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Quasisymmetric toroidal plasmas with large mean flows

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Geometric conditions for quasisymmetric toroidal plasmas with large mean flows on the order of the ion thermal speed are investigated. Equilibrium momentum balance equations including the inertia term due to the large flow velocity are used to show that, for rotating quasisymmetric plasmas with no local currents crossing flux surfaces, all components of the metric tensor should be independent of the toroidal angle in the Boozer coordinates, and consequently these systems need to be rigorously axisymmetric. Unless the local radial currents vanish, the Boozer coordinates do not exist and the toroidal flow velocity cannot take any value other than a very limited class of eigenvalues corresponding to very rapid rotation especially for low beta plasmas.

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I. INTRODUCTION

There have been numerous studies on plasma flows as an attractive means for improving plasma confinement [1]. Sheared $\mathbf{E} \times \mathbf{B}$ flows are considered to cause the reduction of transport observed in internal transport barriers (ITBs) [2] and high-confinement modes (H-modes) [3]. Microscopic oscillatory flow structures called zonal flows also play important roles in regulating microturbulence [4, 5]. For axisymmetric systems, large mean flows on the order of the ion thermal velocity v_T can be produced in the toroidal direction, and the toroidal flows are counted on to avoid dangerous instabilities such as resistive wall modes in tokamaks [6]. Similarly, in nonaxisymmetric or helical systems such as stellarators and heliotrons [7], plasma mean flows and zonal flows are closely connected to both neoclassical and turbulent transport processes [8–15] and it is important to investigate conditions for magnetic geometry to allow large flow velocities.

It was shown by Helander [16] that, in order for the large mean flows of $\mathcal{O}(v_T)$ to occur in general three-dimensional configurations, the magnetic field strength should be given by a function of the flux surface label and the arc length along the field line. This condition is called isometry [17] or omnigenity [18] and it is satisfied by the quasisymmetric magnetic configurations [19, 20] where the magnetic field strength is given as a function of the radial coordinate and one of or a linear combination of the toroidal and poloidal angle coordinates. If flow velocities are of $\mathcal{O}(\delta v_T)$, where the drift ordering parameter δ is given by the ratio of the thermal gyroradius ρ_T to the equilibrium gradient scale length L, bounce-averaged radial drift velocities of ripple-trapped particles vanish and accordingly neoclassical transport is significantly reduced in quasisymmetric and quasi-omnigenous systems [18]. Therefore, these systems have been intensively investigated as advanced concepts of helical devices [21–26]. It is also shown for quasisymmetric toroidal configurations with mean flows of $\mathcal{O}(\delta v_T)$ that neoclassical particle fluxes are intrinsically ambipolar [9, 27, 28], which implies that the radial electric field and accordingly the mean flow velocity in the direction of quasisymmetry can freely take any values even if the background density and temperature gradients are fixed. Recently, Simakov and Helander [29] treated rapid plasma rotation in quasisymmetric systems although they still assumed the plasma flow Mach number to be low in order to neglect effects of the inertia term due to the rapid flow on the equilibrium force balance.

In this paper, geometric conditions for quasisymmetric toroidal plasmas with large mean flows on the order of the ion thermal speed are investigated in detail by including the flow inertia term in the force balance. We hereafter consider quasi-axisymmetric systems although results shown in the present work can be straightforwardly extended to general quasisymmetric systems. Our previous study [30] showed that, for such rapidly-rotating quasisymmetric plasmas, an additional symmetry condition for a component of the metric tensor need to be satisfied in order for the Boozer coordinates [31] to exist.

In Sec. II, it is shown that, for quasisymmetric magnetic configurations with the large mean flows of $\mathcal{O}(v_T)$ in the direction of quasisymmetry, the equilibrium densities, temperatures, electrostatic potential, angular flow velocity, and the Jacobian associated with the flux coordinates are all independent of ζ, where ζ is used as the angle coordinate associated with quasisymmetry. Here, the temperatures and electrostatic potential should be flux-surface functions to the lowest order in δ and the potential needs to be independent of ζ up to the next order, which shows a contrast to the case of Ref. [29] where the potential is assumed to break this symmetry condition. We also find that local currents crossing flux surfaces should vanish or large mean flows cannot take any values except for impractical eigenvalues. For the case of no local radial currents, we can use the Boozer coordinates to represent several geometric constraints imposed by the equilibrium momentum balance equations including the inertia due to the large flows.

In Sec. III, additional conditions for the metric tensor are obtained by using formulas of differential geometry [32] for toroidal flux surfaces. The geometric equations from Sec. III are solved in Sec. IV to show that, for quasisymmetric toroidal plasmas with large mean flows of $\mathcal{O}(v_T)$, all components of the metric tensor should be independent of ζ. This consequence is strongly restrictive and it leads to the conclusion that the rapidly-rotating quasisymmetric systems should be rigorously axisymmetric. Finally, Sec. V summarizes results from the present work and Appendices A, B, and C are added to give explanations about eigenvalues of the flow velocities, transformation of flux coordinates, and basics of differential geometry for surfaces, respectively.

