = 0. \tag{16}$$

We have also solved for the vacuum region $1 \le r$ assuming that ψ_1 and ψ_2 are smoothly connected at the plasma-vacuum boundary $r = a$. We then apply the following normalization

$$r/a \equiv \bar{r}, \qquad \psi_1/\psi_c \equiv \bar{\psi}_1, \qquad \psi_2/\psi_c \equiv \varepsilon\bar{\psi}_2,$$

$$a/R_0 \equiv \varepsilon, \quad \psi_c/B_0 R_0 a \equiv \varepsilon B_p.$$

(For figures 1 - 8, $\varepsilon = 0.1$, $g_c = 4$, $p_{1c} = 3.2$ and $B_p = 1$.) The solution indicates the modification of the magnetic structure and the departure of the pressure surfaces from the magnetic surfaces by sub- or super-poloidal-sonic flows. We have shown that the shift of the magnetic axis from the geometric axis is enhanced by a slightly super-poloidal-sonic flow and it produces a forbidden region of equilibrium by the poloidal-sonic flow (Fig. 1). Figure 2 shows the magnetic structure for $M_{Apc}^2/\gamma p_{1c} = 1.05$, where the equilibrium beta limit is violated since the separatrix appears in the plasma region. The physical mechanism of the shift of the pressure maximum from the magnetic axis due to the poloidal-sonic flow can be explained in analogy to those of the geodesic acoustic mode and the slow magnetosonic wave.

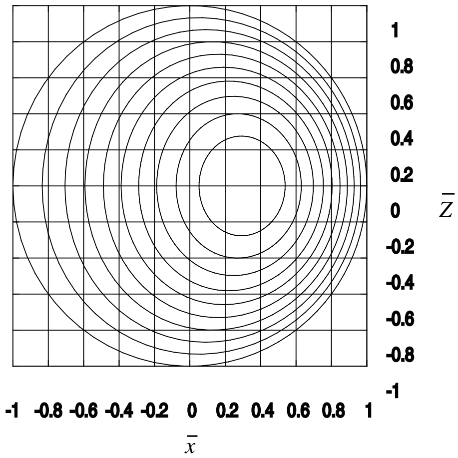

Fig. 3 Isosurfaces of ψ for $V_{Ec} = -\sqrt{\gamma p_{1c}}$ and $V_{dc} = -1$.

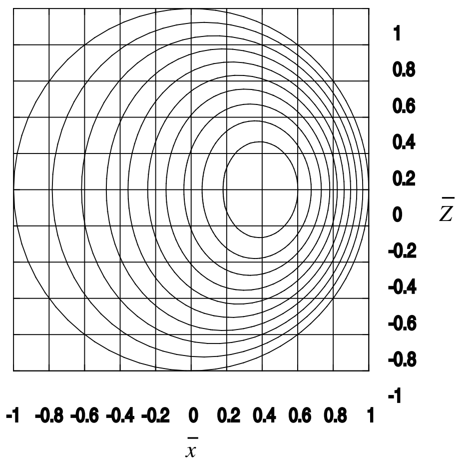

Fig. 4 Isosurfaces of p for $V_{Ec} = -\sqrt{\gamma p_{1c}}$ and $V_{dc} = -1$.

4 Two-fluid equilibria with FLR

Two-fluid equilibria with FLR are obtained by setting $(\lambda_i, \lambda_e, \lambda_H) = (1, 1, 1)$. From Faraday's law (2), we obtain $\mathbf{E} \equiv -\nabla\Phi$. The generalized Ohm's law (4) is rewritten as

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \frac{\lambda_H}{ne}\left(\nabla p_i + m_i n\mathbf{v}\cdot\nabla\mathbf{v} + \lambda_i \nabla\cdot\Pi_i^{gv}\right). \quad (17)$$

The ion flow velocity $\mathbf{v}$ is defined from Eq. (17) up to the second order as

$$\mathbf{v} \equiv \mathbf{v_E} + \lambda_H \mathbf{v}_{di} + v_{\|}(\mathbf{B}/B), \quad (18)$$

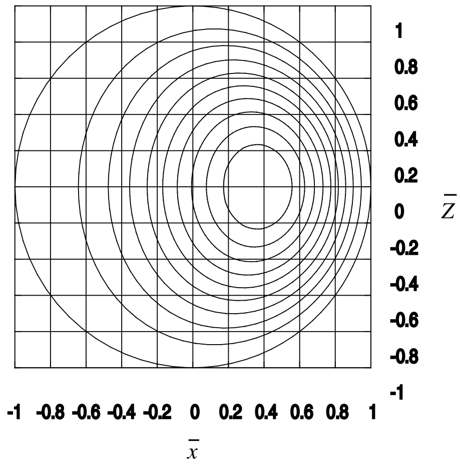

Fig. 5 Isosurfaces of Ψ for $V_{Ec} = -\sqrt{\gamma p_{1c}}$ and $V_{dc} = -1$.

$$\mathbf{v}_E \equiv -\frac{\nabla\Phi\times\mathbf{B}}{B^2}, \quad \mathbf{v}_{di} \equiv -\frac{\nabla p_i \times \mathbf{B}}{enB^2}. \quad (19)$$

