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Langevin Equation for Guiding Center Motion and its Application to Neoclassical Transport Theory

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Neoclassical transport in toroidal plasmas is addressed by the Langevin equation. To ensure the diffusive nature of a stochastic process, the dynamics of test particles is expressed in terms of the Langevin equation with the assumption of sufficiently small radial orbit width. The transport coefficients are evaluated by the time integration of auto- and cross-correlation functions for each pair of time-reversal expressions of microscopic fluxes. As a test of this method, the neoclassical viscosity coefficients are calculated numerically and are shown to agree with analytical formulas.

Keywords: Neoclassical Transport, Langevin Equation, Linear Response Theory

1 Introduction

The Langevin equation[1] is used to describe the transport processes in a system close to thermal equilibrium. An advantage of the Langevin-type description is that it can be easily simulated using a quasi-random number generator. The Monte Carlo methods have been extensively used in the stellarator/heliotron research, in particular, to estimate the $1/\nu$ radial diffusion[2], which gives the irreducible minimum of the transport level in a toroidal configuration. The calculation of parallel transport such as bootstrap currents is also important for predicting the non-inductive currents which are observed in experiments.

Mathematically, the Langevin equation is an example of stochastic differential equations(SDEs)[3]. When we consider a stochastic variable $X(t)$ in a Gaussian random process with t, a time variable, the evolution of $X(t)$ can be written in terms of SDEs by

$$dX(t) = a(X;t)\,dt + b(X;t)\,dW(t), \tag{1}$$

where $a(X;t)$ is the deterministic part of test particle motion, while $b(X;t)$ denotes the random acceleration with the standard Wiener process $W(t)$. In most cases, we need not solve exact trajectories of test particles. Instead, the Wiener increments $dW(t)$ are approximated by relatively simple random variables, such as the two-point or uniform ones. The Monte Carlo collision operator introduced by Boozer and Kuo-Petravic[2] is an example of using the two-point random variables as an approximation of the Wiener increments.

On the other hand, the transport theory in such a stochastic system can be treated quantitatively by the correlation-function method[4]. The transport coefficients are then calculated from the time-integration of correlation function between microscopic fluxes carried by each particle. This method is also known as the Green-Kubo formula[5, 6]. In the linear response theory of stochastic process[7], this approach is valid in a variety of transport phenomena, provided that the evolution of a system is dominated by Gaussian probability distributions.

In the present paper, we propose a method for computing neoclassical transport matrix using the Langevin equation and correlation function. Although it is not easy to calculate the correlation function in general, the neoclassical ordering[8] enables us to evaluate it as a function of magnetic surface label. In Sec. 2, we derive the Langevin equation for guiding center motion. Section 3 describes the correlation-function method for neoclassical transport theory, in which we calculate the neoclassical viscosity coefficients[9] as a numerical test. In Sec.4, we summarize the main finding of this work and comment on future studies.

2 Langevin Equation

We now wish to derive the Langevin equation for describing the neoclassical transport in toroidal plasmas. Here, we note that the phase-space coordinates of guiding center are regarded as stochastic variables. The test particle experiences the deterministic friction and random acceleration together with the complex Hamiltonian motion, which is formulated by the drift Hamiltonian theory[10]. If we denote the guiding center coordinate by $z_i\,(i = 1,\ldots,5)$, its Hamiltonian motion is expressed by the Pois-

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son bracket $\{\bullet, \bullet\}$ or Poisson tensor $\omega_{ij} \equiv \{ z_i, z_j \}$ as

$$z_i = \{ z_i, H \} = \omega_{ij} \frac{\partial H}{\partial z_j}, \tag{2}$$

where H is the drift Hamiltonian[10]. If we take into account the Coulomb collisions, equations of motion must be written in terms of SDEs like Eq.(1). Therefore,

$$dz_i(t) = \left[a_i^{(H)}(t, \mathbf{z}) + a_i^{(F)}(t, \mathbf{z}) \right] dt + b_{ij}(t, \mathbf{z})\, dW_j(t), \tag{3}$$

where we divided the deterministic part a_i into the Hamiltonian $a_i^{(H)} = \omega_{ij}\, \partial H / \partial z_j$ and the friction part $a_i^{(F)}$. We note that Eq.(3) becomes non-Hamiltonian and stochastic because of the friction force and random scattering.