II. EQUILIBRIA OF QUASISYMMETRIC ROTATING PLASMAS

We consider toroidal plasmas, in which the magnetic field $\mathbf{B}$ is written in terms of the flux coordinates (s, θ, ζ) as

$$\mathbf{B} = \psi' \nabla s \times \nabla \theta + \chi' \nabla \zeta \times \nabla s = B_s \nabla s + B_\theta \nabla \theta + B_\zeta \nabla \zeta, \tag{1}$$

where s is an arbitrary label of a flux surface, θ and ζ represent the poloidal and toroidal angles, respectively, and $' \equiv \partial/\partial s$ denotes the derivative with respect to s. The toroidal and poloidal fluxes within the volume inside the surface with the label s are given by $2\pi\psi(s)$ and $2\pi\chi(s)$, respectively. The contravariant components of the magnetic field $\mathbf{B}$ are given from Eq. (1) as

$$B^s = 0, \quad B^\theta = \chi'/\sqrt{g}, \quad B^\zeta = \psi'/\sqrt{g}, \tag{2}$$

where $\sqrt{g} \equiv [\nabla s \cdot (\nabla \theta \times \nabla \zeta)]^{-1}$ represents the Jacobian for the flux coordinates (s, θ, ζ).

Hereafter, we investigate quasisymmetric toroidal systems with large mean flows on the order of the ion thermal velocity v_T. The $\mathcal{O}(v_T)$ equilibrium flow should be tangential to the direction of quasisymmetry, in which the field strength B is uniform. Here, for simplicity, we restrict our consideration to the quasi-axisymmetric case, in which the magnetic field strength B is independent of the toroidal angle ζ,

$$\partial B/\partial \zeta = 0, \quad B = B(s, \theta), \tag{3}$$

although we can treat general quasisymmetric cases such as quasi-poloidally-symmetric and quasi-helically-symmetric ones in the same way as shown below. When using the perturbative expansion in terms of the drift ordering parameter δ defined by the ratio of the ion thermal gyroradius ρ_T to the equilibrium gradient scale length L, the lowest-order momentum balance equation reduces to

$$\mathbf{E}_0 + \frac{\mathbf{V}_0}{c} \times \mathbf{B} = 0. \tag{4}$$

Here, the lowest-order electric field is given by $\mathbf{E}_0 = -\nabla \Phi_0(s) = -\Phi_0'(s) \nabla s$ and the lowest-order electrostatic potential $\Phi_0(s)$ is a flux-surface function satisfying $e_a \Phi / T_a = \mathcal{O}(\delta^{-1})$, where e_a is regarded as a quantity of $\mathcal{O}(\delta^{-1})$. In the same way as shown in Refs. [16, 33] we find from the lowest-order kinetic equation that the equilibrium flow velocity $\mathbf{V}_0$ of $\mathcal{O}(v_T)$, which consists of the $\mathbf{E} \times \mathbf{B}$ drift and the parallel flow components, should be represented by

$$\mathbf{V}_0 = V^\zeta \frac{\partial \mathbf{x}}{\partial \zeta}, \qquad V^\zeta = -c \frac{\Phi_0'(s)}{\chi'(s)}, \tag{5}$$

and that the following incompressibility condition and other constraints hold,

$$\begin{aligned}
&\nabla \cdot \mathbf{V}_0 = \mathbf{b} \cdot \nabla \mathbf{V}_0 \cdot \mathbf{b} = 0, \\
&\nabla \cdot \frac{\partial \mathbf{x}}{\partial \zeta} = 0, \qquad \frac{\partial \sqrt{g}}{\partial \zeta} = 0 \\
&\mathbf{V}_0 \cdot \nabla n_a = \mathbf{V}_0 \cdot \nabla T_a = 0 \\
&\mathbf{B} \cdot \nabla T_a = 0 \\
&n_a = n_a(s, \theta), \qquad T_a = T_a(s)
\end{aligned} \tag{6}$$

where n_a and T_a are the lowest-order density and temperature of the particle species a, respectively. In the presence of the large flow velocity $\mathbf{V}_0$, n_a is not uniform over the flux surface while T_a is still a flux-surface function as in the case of the conventional equilibrium without $\mathbf{V}_0$. We also see from Eq. (6) that the density n_a and the Jacobian $\sqrt{g}$ are independent of ζ like the field strength B.

