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Preliminary Study on Uncertainty-Driven Plasma Diffusion

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Quantum mechanical plasma diffusion is studied using a semi-classical model with two different characteristic lengths; one is the average interparticle separation, and the other is the magnetic length. The diffusion coefficients D by the model give similar dependence to experiments on many parameters, such as the temperature T, the mass m, the density n, and the magnetic field B. The numerical values of D is larger than that of the neo-classical diffusion.

Keywords: neo-classical theory, anomalous diffusion, quantum mechanical diffusion, uncertainty.

1 Introduction

A classical particle obeys the deterministic equation of motion which gives the particle trajectory in the phase space $(\boldsymbol{r}, \boldsymbol{v})$ at a time t. The actual *trajectory* of a particle with a mass m, however, is stochastic in the phase space with uncertainties in position $\Delta\boldsymbol{r}$, in velocity $\Delta\boldsymbol{v}$, and in energy ΔE in a time interval Δt because of the uncertainty relation:

$$\Delta r \Delta v > \frac{\hbar}{m}, \ \Delta E > \frac{\hbar}{\Delta t}, \tag{1}$$

where $\hbar = 1.05457 \times 10^{-34}$ Joule·sec stands for Planck constant. Equation (1) tells us that (i) lighter particle has larger uncertainty in phase space, and (ii) the uncertainty in energy ΔE is larger for shorter time intervals.

Since, for a given time interval Δt, there are three unknowns Δr, Δv, and ΔE in Eq. (1), we need to find/impose another relation among these uncertainties. For this purpose, let L be a length that the particle travels during some characteristic time interval, i.e. $L \equiv v_0 \Delta t$, where v_0 is the initial particle speed.

In the presence of a uniform magnetic field $\boldsymbol{B}$, the classical particle's energy $E = mv^2/2$ is a constant of the motion:

$$\Delta E = m\Delta\boldsymbol{v} \cdot \left(\boldsymbol{v}_0 + \frac{\Delta\boldsymbol{v}}{2}\right) = 0. \tag{2}$$

In the case of a quantum mechanical particle, ΔE is not necessarily zero, as

$$\Delta E \sim m\boldsymbol{v}_0 \cdot \Delta\boldsymbol{v} > \frac{\hbar}{\Delta t}. \tag{3}$$

Comparing the above with the uncertainty relation in Eq. (1), we have

$$\Delta r < L, \quad \Delta v > \frac{\hbar}{mL}. \tag{4}$$

Thus, the square of the uncertainty in the cyclotron center $\boldsymbol{r}_G = \boldsymbol{r} + \boldsymbol{v} \times \boldsymbol{\omega}/\omega^2$ is given by

$$(\Delta\boldsymbol{r}_G)^2 = (\Delta\boldsymbol{r})^2 + \left(\frac{m\Delta\boldsymbol{v}}{qB}\right)^2 \sim L^2 + \left(\frac{\hbar}{qBL}\right)^2. \tag{5}$$

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2 Semi-Classical Model for Motion

Let us assume that a charged particle with a positive charge $q > 0$ is moving in the presence of a uniform magnetic field $\boldsymbol{B} = (0, 0, B)$ in the z-direction. First, we integrate the equation of motion for the classical particle for the time interval of Δt to get the classical position in the phase space $(\boldsymbol{r}(\Delta t), \boldsymbol{v}(\Delta t))$.

$$\boldsymbol{r}(\Delta t) = \boldsymbol{r}(0) + \int_0^{\Delta t} \boldsymbol{v}(t)\ dt, \tag{6}$$

$$\boldsymbol{v}(\Delta t) = \boldsymbol{v}(0) + \int_0^{\Delta t} \boldsymbol{v}(t) \times \boldsymbol{\omega}\ dt, \tag{7}$$

where $\boldsymbol{\omega} = q\boldsymbol{B}/m$ is the cyclotron frequency vector.

