Proceedings of ITC18, 2008

Algebraic analysis approach for multibody problems II: Variance of velocity changes

Shun-ichi OIKAWA, Koichiro HIGASHI, Hideo FUNASAKA, and Yu KITAGAWA

Graduate School of Engineering, Hokkaido University, Sapporo 060-8628, Japan

The algebraic model (ALG) proposed by the authors has sufficiently high accuracy in calculating the motion of a test particle with all the field particles at rest. When all the field particles are moving, hoewver, the ALG has poor prediction ability on the motion of the test particle initially at rest. None the less, the ALG approximation gives a good results for the statistical quantities, such as variance of velocity changes or the scattering cross section, for a sufficiently large number of Monte Carlo trials.

Keywords: multibody problems, algebraic model

1 Introduction

Since it is difficult to rigorously deal with multibody Coulomb and gravitational collisions, the current classical theory considers them as a series of temporally-isolated binary Coulomb collisions within the Debye sphere. The efficient and fast algorithms to calculate inter-particle forces include the tree method [2, 3], the fast multipole expansion method (FMM) and the particle-mesh Ewalt (PPPM) method [4]. Efforts have been made to use parallel computers, and/or to develop special purpose hardware to calculate interparticle forces, e.g. the GRAPE (GRAvity PipE) project [1].

Some of the authors have developed an algebraic model for multibody problems, and have shown that the momentum transfer cross-section with our model is in good agreement with the exact one [1]. As shown in Fig. 1

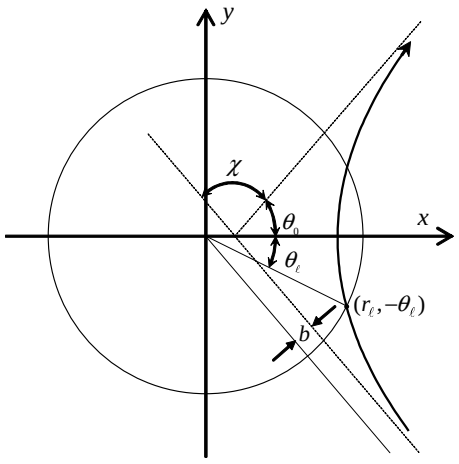

Fig. 1 Unperturbed relative trajectory $r = r(\theta)$ in an orbital plane. The scattering center is at the origin. An impact parameter is $b = b_0 \tan\theta_0$. Interaction region is inside the circle with a radius $r_\ell = \Delta\ell/2$.

author's e-mail: oikawa@fusion.eng.qe.hokudai.ac.jp

its scattering angle, $\chi \equiv \pi - 2\theta_0$, is given by $b = b_0 \tan\theta_0$, where b is the impact parameter, $b_0 \equiv e^2/4\pi\varepsilon_0\mu g_0^2$ corresponds to $\chi = \pi/2$ scattering, and g_0 the initial relative speed at $r = \infty$ and $\theta = -\theta_0$.

The angular component of the equation motion gives the well-known invariant of

$$r^2 \frac{\mathrm{d}\theta}{\mathrm{d}t} = \text{const} = b g_0, \tag{1}$$

and the radial component is given by

$$\frac{\mathrm{d}g_r}{\mathrm{d}t} = \frac{g_0^2 b_0}{r^2}\left(1 + \frac{b_0}{r}\tan^2\theta_0\right), \tag{2}$$

where $g_r \equiv \dot{r}$ denotes the radial velocity. The first term in the parentheses on the right hand side of Eq. (2) stands for the Coulomb force $F_\mathrm{c} \propto r^{-2}$. This force is much smaller for small angle scatterings, i.e. $\chi \ll 1$, than the second term F_a which scales as $\propto r^{-3}$ and results from the conservation of angular momentum Eq.(1), since, at the closest point $r_\mathrm{min} = r(\theta = 0)$ shown in Fig. 1, we have

$$\frac{b_0 \tan^2\theta_0}{r_\mathrm{min}} \simeq \frac{2}{\chi} \gg 1. \tag{3}$$

Thus the main force on the particle is not the Coulomb force F_c, but F_a due to the conservation of angular momentum.