Using Eqs. (4) and (5) we have

$$\nabla \times \left(\frac{\partial \mathbf{x}}{\partial \zeta} \times \mathbf{B} \right) = c \nabla \times \left(\frac{\Phi_0'(s)}{V^\zeta(s)} \nabla s \right) = 0, \tag{7}$$

which leads to

$$\frac{\partial \mathbf{x}}{\partial \zeta} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \frac{\partial \mathbf{x}}{\partial \zeta}. \tag{8}$$

Equation (8) is also derived from

$$\frac{\partial B^\theta}{\partial \zeta} = \frac{\partial B^\zeta}{\partial \zeta} = 0 \tag{9}$$

which are immediately obtained from Eq. (2) and $\partial \sqrt{g}/\partial \zeta = 0$ in Eq. (6). Then, using $B^2 = B^\theta B_\theta + B^\zeta B_\zeta$, $\partial B/\partial \zeta = 0$, and Eq. (9), we find

$$\frac{\partial B_\theta}{\partial \zeta} + q \frac{\partial B_\zeta}{\partial \zeta} = 0, \tag{10}$$

where $q \equiv B^\zeta / B^\theta = \psi'(s)/\chi'(s)$ represents the safety factor.

Each term of the lowest-order momentum balance equation shown in Eq. (4) has no component in the direction parallel to the magnetic field. To the next order, the parallel momentum balance equation for the particle species a is written as

$$\mathbf{B} \cdot (n_a e_a \mathbf{E}_1 - \nabla p_a - n_a m_a \mathbf{V}_0 \cdot \nabla \mathbf{V}_0) = 0, \tag{11}$$

where m_a denotes the mass of the particle species a, the electric field $\mathbf{E}_1 = -\nabla\Phi_1$ is given by the first-order electrostatic potential Φ_1 and the pressure is represented by $p_a = n_a T_a$. Here, $e_a\Phi_1/T_a = \mathcal{O}(1)$ and we generally have $\mathbf{B}\cdot\nabla\Phi_1 \neq 0$ in contrast to $\mathbf{B}\cdot\nabla\Phi_0 = 0$. Using Eqs. (6), (8), and (11), we get

$$\mathbf{B}\cdot\nabla\left(\ln n_a + \frac{e_a}{T_a}\Phi_1 - \frac{m_a}{2T_a}(V^\zeta)^2 g_{\zeta\zeta}\right) + \frac{m_a}{T_a}(V^\zeta)^2\frac{\partial B_\zeta}{\partial\zeta} = 0, \tag{12}$$

where $g_{\zeta\zeta} \equiv |\partial\mathbf{x}/\partial\zeta|^2$. For electrons ($a = e$), Eq. (12) is written by neglecting the electron mass m_e as

$$\mathbf{B}\cdot\nabla\left(\ln n_e - \frac{e}{T_e}\Phi_1\right) = 0, \tag{13}$$

which implies that $(\ln n_e - e\Phi_1/T_e)$ is a flux-surface function. Therefore, Φ_1 is independent of ζ like n_e,

$$\partial\Phi_1/\partial\zeta = 0, \qquad \Phi_1 = \Phi_1(s,\theta). \tag{14}$$

This result is contrastive to the study by Simakov and Helander [29] for rotating quasisymmetric plasmas with low Mach number flows, in which non-intrinsically ambipolar particle fluxes due to the symmetry-breaking electrostatic potential are considered. We now define the ζ-averaged part $\overline{A} = (2\pi)^{-1}\oint A d\zeta$ and the ζ-dependent part $\widetilde{A} = A - \overline{A}$ for an arbitrary function A of ζ. Then, Eq. (12) is rewritten as

$$\begin{aligned} B^\theta\frac{\partial}{\partial\theta}&\left(\ln n_a + \frac{e_a}{T_a}\Phi_1 - \frac{m_a}{2T_a}(V^\zeta)^2\overline{g_{\zeta\zeta}}\right) \\ &+ \frac{m_a}{T_a}(V^\zeta)^2\left(-\frac{1}{2}\mathbf{B}\cdot\nabla\widetilde{g_{\zeta\zeta}} + \frac{\partial B_\zeta}{\partial\zeta}\right) = 0, \end{aligned} \tag{15}$$

which is separated into the ζ-averaged and ζ-dependent parts,

$$\begin{aligned} &B^\theta\frac{\partial}{\partial\theta}\left(\ln n_a + \frac{e_a}{T_a}\Phi_1 - \frac{m_a}{2T_a}(V^\zeta)^2\overline{g_{\zeta\zeta}}\right) = 0, \\ &-\frac{1}{2}\mathbf{B}\cdot\nabla\widetilde{g_{\zeta\zeta}} + \frac{\partial B_\zeta}{\partial\zeta} = 0. \end{aligned} \tag{16}$$