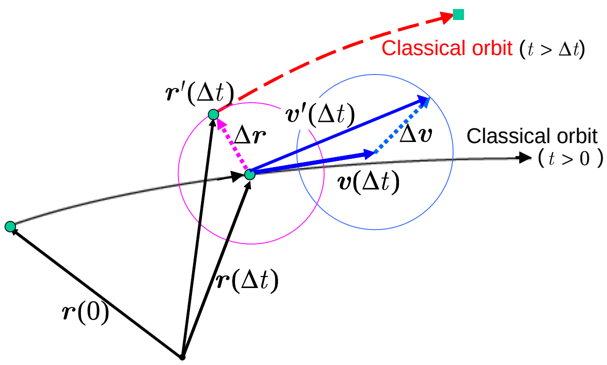

Fig. 1 Semi classical model for quantum motion. The particle initially at $r(0)$ *classically* moves to $\boldsymbol{r}(\Delta t)$ with a velocity $\boldsymbol{v}(\Delta t)$ at $t = \Delta t - 0$. At this time, it suffers the quantum mechanical deviations in position, $\Delta\boldsymbol{r}$, and in velocity $\Delta\boldsymbol{v}$. The particle is at $\boldsymbol{r}'(\Delta t)$ with a velocity $\boldsymbol{v}'(\Delta t)$ at the time $t = \Delta t + 0$.

As shown in Fig. 1, next we add the randomly-oriented uncertainties $\Delta\boldsymbol{r}$, and $\Delta\boldsymbol{v}$ to $\boldsymbol{r}(\Delta t)$, and $\boldsymbol{v}(\Delta t)$, the magnitude of which is given by Eq. (4), as

$$\boldsymbol{r}'(\Delta t) = \boldsymbol{r}(\Delta t) + \Delta \boldsymbol{r}, \quad (8)$$

$$\boldsymbol{r}'(\Delta t) = \boldsymbol{v}(\Delta t) + \Delta \boldsymbol{v} \quad (9)$$

This procedure is repeated until the time t reaches $\tau_c \equiv 2\pi/\omega$, i.e. the cyclotron period.

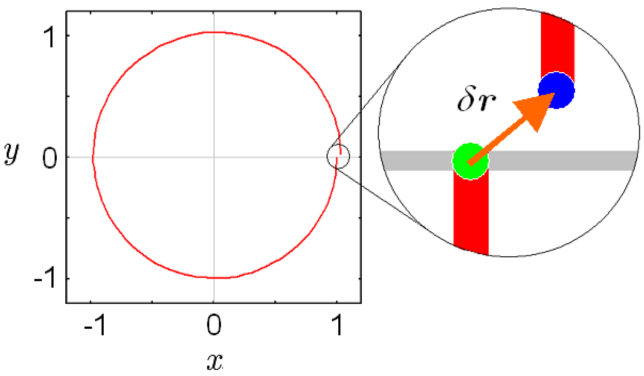

Fig. 2 Deviation of cyclotron motion, $\delta \boldsymbol{r} \equiv \boldsymbol{r}(\tau_c) - r(0)$, due to uncertainty in one gyration for a given characteristic length $L = v_0 \Delta t$. Lengths are normalized by the cyclotron radius $\rho = m v_0 / qB$.

Figure 2 shows the particle trajectory during one cyclotron period, in which a deviation $\delta \boldsymbol{r} \equiv \boldsymbol{r}(\tau_c) - \boldsymbol{r}(0)$ from the classical motion is seen.

In the following subsections we will choose the average interparticle separation, $\Delta \ell \equiv n^{-1/3}$, and the magnetic length [1], $\ell_B \equiv \sqrt{\hbar/qB}$, as the characteristic length L.

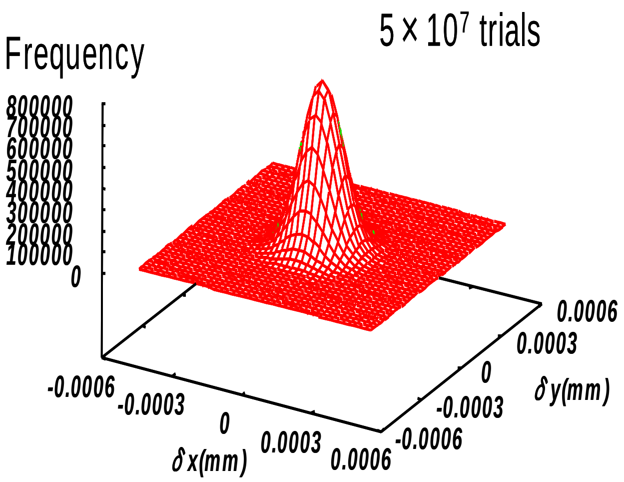

Fig. 3 Histgram for $\delta \boldsymbol{r} = (\delta x, \delta y)$ for $N_{\mathrm{MC}} = 5 \times 10^7$ Monte Carlo trials.