The ion gyroviscous force is needed only in their leading orders [5],

$$\nabla\cdot\Pi_i^{gv} \simeq -\frac{m_i}{eB_0}\left(R_0\nabla\varphi\times\nabla p_{i1}\right)\cdot\nabla\mathbf{v} - \nabla\left(\chi_v + \chi_q\right), \quad (20)$$

Unlike for single-fluid equilibria, the profile of n_0 must be specified in order to give the profiles of $E\times B$ and diamagnetic flows for two-fluid equilibria. We assume the linear profile of n_0,

$$n_0 = n_c\overline{\psi}_1. \quad (21)$$

Then the $E\times B$ and the diamagnetic flows are given by

$$V_E/V_{Ap} = \sqrt{\varepsilon}V_{Ec}\overline{\psi}_1^{1/2}, \quad (22)$$

$$V_{di}/V_{Ap} = \sqrt{\varepsilon}V_{dc}p_{i1c}\overline{\psi}_1^{-1/2}, \quad (23)$$

$$V_{de}/V_{Ap} = -\sqrt{\varepsilon}V_{dc}p_{e1c}\overline{\psi}_1^{-1/2}, \quad (24)$$

where

$$V_{dc} \equiv -\sqrt{\varepsilon\mu_0 m_i n_{0c}}\frac{R_0\left(B_0^2/\mu_0\right)}{en_{0c}B_0\psi_c}. \quad (25)$$

We solve the GS equation for ψ_2 by using the finite element method with 40×40 grid points. Figures 3 - 5 show the isosurfaces of ψ, p and the ion stream function Ψ respectively for $V_{Ec} = -1$ and $V_{dc} = -1$. The single-fluid equilibria with $V_{Ec}^2 = \gamma p_{1c}$ is singular because of the poloidal-sonic singularity. The Hall-MHD equilibria also have singularity at $\bar{r} = 1$ because the convective term in the equation for ψ_2,

$$M_{Ap}\left(M_{Ap} - \lambda_i\frac{V_{di}}{V_{Ap}}\right)$$

goes to infinity for the given profiles. Thus, this solution is regular in the presence of both two-fluid and FLR effects. Figures 3 - 5 also show that the isosurfaces of ψ, p and Ψ do not coincide with each other due to the two-fluid effect.

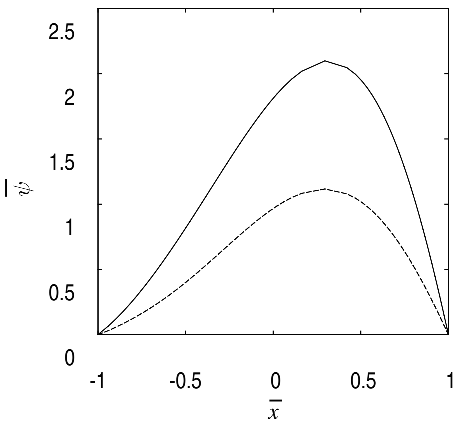

Fig. 6 The profiles of ψ in the mid plane.

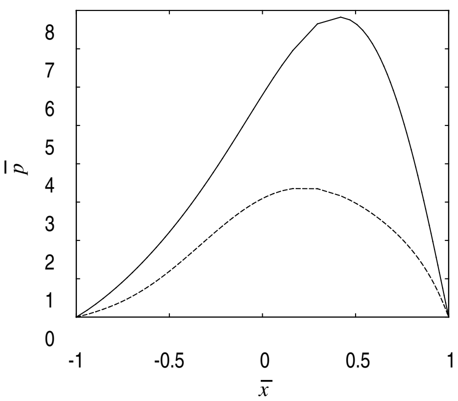

Fig. 7 The profiles of p in the midplane.

Figures 6 - 8 show the profiles of ψ, p and the ion stream function Ψ respectively in the midplane for $V_{Ec} = \pm 1$ and $V_{dc} = -1$. The solid lines are for $V_{Ec} = V_{dc}$ and the dashed line are for $V_{Ec} = -V_{dc}$. These results show that the equilibrium solutions for two fluid equilibria with FLR depend on the sign of the $E \times B$ flow.

5 Summary

We have solved a reduced set of equations for high-beta tokamak equilibria with flow comparable to the poloidal sound velocity analytically for the single-fluid MHD case and numerically for the case of two-fluid model with ion Finite Larmor radius (FLR). The analytical solution for single-fluid MHD equilibria shows that the shift of the magnetic axis from the geometric axis is enhanced by a slightly super-poloidal-sonic flow and it produces a forbidden region of equilibrium by the poloidal-sonic flow. Numerical analysis shows that there are regular solutions for the two-fluid model with FLR that are singular for the single-fluid and Hall MHD models, and that the solutions depend on the sign of the $E \times B$ flow compared to that of the ion diamagnetic flow.

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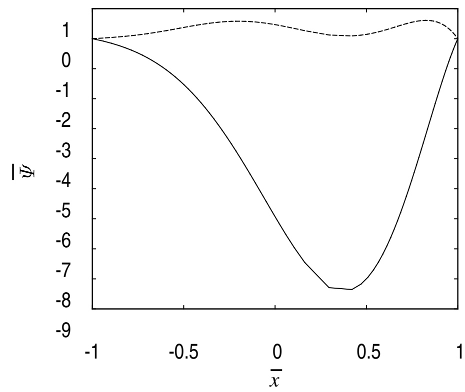

Fig. 8 The profiles of Ψ in the midplane.

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