The species summation of the equilibrium force balance is written to the lowest order as

$$\left(\sum_a n_a m_a\right)\mathbf{V}_0\cdot\nabla\mathbf{V}_0 = \frac{1}{c}\mathbf{J}\times\mathbf{B} - \nabla p, \tag{17}$$

where the current density $\mathbf{J}$ is given by Ampère's law $\nabla\times\mathbf{B} = (4\pi/c)\mathbf{J}$ and $p = \sum_a p_a$ is the total kinetic pressure. The inertia term resulting from the flow velocity $\mathbf{V}_0$ is included in Eq. (17). Then, taking the inner product between Eq. (17) and $\partial\mathbf{x}/\partial\zeta$, we obtain

$$\frac{1}{2}\left(\sum_a n_a m_a\right)(V^\zeta)^2\frac{\partial g_{\zeta\zeta}}{\partial\zeta} = \frac{\chi'}{c}J^s = \frac{B^\theta}{4\pi}\left(\frac{\partial B_\zeta}{\partial\theta} - \frac{\partial B_\theta}{\partial\zeta}\right), \tag{18}$$

In the rigorous axisymmetric case, $\partial g_{\zeta\zeta}/\partial\zeta = 0$ holds although this condition is not trivially satisfied for the quasi-axisymmetric case. If $\partial g_{\zeta\zeta}/\partial\zeta \neq 0$, then Eq. (18) leads to nonzero local radial current $J^s \neq 0$. This gives rise to a serious problem because the quasisymmetric system is considered usually by using the Boozer coordinates while the Boozer coordinates cannot be constructed for the case of $J^s \neq 0$. Let us consider this point below in more detail. Differentiating Eq. (18) with respect to ζ and using Eqs. (10) and (16), we obtain

$$(\mathbf{B}\cdot\nabla)^2\widetilde{g_{\zeta\zeta}} = 4\pi\left(\sum_a n_a m_a\right)(V^\zeta)^2\frac{\partial^2\widetilde{g_{\zeta\zeta}}}{\partial\zeta^2}. \tag{19}$$

It is shown in Appendix A that nontrivial solutions of Eq. (19) exist only if the toroidal flow velocity $V^\zeta(s)$ takes one of eigenvalues determined by Eq. (A9) for each flux surface. As discussed in Appendix A, this constraint on V^ζ is so restrictive that we hereafter assume a trivial solution $\widetilde{g_{\zeta\zeta}} = 0$ of Eq. (19) and accordingly

$$\partial g_{\zeta\zeta}/\partial\zeta = 0. \tag{20}$$

Then, we find from Eq. (18) that the radial current vanishes,

$$J^s = 0. \tag{21}$$

Appendix B shows that, for the case of $J^s = 0$, there exist the Boozer coordinates in which the covariant poloidal and toroidal components, $B_\theta(s)$ and $B_\zeta(s)$, of the magnetic field are flux-surface functions. It is also shown in Appendix B that, without loss of generality, we can regard the flux surface (s,θ,ζ) as the Boozer coordinates from the beginning for systems considered in this section. Thus, we hereafter assume (s,θ,ζ) to be the Boozer coordinates. Using Eq. (20) and the relations between the covariant and contravariant components of the magnetic field,

$$B_\theta(s) = g_{\theta\theta}B^\theta + g_{\theta\zeta}B^\zeta, \qquad B_\zeta(s) = g_{\theta\zeta}B^\theta + g_{\zeta\zeta}B^\zeta \tag{22}$$

we find that the components $g_{\theta\theta}$, $g_{\theta\zeta}$, and $g_{\zeta\zeta}$ of the metric tensor are all independent of ζ,

$$\partial g_{\theta\theta}/\partial\zeta = \partial g_{\theta\zeta}/\partial\zeta = \partial g_{\zeta\zeta}/\partial\zeta = 0. \tag{23}$$

Then, the contravariant metric tensor component $g^{ss} = [g_{\theta\theta}g_{\zeta\zeta} - (g_{\theta\zeta})^2]/g$ is independent of ζ, too,

$$\partial g^{ss}/\partial\zeta = 0. \tag{24}$$

Using Eq. (23) and taking the covariant poloidal and radial components of Eq. (17), we obtain

$$\frac{1}{2}\left(\sum_a n_a m_a\right)(V^\zeta)^2\frac{\partial g_{\zeta\zeta}}{\partial\theta} = \frac{\partial p}{\partial\theta}, \tag{25}$$

and

$$\left(\sum_a n_a m_a\right)(V^\zeta)^2\left(\frac{\partial g_{s\zeta}}{\partial\zeta}-\frac{1}{2}\frac{\partial g_{\zeta\zeta}}{\partial s}\right)$$
$$= \frac{\chi'}{4\pi\sqrt{g}}\left[\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\zeta}\right)B_s-\left(\frac{\partial B_\theta}{\partial s}+q\frac{\partial B_\zeta}{\partial s}\right)\right]-\frac{\partial p}{\partial s}, \tag{26}$$

respectively. Furthermore, taking only ζ-dependent part $\widetilde{\cdots}$ of Eq. (26) yields

$$\left(\sum_a n_a m_a\right)(V^\zeta)^2\frac{\partial\widetilde{g_{s\zeta}}}{\partial\zeta}=\frac{\chi'}{4\pi\sqrt{g}}\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\zeta}\right)\widetilde{B_s}. \tag{27}$$