It should be noted that the magnetic length $\ell_B = \sqrt{\hbar/qB}$ is the spatial size of wave packet in the plane perpendicular to the magnetic field [1], i.e.

$$|\psi(\boldsymbol{r}_\perp, t)|^2 = \frac{1}{\pi \ell_B^2} \exp\left[-\frac{(\boldsymbol{r}_\perp - \langle \boldsymbol{r}_\perp(t) \rangle)^2}{\ell_B^2} \right], \quad (10)$$

where $\psi(\boldsymbol{r}_\perp, t)$ stands for the wavefunction, $\langle \boldsymbol{r}_\perp(t) \rangle$ the classical position of the particle in the plane perpendicular to $\boldsymbol{B}$.

From many Monte Carlo calculations (typipically $N_{\mathrm{MC}} \sim 10^4$ turns out to be enough in this study for convergence) of such the diffusion coefficient

$$D \sim \frac{\left\langle (\delta \boldsymbol{r})^2 \right\rangle}{\tau_c} \quad (11)$$

will be obtained for a particular choice of the characteristic length L, where $\langle \cdot \rangle$ stands for the ensemble average. Figure 3 shows the histgram of $\delta \boldsymbol{r}$ for $N_{\mathrm{MC}} = 5 \times 10^7$ Monte Carlo trials, which resembles the probability density function of a wavefunction in quantum mechanics.

2.1 CASE-A: L = interparticle separation, $\Delta \ell$

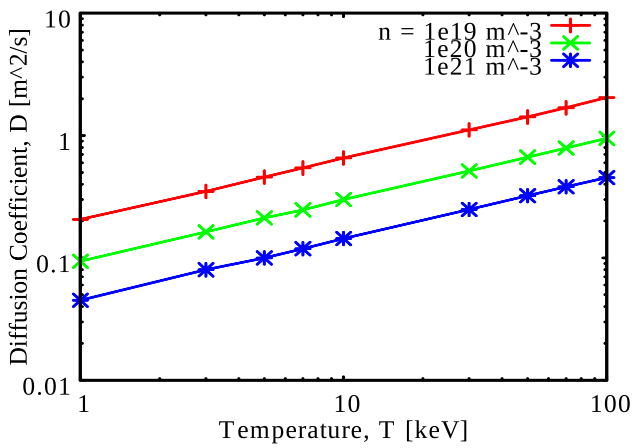

Fig. 4 CASE-A: Temperature T and density n dependence of diffusion coefficient D [m^2/s] in the case of $L = \Delta \ell$, the interparticle separation.

If we choose the characteristic length $L \equiv \Delta \ell$, where $\Delta \ell \equiv n^{-1/3}$ stands for the average interparticle separation, then the uncertainty in energy is given as $\Delta E \sim m v_0 \Delta v$. Thus, from Eq. (4), we have

$$\Delta r \sim \Delta \ell, \text{ and } \Delta v \sim \frac{\hbar}{m \Delta \ell}. \quad (12)$$

The particle is assumed to be in typical fusion plasmas of $T = 1$–100 keV, and $n = 10^{19}$–10^{21} m^{-3}, and $B = 1$–10 Tesla. The initial particle speed v_0 is selected as the thermal speed $v_{\mathrm{th}} = \sqrt{2T/m}$. The above calculation for a fixed T, n, and B is repeated $N_{\mathrm{MC}} = 10^4$ times.

Figure 4 shows the temperature and density dependence of the diffusion coefficient $D = D(T, n)$, which leads to the scaling of

$$D_{\mathrm{CASE\text{-}A}} \sim 0.094 \sqrt{\frac{T_{\mathrm{keV}}}{A}} \left(\frac{10^{20}}{n} \right)^{\frac{1}{3}} \propto \sqrt{\frac{T}{m}} n^{-1/3}, \quad (13)$$

where $A = m/m_p$ is the mass number with m_p being proton mass. It is interesting to note that the diffusion coefficient D does not depend on the magnetic field B, but on the particle mass $m^{-1/2}$. The latter is known as the isotope effect [3, 4] .

2.2 CASE-B: L = magnetic length, ℓ_B.

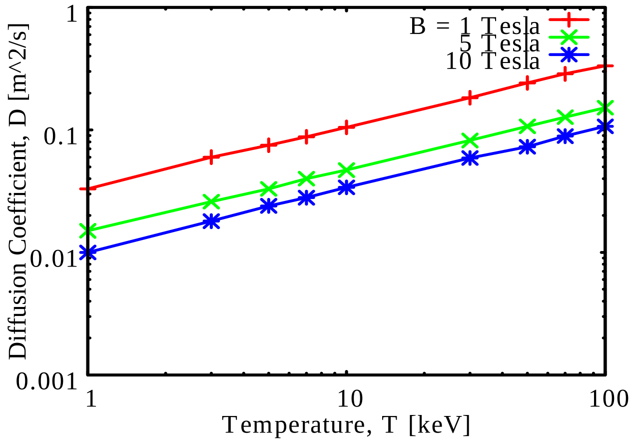

Fig. 5 CASE-B: Temperature T and the magnetic field B dependence of diffusion coefficient D [m^2/s]in the case of $L = \ell_B$, the magnetic length.