In the present work, we retain only the pitch angle scattering in the collision operator. We used the coordinate system $(s, \theta, \zeta, v, \xi)$, where s is the surface label, θ and ζ are the poloidal and toroidal angles, respectively, v is particle velocity, and ξ is the pitch variables. This choice is convenient because, in our case, the particle energy becomes manifestly a constant of motion, and the non-Hamiltonian terms are included only in ξ. We also note that the Boozer coordinates[11] were used to describe the guiding center position. From Eq.(3), we can write the stochastic motion of guiding center by

$$\begin{aligned} ds &= \{ s, H \}\, dt, \\ d\theta &= \{ \theta, H \}\, dt, \\ d\zeta &= \{ \zeta, H \}\, dt, \\ dv &= 0, \\ d\xi &= \{ \xi, H \}\, dt - \nu_D \xi\, dt + \sqrt{(1-\xi^2)\nu_d}\, dW_t, \end{aligned} \tag{4}$$

where ν_D is the deflection frequency. The Poisson brackets appeared in Eqs.(4) represent the guiding center equations used in the orbit calculation. Equations (4) contain the guiding center drifts up to the first order of dimensionless parameter $\epsilon \equiv \rho_L / L$, where ρ_L is the Larmor radius and L_c is the characteristic scale length in the radial direction. In the local and diffusive picture of neoclassical transport[8], however, the effect of finite radial orbit width is neglected so that the particle dynamics relevant to the transport can be expressed only by the terms up to zeroth order. Therefore, we approximate the Poisson brackets up to the lowest order:

$$\begin{aligned} ds &= 0, \\ d\theta &= \frac{v\xi}{B} \frac{\chi'}{\sqrt{g_B}}\, dt, \\ d\zeta &= \frac{v\xi}{B} \frac{\psi'}{\sqrt{g_B}}\, dt, \\ d\xi &= -\frac{v(1-\xi^2)}{2B} \left[\frac{\chi'}{\sqrt{g_B}} \frac{\partial \ln B}{\partial \theta} + \frac{\psi'}{\sqrt{g_B}} \frac{\partial \ln B}{\partial \zeta} \right] dt, \\ &\quad -\nu_D \xi\, dt + \sqrt{(1-\xi^2)\nu_D}\, dW_t, \end{aligned} \tag{5}$$

where χ and ψ are the poloidal and toroidal flux function, respectively, the prime denotes the derivative with respect to the surface label s, and the Jacobian $\sqrt{g_B}$ is of Boozer coordinates. For simplicity, we did not consider the poloidal and toroidal drifts induced by the radial electric field in Eqs.(5). As discussed in the next section, the radial drift term $\{s, H\}$ affects the transport only through the linear response to the radial thermodynamic force. Accordingly, the dynamics of test particles involves only the motion along magnetic field lines. In this approximation, the treatment using the Poisson bracket guarantees conservation of energy.

The numerical solution of Eqs.(5) gives the stochastic motion of test particles. Those ensemble relax to the stationary probability distribution, P_0, within the statistical error due to the finite number of test particles. We use P_0 as the initial condition of test particle distribution when we calculate the correlation function.

3 Correlation Function Method

In this section, we discuss the correlation-function method for calculating the neoclassical transport matrix. The transport matrix determines the phenomenological relation between the macroscopic fluxes and thermodynamic forces[12], and it can be evaluated through the auto- and cross-correlations of the microscopic fluxes.

The correlation function of microscopic fluxes, $\sigma_i(\mathbf{z})$ and $\sigma_j(\mathbf{z})$, is given by

$$R_{ij}(t) = \int d\mathbf{z}\, P_0(\mathbf{z})\, \sigma_i(\mathbf{z}(0))\, \sigma_j(\mathbf{z}(t)), \tag{6}$$

where we assumed that $\sigma_i(\mathbf{z})$ and $\sigma_j(\mathbf{z})$ have the time-reversal symmetry (an even function of pitch variable) and the test particles are distributed with P_0 at $t = 0$. If the thermodynamic force is switched on at $t = 0$, the first-order flux such as radial drifts will be driven as a linear response. The time-dependent transport coefficient, which is called as the running transport coefficient[13], is evaluated by

$$D_{ij}(t) = \int_0^t d\tau\, R_{ij}(\tau). \tag{7}$$

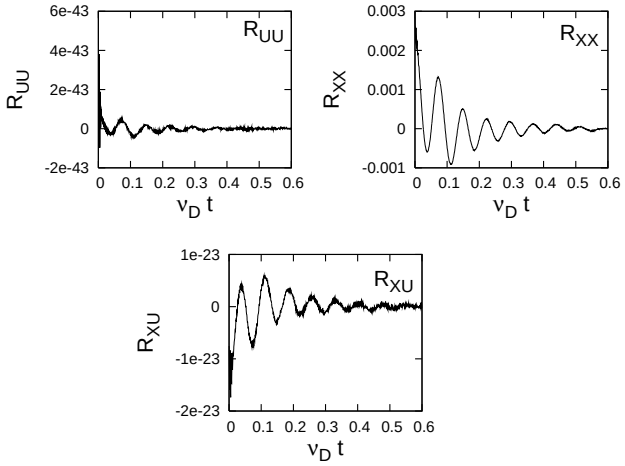

Fig. 1 The correlation functions R_{UU}, R_{XX} and R_{XU} numerically calculated by Eq.(6). The collisionality is $\nu_D/v = 1 \times 10^{-3}$, the banana regime. The horizontal axis denotes the time normalized by ν_D.