Noting that $B_s = g_{s\theta}B^\theta + g_{s\zeta}B^\zeta$, we regard Eq. (27) as the constraint imposed on the components $g_{s\theta}$ and $g_{s\zeta}$ of the metric tensor.

III. GEOMETRIC CONDITIONS FOR TOROIDAL FLUX SURFACES

In this section, we further investigate the conditions considered in Sec. II for quasisymmetric toroidal plasmas with large mean flows on the order of the ion thermal speed. Here, the quasi-axisymmetric case is considered again as an example even though general quasisymmetric cases can be treated similarly. We also recall that the Boozer coordinates (s,θ,ζ) are used and that the field strength B, the Jacobian $\sqrt{g}=[\nabla s\cdot(\nabla\theta\times\nabla\zeta)]^{-1}$ and the components $g_{\alpha\beta}$ $(\alpha,\beta=\theta,\zeta)$ of the metric tensor are all independent of ζ as shown in the previous section. Accordingly, the contravariant component $g^{ss}\equiv[g_{\theta\theta}g_{\zeta\zeta}-(g_{\theta\zeta})^2]/g$ of the metric tensor is independent of ζ, too.

In order to examine properties of toroidal flux surfaces, we hereafter use several quantities defined in the differential geometry of surfaces such as the Christoffel symbols and the Riemann curvature tensor [32]. Definitions of these quantities for general surfaces are briefly described in Appendix C. Using the components $g_{\alpha\beta}$ $(\alpha,\beta=\theta,\zeta)$ of the metric tensor for each flux surface, which are independent of ζ, the Christoffel symbols $\Gamma^\gamma_{\alpha\beta}$ $(\alpha,\beta,\gamma=\theta,\zeta)$ are given from Eq. (C2) as

$$\begin{aligned}
\Gamma^\theta_{\theta\theta} &= \frac{g_{\zeta\zeta}\partial_\theta g_{\theta\theta}-2g_{\theta\zeta}\partial_\theta g_{\theta\zeta}}{2gg^{ss}},\\
\Gamma^\theta_{\theta\zeta} &= -\Gamma^\zeta_{\zeta\zeta} = -\frac{g_{\theta\zeta}\partial_\theta g_{\zeta\zeta}}{2gg^{ss}},\\
\Gamma^\theta_{\zeta\zeta} &= -\frac{g_{\zeta\zeta}\partial_\theta g_{\zeta\zeta}}{2gg^{ss}},\\
\Gamma^\zeta_{\theta\theta} &= \frac{-g_{\theta\zeta}\partial_\theta g_{\theta\theta}+2g_{\theta\theta}\partial_\theta g_{\theta\zeta}}{2gg^{ss}},\\
\Gamma^\zeta_{\theta\zeta} &= \frac{g_{\theta\theta}\partial_\theta g_{\zeta\zeta}}{2gg^{ss}},
\end{aligned} \tag{28}$$

where $\partial_\theta=\partial/\partial\theta$ and $gg^{ss}=g_{\theta\theta}g_{\zeta\zeta}-(g_{\theta\zeta})^2$ are used. The nonzero component $R_{\theta\zeta\theta\zeta}$ of the Riemann curvature tensor is derived from using Eq. (C3) and the Christoffel symbols in Eq. (28) as

$$R_{\theta\zeta\theta\zeta}=-\frac{\sqrt{gg^{ss}}}{2}\frac{\partial}{\partial\theta}\left(\frac{\partial_\theta g_{\zeta\zeta}}{\sqrt{gg^{ss}}}\right). \tag{29}$$