2 Algebraic Approximation for Multibody Interaction

Since the r-dependence on $F_\mathrm{a} \propto r^{-3}$ is steeper than that on $F_\mathrm{c} \propto r^{-2}$, the momentum change in $\mu\boldsymbol{g}$ is almost due solely to F_a near $r = r_\mathrm{min}$. As a consequence, the exact hyperbolic trajectory for the particle can be approximated as a broken line with an impulse force of

$$\mu\Delta\boldsymbol{g} = 2\mu g_0 \cos\theta_0 \boldsymbol{e}_x \tag{4}$$

near the closest point as shown in Fig. 2.

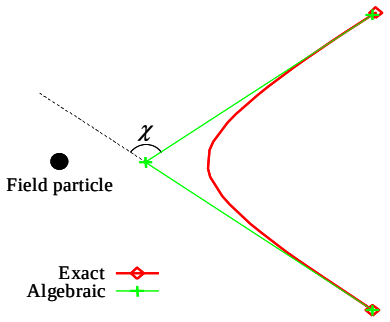

Fig. 2 Algebraic trajectory (broken line) and exact trajectory (curved line) which is a part of a hyperbola. A field particle (black circle) is on the left.

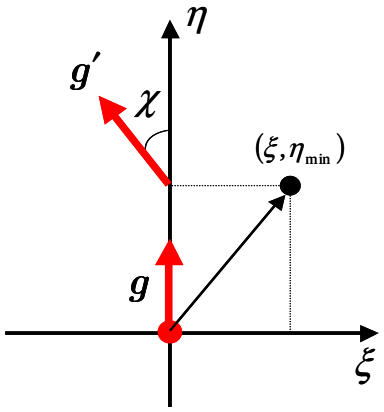

Fig. 3 Coordinate transform from the (x, y) to (ξ, η). In this coordinate system, the scattering angle χ, i.e. the impact parameter b and the time of the interaction Δt are approximately given by ξ and η, respectively.

With this in mind, we have approximated a multibody problem to a series of binary deflections near their closest point as shown in Fig. 2, in which a test particle starts at the lower-right point, and its final point is at the upper-right point due to the interaction with a field particle at rest.

2.1 Coordinate transformation

In order to apply the above binary interaction approximation (ALG) shown in Fig. 2 to multibody cases, first we seek for a field particle that gives the test particle an impulse force *at the earliest time*. For this purpose, it is convenient to transform the coordinate system from $(x,\ y)$ to (ξ, η), in such a way that the initial position of the test particle is at the origin $(\xi, \eta) = (0, 0)$ and the relative velocity $\boldsymbol{g} \equiv \boldsymbol{v}_i - \boldsymbol{v}_j$ is $(g_\xi, g_\eta) = (0, g)$. Then the relative position $\boldsymbol{r}_{ij}$ has an η-coordinate of

$$\eta_{ij} = \left(\boldsymbol{r}_i - \boldsymbol{r}_j\right) \cdot \boldsymbol{g}/g. \tag{5}$$

The particle moves along the η-axis with a constant velocity of g, and is to interact at $\left(0, \eta_{ij}\right)$ with this field particle in a time interval of $\Delta t_{ij} \equiv \eta_{ij}/g$ sec. Accordingly, the field particle that the test particle is given an impulse force at the earliest time has the smallest positive η_{ij}, i.e.

$$\eta_{\min} \equiv \min(\max(0, \eta_{ij})), \text{ for } 1 \le i, j \le N, \tag{6}$$

We have ignored the effect of field particles with $\eta_{ij} < 0$, since the interaction is completed at $\eta = 0$ in our approximation. In other words, such field particles have already interacted with the test particle in the past.

When the test particle moves to the position of $(0, \eta_{\min})$, it changes the relative velocity by $\Delta \boldsymbol{g}_{ij}$ as

$$\Delta \boldsymbol{g}_{ij} = -2g \sin\frac{\chi_{ij}}{2}\, \boldsymbol{e}_\xi, \tag{7}$$

$$\chi_{ij} \simeq 2 \arctan\frac{b_0}{\xi_{ij}}, \tag{8}$$

where the pair i and j satisfies Eq. (6), and we have approximated that the impact parameter is given by $b = \xi_{ij}$ in Eq. (4) as shown in Fig. 3. Thus, in the (ξ, η) coordinate system, the field particle position ξ_{ij} and η_{ij} correspond to the velocity change $\Delta \boldsymbol{g}_{ij}$ and the time of the interaction Δt_{ij}, respectively. This procedure will be repeated until the test particle leaves the prescribed interaction region, i.e. $r < \Delta\ell/2$ as depicted in Fig. 1.