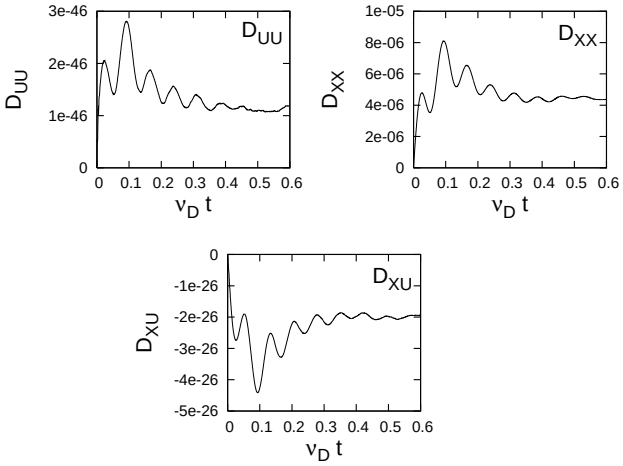

Fig. 2 The running transport coefficients D_{UU}, D_{XX}, and D_{XU} calculated by the time integration in Eq.(7) for the correlation functions in Fig.(1). The horizontal axis denotes the time normalized by ν_D.

This quantity will converge to a finite value if the transport process is characterized by diffusive nature[13].

In Eq.(6), we choose $\sigma_i(\mathbf{z})$ and $\sigma_j(\mathbf{z})$ by[9]

$$\sigma_U \equiv -mv^2 P_2(\xi)\,\mathbf{B}\cdot\nabla \ln B, \quad (8)$$

$$\sigma_X \equiv -v^2 P_2(\xi)\,\frac{B}{\Omega}\left(\tilde{U}\mathbf{b} + \frac{\nabla s\times\mathbf{b}}{B}\right)\cdot\nabla \ln B, \quad (9)$$

where m is the particle mass, $P_2(\xi)$ is the second order Legendre polynomial and $\mathbf{B}$ is the magnetic field, $B = |\mathbf{B}|$, $\mathbf{b} = \mathbf{B}/B$, and Ω is the Larmor frequency. The quantity $\tilde{U}$ is defined by the solution of

$$\mathbf{B}\cdot\nabla(\tilde{U}/B) = \mathbf{B}\times\nabla s\cdot\nabla(1/B^2), \quad \langle B\tilde{U}\rangle = 0. \quad (10)$$

The bracket denotes the surface averaging operation. We calculate the auto- and cross-correlation functions of σ_U and σ_X as

$$\begin{aligned} R_{UU}(t) &= \int d\theta d\zeta d\xi\, P_0\,\sigma_U(t=0)\,\sigma_U(t), \\ R_{XU}(t) &= \int d\theta d\zeta d\xi\, P_0\,\sigma_X(t=0)\,\sigma_U(t), \\ R_{XX}(t) &= \int d\theta d\zeta d\xi\, P_0\,\sigma_X(t=0)\,\sigma_X(t), \end{aligned} \quad (11)$$

where the integrals are evaluated by the Monte Carlo methods. We note that the correlation functions parametrically depend on the initial surface label and particle velocity because $ds = 0$ and $dv = 0$ in Eqs.(5).

The evolution of correlation functions for each pair of fluxes are given in Figs. (1). The magnetic field model used here is the same with that in Ref.[9]. For this case, magnetic field strength is given by $B = B_0[1 - \epsilon_t\cos\theta - \epsilon_h\cos(l\theta - m\zeta)]$ with $\epsilon_t = 0.1$ and $\epsilon_h = 0.01$, where l and m are the poloidal and toroidal field periods, respectively. The collisionality is set as $\nu_D/v = 1\times10^{-3}$ (Banana regime). We observed that the correlation functions decayed within the relaxation time. Figure (2) shows the running transport coefficients, which are denoted by D_{UU}, D_{XU}, and D_{XX}, respectively, were converged to the finite values, asymptotically.