We now represent the Fourier series expansion of an arbitrary function Q of ζ by

$$Q=\sum_{n=-\infty}^{\infty}Q_n\exp(-in\zeta). \tag{30}$$

The components L, M, and N of the second fundamental form for flux surfaces are evaluated by using Eq. (C7). The Fourier coefficients of their ζ-dependent parts $\widetilde{L}$, $\widetilde{M}$, and $\widetilde{N}$ are written as

$$\begin{aligned}
\widetilde{L}_n &= \sqrt{g^{ss}}\partial_\theta(\widetilde{g_{s\theta}})_n\\
&\quad+(g\sqrt{g^{ss}})^{-1}\Big[(\widetilde{g_{s\theta}})_n\left(-\frac{1}{2}g_{\zeta\zeta}\partial_\theta g_{\theta\theta}+g_{\theta\zeta}\partial_\theta g_{\theta\zeta}\right)\\
&\quad+(\widetilde{g_{s\zeta}})_n\left(\frac{1}{2}g_{\theta\zeta}\partial_\theta g_{\theta\theta}-g_{\theta\theta}\partial_\theta g_{\theta\zeta}\right)\Big],\\
\widetilde{M}_n &= \frac{1}{2}\sqrt{g^{ss}}[\partial_\theta(\widetilde{g_{s\zeta}})_n-in(\widetilde{g_{s\theta}})_n]\\
&\quad+\frac{1}{2}(g\sqrt{g^{ss}})^{-1}(\partial_\theta g_{\zeta\zeta})[g_{\theta\zeta}(\widetilde{g_{s\theta}})_n-g_{\theta\theta}(\widetilde{g_{s\zeta}})_n],\\
\widetilde{N}_n &= -in\sqrt{g^{ss}}(\widetilde{g_{s\zeta}})_n\\
&\quad+\frac{1}{2}(g\sqrt{g^{ss}})^{-1}(\partial_\theta g_{\zeta\zeta})[g_{\zeta\zeta}(\widetilde{g_{s\theta}})_n-g_{\theta\zeta}(\widetilde{g_{s\zeta}})_n].
\end{aligned} \tag{31}$$

Since the $n=0$ Fourier component of the ζ-dependent part $\widetilde{\cdots}$ vanishes, we hereafter consider only the case of $n\neq 0$. We find from Eq. (C9) that $\widetilde{L}_n$, $\widetilde{M}_n$, and $\widetilde{N}_n$, satisfy the Fourier-transformed version of the Codazzi-Mainardi equations,

$$\begin{aligned}
-in\widetilde{L}_n-\partial_\theta\widetilde{M}_n &= \Gamma^\theta_{\theta\zeta}\widetilde{L}_n+(\Gamma^\zeta_{\theta\zeta}-\Gamma^\theta_{\theta\theta})\widetilde{M}_n-\Gamma^\zeta_{\theta\theta}\widetilde{N}_n,\\
-in\widetilde{M}_n-\partial_\theta\widetilde{N}_n &= \Gamma^\theta_{\zeta\zeta}\widetilde{L}_n+(\Gamma^\zeta_{\zeta\zeta}-\Gamma^\theta_{\theta\zeta})\widetilde{M}_n-\Gamma^\zeta_{\theta\zeta}\widetilde{N}_n,
\end{aligned} \tag{32}$$

which impose constraints on $(\widetilde{g_{s\theta}})_n$ and $(\widetilde{g_{s\zeta}})_n$ through the expressions in Eq. (31).

Substituting Eq. (31) into the second equation in Eq. (32), we obtain the first-order ordinary differential equation in θ,

$$\partial_\theta(\widetilde{g_{s\zeta}})_n=Y_n(\widetilde{g_{s\theta}})_n+Z_n(\widetilde{g_{s\zeta}})_n \tag{33}$$

where

$$\begin{aligned}
Y_n &\equiv -in-\frac{i}{n}\frac{g_{\zeta\zeta}}{g^{ss}\sqrt{g}}\frac{\partial}{\partial\theta}\left(\frac{\partial_\theta g_{\zeta\zeta}}{\sqrt{g}}\right),\\
Z_n &\equiv -\frac{\partial_\theta g^{ss}}{g^{ss}}+\frac{i}{n}\frac{g_{\theta\zeta}}{g^{ss}\sqrt{g}}\frac{\partial}{\partial\theta}\left(\frac{\partial_\theta g_{\zeta\zeta}}{\sqrt{g}}\right).
\end{aligned} \tag{34}$$

Another first-order differential equation is given by substituting Eq.(31) into the first equation in Eq. (32) and using Eq. (33) as

$$\partial_\theta(\widetilde{g_{s\theta}})_n = W_n(\widetilde{g_{s\theta}})_n + X_n(\widetilde{g_{s\zeta}})_n \tag{35}$$