3 Calculation

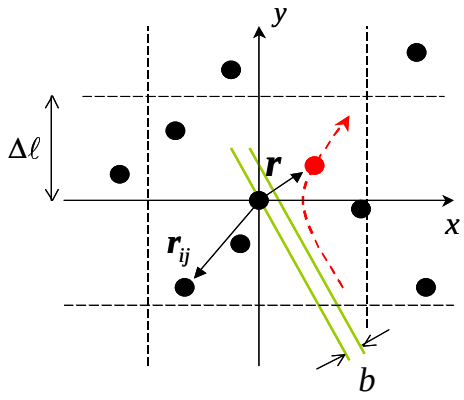

Fig. 4 Typical initial particle positions. The almost-uniformly-distributed black circles are field particles. For Case 1, a gray circle (or red in color) is the test particle at $\boldsymbol{r}$ with an impact parameter b. Initial conditions for Case 2 is that all the field particles are moving, and the test particle locates at the orign at rest.

The numerical results with using the *direct integration method*, DIM, hereafter refers to that obtained by solving the following equation of motion a particle-i with a charge q_i, a mass m_i, and velocity $\boldsymbol{v}_i$ at a position $\boldsymbol{r}_i$

$$m_i \frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = q_i \sum_{j \ne i}^{N} \frac{q_j}{4\pi\varepsilon_0} \frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{\left|\boldsymbol{r}_i - \boldsymbol{r}_j\right|^3}, \tag{9}$$

where $\boldsymbol{r}_j$ are the field particles' positions. As the DIM in this study, we will use the 6-stage 5-th order Runge-Kutta-Fehlberg method known as the RKF65 [7, 8] with the absolute numerical error tolerance of 10^{-16}.

In the following, we will assume that, except a test particle, the field particles on the average are randomly distributed in the phase space $(\boldsymbol{r}, \boldsymbol{v})$. In configuration space, field particles are distributed with the average interparticle separation, $\Delta\ell$. We will consider two cases: all the field particles are fixed at their initial positions, the Case 1, and moving field particles, Case-2. The typical initial condition for Case 1 are depicted in Fig. (4)

3.1 Case 1: All the field particles at rest [5]

In Case 1, all the field particles are at rest, and one of them locates at the origin. The test particle starts from the position of $(b, -\Delta\ell/2)$ with a velocity of $(0, v_0)$. Thus b is the impact parameter against the field particle initially at the origin.

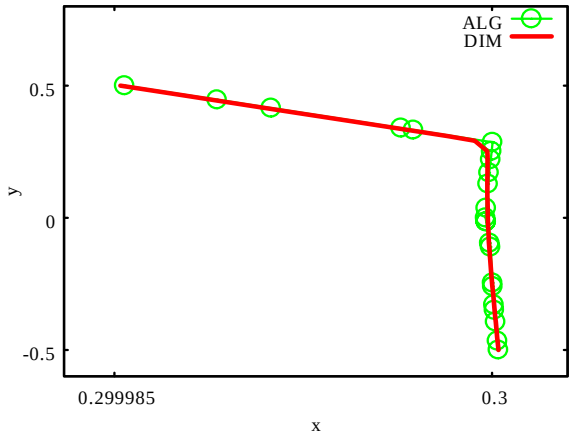

Fig. 5 Comparison of algebraic trajectory (denoted by ALG) and the exact trajectory (denoted by DIM, direc inegration method) in the case of 442-body Coulomb collisions with an impact parameter $b = 0.3\Delta\ell$. Coordinates (x, y) are normalized by $\Delta\ell$. The circles in the figures for the algebraic trajectories stand for the positions at which the test partcile is given the impulse force by one of 441 field particles. See Ref. [5] for more detail.