The choice of microscopic fluxes given in Eqs.(8) and (9) is useful because the neoclassical viscosity coefficients[9] can be evaluated by D_{UU}, D_{XU}, and D_{XX}. We calculate the neoclassical viscosity coefficients, L, M, and N by

$$M = \frac{1}{T}D_{UU}\left[1 - \frac{3D_{UU}}{2mT\nu_D K\langle B^2\rangle}\right], \quad (12)$$

$$N = \frac{1}{T}D_{XU}\left[1 - \frac{3D_{UU}}{2mT\nu_D K\langle B^2\rangle}\right], \quad (13)$$

$$L = \frac{1}{T}D_{XX} + \frac{3D_{XU}^2}{2mT^2\nu_D K\langle B^2\rangle} \times\left[1 - \frac{3D_{UU}}{2mT\nu_D K\langle B^2\rangle}\right], \quad (14)$$

where T is the temperature in energy units and $K \equiv mv^2/2T$. Once we obtain the collisionality dependence of L, M, and N (or the geometric factor instead of N, defined by $G_{\rm BS} = -e\langle B^2\rangle N/M$), we can calculate the Onsager-symmetric viscosity matrix using the energy integral[9].

Figures (3), (4) and (5) show the collisionality dependence of normalized viscosity coefficients L^*, M^*, and $G_{\rm BS}$ for the same magnetic field strength with that used in Fig.(1) and (2). The solid lines represent the analytical

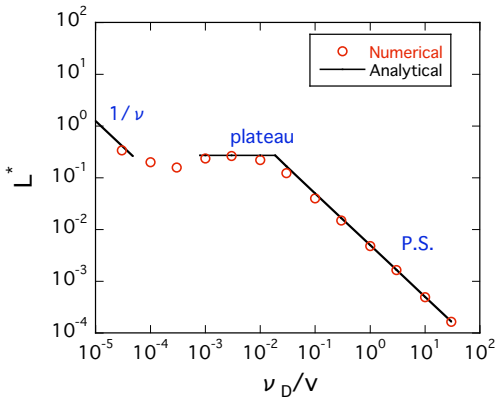

Fig. 3 The collisionality dependence of radial diffusion coefficients $L^* \equiv L/[\frac{1}{2}(v_T/T)(Bv_T/\Omega)^2K^{3/2}]$, where $v_T = \sqrt{2T/m}$ is the thermal velocity.

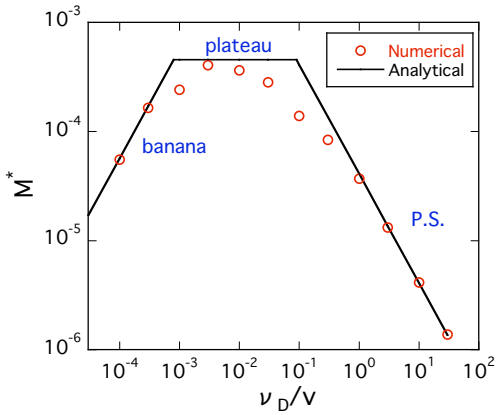

Fig. 4 The collisionality dependence of Parallel viscosity coefficients, $M^* \equiv M/(mv_TK^{3/2})$, against parallel flows.

asymptotic expressions given in Ref.[9]. In these figures, the numerical results obtained by the correlation-function method show reasonable agreements with the analytical values. Therefore, we have concluded that the correlation-function method in Eq.(11) has been successfully verified through the calculation of neoclassical viscosity coefficients.

4 Conclusions

In the present work, we derived the Langevin equation of guiding center motion and developed the method for computing neoclassical transport using the correlation-function method. As noted in Sec.2 and 3, the neoclassical ordering for the Langevin equation is important to ensure the local and diffusive nature of neoclassical transport. We should also mention that, owing to this assumption, the explicit calculation of correlation functions becomes possible.

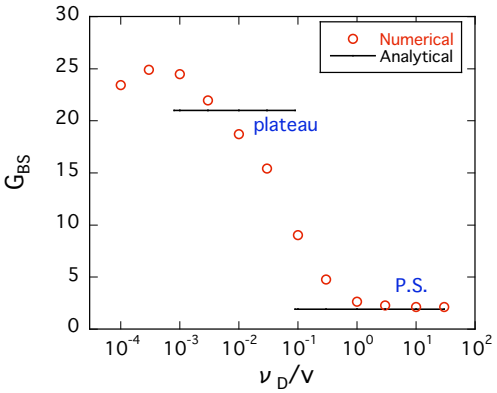

Fig. 5 The collisionality dependence of geometric factor $G_{\rm BS}$.

As a specific example, we evaluated the neoclassical viscosity coefficients, L^*, M^* and $G_{\rm BS}$, which are used in the moment-equation method[9] for obtaining the viscosity-flow relation. We showed that the viscosity coefficients can also be calculated by the correlation-function method, which is based not on the kinetic but on the stochastic approach using the Langevin equation.

In future work, the effect of radial electric field should be taken into account in Eqs.(5) and we will test the method for realistic toroidal MHD equilibria.

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