where

$$\begin{aligned} W_n &\equiv -2in\frac{g_{\theta\zeta}}{g_{\zeta\zeta}} - \frac{i}{n}\frac{g_{\theta\zeta}}{g^{ss}\sqrt{g}}\frac{\partial}{\partial\theta}\left(\frac{\partial_\theta g_{\zeta\zeta}}{\sqrt{g}}\right) \\ &\quad - \frac{g(g^{ss})^2}{g_{\zeta\zeta}}\left[\frac{\partial}{\partial\theta}\left(\frac{\partial_\theta g_{\zeta\zeta}}{\sqrt{g}}\right)\right]^{-1} \\ &\quad \times \frac{\partial}{\partial\theta}\left[\frac{g_{\zeta\zeta}}{g(g^{ss})^2}\frac{\partial}{\partial\theta}\left(\frac{\partial_\theta g_{\zeta\zeta}}{\sqrt{g}}\right)\right], \\ X_n &\equiv in\frac{g_{\theta\theta}}{g_{\zeta\zeta}} + in\frac{g(g^{ss})^2}{g_{\zeta\zeta}}\left[\frac{\partial}{\partial\theta}\left(\frac{\partial_\theta g_{\zeta\zeta}}{\sqrt{g}}\right)\right]^{-1} \\ &\quad \times \frac{\partial}{\partial\theta}\left(\frac{\partial_\theta\sqrt{g^{ss}}}{(g^{ss})^{3/2}\sqrt{g}}\right) \\ &\quad + \frac{i}{n}\frac{(g_{\theta\zeta})^2}{g_{\zeta\zeta}g^{ss}\sqrt{g}}\frac{\partial}{\partial\theta}\left(\frac{\partial_\theta g_{\zeta\zeta}}{\sqrt{g}}\right) \\ &\quad + \frac{g(g^{ss})^3}{g_{\zeta\zeta}}\left[\frac{\partial}{\partial\theta}\left(\frac{\partial_\theta g_{\zeta\zeta}}{\sqrt{g}}\right)\right]^{-1} \\ &\quad \times \frac{\partial}{\partial\theta}\left[\frac{g_{\theta\zeta}}{g(g^{ss})^3}\frac{\partial}{\partial\theta}\left(\frac{\partial_\theta g_{\zeta\zeta}}{\sqrt{g}}\right)\right]. \end{aligned} \tag{36}$$

We also find that another condition for $(\widetilde{g_{s\theta}})_n$ and $(\widetilde{g_{s\zeta}})_n$ is derived from the Fourier transform of the radial equilibrium force balance in Eq. (27) as

$$\begin{aligned} &\sqrt{g}\left(\frac{\partial}{\partial\theta} - inq\right)\left(\frac{(\widetilde{g_{s\theta}})_n + q(\widetilde{g_{s\zeta}})_n}{\sqrt{g}}\right) \\ &= -in\frac{4\pi g(\sum_a n_a m_a)(V^\zeta)^2}{(\chi')^2}(\widetilde{g_{s\zeta}})_n \end{aligned} \tag{37}$$

Substituting Eqs. (33) and (35) into Eq. (37) yields the linear relation between $(\widetilde{g_{s\theta}})_n$ and $(\widetilde{g_{s\zeta}})_n$,

$$U_n(\widetilde{g_{s\theta}})_n + V_n(\widetilde{g_{s\zeta}})_n = 0, \tag{38}$$

where

$$\begin{aligned} U_n &\equiv W_n + qY_n - inq - \frac{1}{2}\frac{\partial_\theta g}{g}, \\ V_n &\equiv X_n + qZ_n - inq^2 - \frac{q}{2}\frac{\partial_\theta g}{g} \\ &\quad + in\frac{4\pi g(\sum_a n_a m_a)(V^\zeta)^2}{(\chi')^2}. \end{aligned} \tag{39}$$

Combining Eqs. (33) and (38) gives the first-order ordinary linear differential equation for $(\widetilde{g_{s\zeta}})_n$ with $(\widetilde{g_{s\theta}})_n$ eliminated. We can also obtain the first-order ordinary linear differential equation for $(\widetilde{g_{s\theta}})_n$ without $(\widetilde{g_{s\zeta}})_n$ by using Eqs. (35) and (38). Now, it is questioned whether nontrivial solutions $(\widetilde{g_{s\theta}})_n$ and $(\widetilde{g_{s\zeta}})_n$ of these linear ordinary differential equations exist or not. Since these equations contain the components $g_{\alpha\beta}$ $(\alpha, \beta = \theta, \zeta)$ of the metric tensor and the Jacobian $\sqrt{g}$ in very complicated ways, we first consider the neighborhood of the magnetic axis $s = 0$ to simplify the equations in the next section, and then discuss the solutions in the whole toroidal volume region.

IV. SEARCH FOR THE METRIC TENSOR IN QUASISYMMETRIC ROTATING PLASMAS

Here, we examine whether or not there exist nontrivial solutions $(\widetilde{g_{s\theta}})_n$ and $(\widetilde{g_{s\zeta}})_n$ of the linear ordinary differential equations derived in the previous section. For this purpose, we first investigate the neighborhood of the magnetic axis and then consider the whole volume region of the toroidal system. It should be recalled that we assume quasi-axisymmetry $\partial B/\partial\zeta = 0$ with the Boozer coordinates (s, θ, ζ) used for simplicity even though general quasisymmetric cases can be treated in a similar way.