The uncertainties for $L = \ell_B$, say the CASE-B, are

$$\Delta r \sim \ell_B, \text{ and } \Delta v \sim \frac{\hbar}{m\ell_B}. \tag{14}$$

The particle is assumed to be in typical fusion plasmas of $T = 1 - 100$ keV, and $B = 1 - 10$ Tesla. Note that the density n does not enter into this case. Monte Carlo calculations similar to the CASE-A are made. Figure 5 shows the temperature T and the magnetic field B dependencies of the diffusion coefficient D, which leads to the scaling of

$$D_{\mathrm{CASE-B}} \sim 0.033\sqrt{\frac{T_{\mathrm{keV}}}{AB}} \propto \sqrt{\frac{T}{mB}}, \tag{15}$$

in which $\sqrt{T/m}$ scaling is the same as Eq. (13) for the CASE-A.

3 Discussion

Table 1 summarizes the dependence of D on plasma parameters, such as T, n and B. The parameters' ranges are $1 \leq T \leq 100$ keV, $1 \leq B \leq 10$ Tesla, and if applicable $10^{19} \leq n \leq 10^{21}$ m^{-3}. The ITER-89 dependence of D in the table assumes $D \propto \rho^2/\tau_E$, where τ_E is the energy confinement time in the ITER-89 scaling law. The diffusion coefficients D by these models give similar dependence to experiments on many parameters, such as the temperature T, the mass m, the density n, and the magnetic field B, as well as its values of the order of the anomalous diffusion.

Table 1 Parameter dependence of D, The parameters' ranges are $1 \leq T \leq 100$ keV, $10^{19} \leq n \leq 10^{21}$ m^{-3}, $1 \leq B \leq 10$ Tesla. The ITER-89 dependence of D in the table assumes $D \sim \rho^2/\tau_E$.

Model	$D \propto T^\alpha m^\beta n^\gamma B^\delta \cdots$	Values
ITER-89	$\sqrt{\frac{P}{m}}\, n^{-0.1} B^{-0.2}$	~ 1 m^2/s
CASE-A: $\Delta\ell = n^{-1/3}$	$\sqrt{\frac{T}{m}}\, n^{-1/3} B^0$	0.1-2
CASE-B: $\ell_B = \sqrt{\frac{\hbar}{qB}}$	$\sqrt{\frac{T}{m}}\, n^0\, B^{-0.5}$	0.01-0.3
Neo-classical	$\sqrt{\frac{m}{T}}\, n\, B^{-2}$	~ 0.01

In the case of the most common fusion plasma (e.g. $T = 10$ keV, $n = 10^{20}$ m^{-3} and $B = 3$ T), however, the value of the diffusion coefficients by our model are $D_{\mathrm{CASE-A}} \sim 0.30$ m^2/s and $D_{\mathrm{CASE-B}} \sim 0.08$ m^2/s, both of which are one order smaller than the anomalous diffusion. Moreover, which characteristic length L is right one is an open question.

From Eq. (5), the diffusion coefficient D_G of the gyration center is

$$D_G \sim \frac{\langle(\Delta \boldsymbol{r}_G)^2\rangle}{\tau_c} = v_0 L \left\{1 + \left(\frac{\hbar}{qBL^2}\right)^2\right\}. \tag{16}$$

For $L \gg \ell_B = \sqrt{\hbar/qB}$, the diffusion of the gyration center is determined by L, i.e. $D_G \sim v_0 L$, otherwise $D_G \sim v_0 \ell_B^4/L^3$. In most fusion plasmas, the interparticle separation is much shorter than the magnetic length, i.e. $\Delta\ell \gg \ell_B$, so that $D_G \sim v_0 \ell_B$. This leads to $\sqrt{T/m}$ scaling of the diffusion coefficient D, irrespective of the selection of the length scale L.

4 Conclusion

The diffusion coefficients D by our model give similar dependence to experiments on many parameters, such as the temperature T, the mass m, the density n, and the magnetic field B, as well as its values larger than the neo-classical diffusion.

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