Figure 5 is an example out of 10^5 Monte Carlo calculations for an impact parameter $b = 0.3\Delta\ell$, and compares the algebraic (ALG) trajectory and the exact (DIM) trajectory normalized by the interparticle separation $\Delta\ell$. Note that the DIM results are accurate up to of the order of 10^{-16} which is the absolute error tolerance adopted. The circles in the figure indicate the positions at which the test partcile is given the impulse force by one of 441 field particles. The algebraic (ALG) approximation agrees well with the direct integration method, DIM, in most cases as shown in Fig. 5.

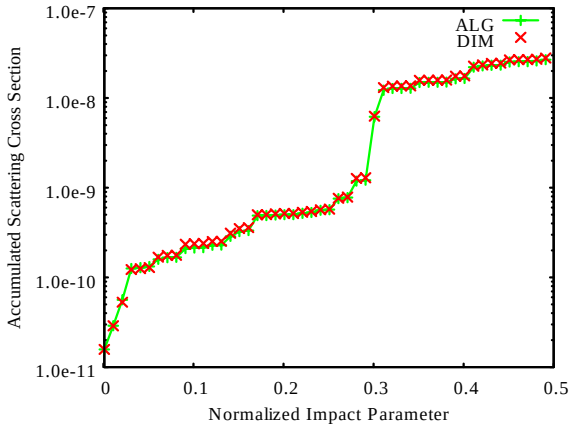

Fig. 6 Accumulated Coulomb scattering cross section $\sigma_{\rm acc}(b)/\Delta\ell^2$ vs normalized impact parameter $\bar{b} = b/\Delta\ell$ in the case of $N = 442$-boby.

Depicted in Fig. 6 is the *accumulated* scattering cross section $\sigma_{\rm acc}(b)$ as a function of the impact parameter b defined by

$$\sigma_{\rm acc}(b) = \int_0^b \left(\frac{\Delta g}{g}\right)^2 \pi b db. \tag{10}$$

The agreement with the exact one is also excellent. It should, however, be noted that all the field particles are at rest throughout the calculation in this case [5]. The CPU time required for the algebraic approximation is only about 20 min using a personal computer, whereas the exact analysis requires 15 hours to integrate the entire set of multibody equations of motion.

3.2 Case 2: Moving field particles.

In Case 2, we will loosen the above restriction on the field particle motion, and have applied the algebraic model to the 10-body problem, in which there are 9 moving field particles and a test particle initially at rest. The change in position $\Delta\boldsymbol{r}$ (results not shown) of the field particles are in good agreement with the exact one, since they are moving so that $\Delta\boldsymbol{r}_i \sim \boldsymbol{v}_i(0)\,\Delta t$ to a good approximation. Although, the absolute value of the change in velocity $|\Delta\boldsymbol{v}|$ of each particle by the ALG are of the same order as the exact one, the orientation of $\Delta\boldsymbol{v}$ are not correct as shown in Fig. 7, in which The test particle is given impluse forces as marked with circles. Also depicted in Fig. 7 is the final point at $t = \Delta t$ by using the BIA, the binary ineraction approximation, proposed by some of the authors [6]. Note that the BIA accurately predicts the final point of the DIM with the absolute error tolerance of 10^{-16}.

In spite of poor accuracy in the individual particle motion, the ALG approximation gives a good results for the statistical quantities, such as variance of velocity changes for a sufficiently large number of Monte Carlo trials. Figure 8 shows the variance of changes in velocity, $\left\langle(\Delta\boldsymbol{g})^2\right\rangle$, of

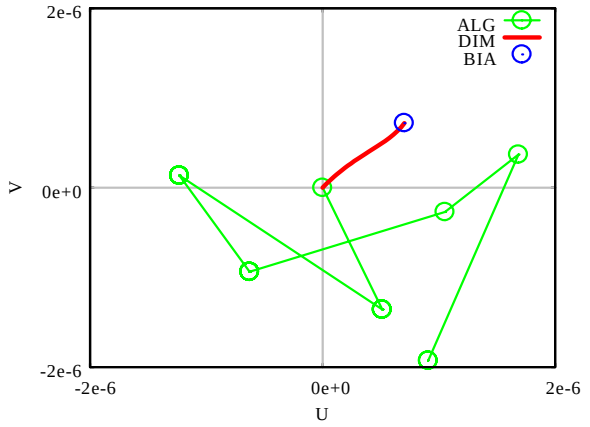

Fig. 7 Case 2: Trajectory of the test particle initially at rest, in the velocity space (U, V) normalized by a thermal speed. There are 9 moving field particles.

the test particle initially at rest, in the case of $N = 10$-boby. For small numbers of trials $N_{\rm MC}$, such as $N_{\rm MC} \sim 2 \times 10^4$ in Fig. 8, the ALG differes significantly from the DIM. Several jumps seen in the figure are due to the close encounters, i.e. the large angle scatterings. The ALG sometimes results in the false close encounters, especially at $N_{\rm MC} \sim 2 \times 10^4$, which have led to numerical errors. Such errors in variance by the ALG becomes smaller for larger $N_{\rm MC} \sim 10^6$, since the large angle scattering seldom occurs in plasmas. The variance calculated by using the BIA, perfectly agrees with the DIM as was shown in Fig. 8.

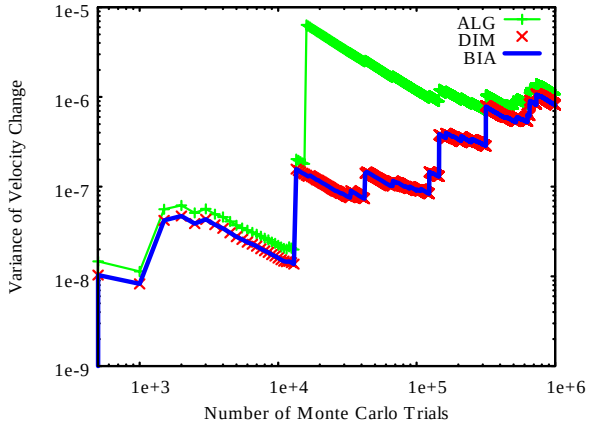

Fig. 8 Variance of the change in velocity of the test particle initially at rest, $\left\langle (\Delta \boldsymbol{g})^2 \right\rangle$, in the case of $N = 10$-boby. The DIA stands for the direct integration method, ALG the algebraic approximation, and BIA the binary ineraction approximation [6].

4 Conclusion

The algebraic model (ALG) proposed by the authors has sufficiently high accuracy in calculating the motion of a test particle with all the field particles at rest. When all the field particles are moving, hoewver, the ALG has poor prediction ability on the motion of the test particle initially at rest. None the less, the ALG approximation gives a good results for the statistical quantities, such as variance of velocity changes or the scattering cross section, for a sufficiently large number of Monte Carlo trials.

Acknowledgement

The author would like to thank Dr. A. Wakasa, Prof. Y. Matsumoto and Prof. M. Itagaki for their fruitful discussions on the subject. The author would also acknowledge the continuous encouragement of the late Prof. T. Yamashina.

[1] J. Makino and M. Taiji, GRAPE-4: A MASSIVELY PARALLEL SPECIAL-PURPOSE COMPUTER FOR COLLISIONAL *N*-BODY SIMULATIONS. APJ **480**, 432–446(1997).

[2] A. W. Appel, An Efficient Program for Many-Body Simulation, *SIAM J. Sci. Stat. Comput.* **6** 85–103 (1985)

[3] J. E. Barnes and P. Hutt, A hierarchical $O(N\log N)$ force-calculation algorithm, *Nature* **324** 446–449 (1986)

[4] P. P. Brieu, F. J. Summers and J. P. Ostriker, Cosmological Simulations Using Special Purpose Computers: Implementing P^3M on GRAPE *ApJ* **453** 566 (1995)

[5] S. Oikawa, H. Funasaka, Algebraic analysis approach for multibody problems. J. Plasma Fusion Res. **36** (2008) S1073.

[6] H. Funasaka, S. Oikawa, K. Higashi, and Y. Kitagawa, Binary Interaction Approximation to *N*-Body Problems, *this conference*, P2-61.

[7] E. Fehlberg. Low-order classical Runge-Kutta formulas with step size control and their application to some heat transfer problems. NASA Technical Report 315 (1969).

[8] E. Fehlberg. Klassische Runge-Kutta-Formeln vierter und niedrigerer Ordnung mit Schrittweiten-Kontrolle und ihre Anwendung auf Warmeleitungsprobleme, *Computing* (Arch. Elektron. Rechnen), vol. 6, (1970) pp. 61–71.