As shown in Eq. (B11), the Jacobian associated with the Boozer coordinates is given by

$$\sqrt{g} = \frac{dV(s)/ds}{4\pi^2}\frac{\langle B^2\rangle}{B^2}, \tag{40}$$

where $V(s)$ represents the volume within the flux surface labeled s. We find in Sec. II that $\sqrt{g}$ is independent of ζ when there exist toroidal flows on the order of ion thermal speed. Along the magnetic axis $s = 0$, $\sqrt{g}$ is independent of θ, too, and it takes a constant value which is denoted by $R_0 \equiv \sqrt{g_{\zeta\zeta}(s=0)}$. Then, the total length of the magnetic axis is represented by $2\pi R_0$. The field strength $B(s=0) \equiv B_0$ at the magnetic axis is another constant. In addition, $g_{\alpha\beta}$ $(\alpha, \beta = \theta, \zeta)$ and $g^{ss} \equiv |\nabla s|^2$ are independent of ζ so that they take constant values along the magnetic axis. Therefore, toroidal cross sections of flux surfaces near the magnetic axis are concentric circles, the center of which is located at the magnetic axis. The unit vector $\mathbf{b} = \mathbf{B}/B$ along the magnetic field line at $s = 0$ is in the toroidal direction and it is written as $\mathbf{b}(s=0) = R_0^{-1}\partial\mathbf{x}(s=0,\zeta)/\partial\zeta$. In the present section, we define the radial coordinate s by $\psi = \frac{1}{2}B_0 s^2$, from which $g^{ss}(s=0) = 1$ is derived. Therefore, s is the radius length of the circular toroidal cross section of the flux surface.

We now consider the neighborhood of a certain point on the magnetic axis. This small local region is represented by $r < \epsilon$ and $|\Delta z| < \epsilon$. Here, the cylindrical coordinates (r, ϑ, z) is chosen such that the magnetic field is written as $\mathbf{B} = B_r\hat{\mathbf{r}} + B_\vartheta\hat{\boldsymbol{\vartheta}} + B_z\hat{\mathbf{z}}$, where $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\vartheta}}$, and $\hat{\mathbf{z}}$ are the orthogonal unit vectors in the r-, ϑ-, and z-directions, respectively, and $\hat{\mathbf{z}}$ coincides with $\mathbf{b}$ at the origin $(r, z) = (0, 0)$. An arbitrary function defined in the local region can be expanded with respect to the small radial coordinate s or r, where $s \sim r \sim \epsilon$ and $s = r + \mathcal{O}(\epsilon^2)$. Flux surfaces are approximately given by $r = \text{const.}$

It is shown by using the solenoidal field condition $\nabla\cdot\mathbf{B} = 0$ and Ampère's law $\nabla\times\mathbf{B} = (4\pi/c)\mathbf{J}$ that the components of the magnetic field in the cylindrical coordinates are represented by

$$\begin{aligned} B_r &= \mathcal{O}(r^2), \\ B_\vartheta &= rB_\vartheta^{(1)}(z) + \mathcal{O}(r^2) \\ B_z &= B_0 + \mathcal{O}(r^2), \end{aligned} \tag{41}$$

while the components of the current density are written as $J_r = \mathcal{O}(r)$, $J_\vartheta = \mathcal{O}(r)$, and $J_z = J_{z0} + \mathcal{O}(r)$. Using Eq. (41) and $r \sim s$, we obtain $B^2 = B_0^2 + \mathcal{O}(s^2)$. We see that $V(s)$ is approximated by the volume of the tube which has the length $2\pi R_0$ and the circular cross section with the radius s. Then, we get

$$\begin{aligned} dV/ds &= 4\pi^2 s R_0 + \mathcal{O}(s^2), \\ \sqrt{g} &= sR_0 + \mathcal{O}(s^2), \end{aligned} \tag{42}$$

where Eq. (40) is used. The definition of s immediately gives $d\psi/ds = B_0 s$, and the safety factor is written as $q(s) = q_0 + \mathcal{O}(s)$. Therefore, we have $d\chi/ds = (d\psi/ds)/q(s) = (B_0/q_0)s + \mathcal{O}(s^2)$. The poloidal and toroidal contravariant components of the magnetic field given in Eq. (2) are rewritten as

$$\begin{aligned} B^\theta &= B_0^\theta + \mathcal{O}(s), \\ B^\zeta &= B_0^\zeta + \mathcal{O}(s), \end{aligned} \tag{43}$$

where $B_0^\theta = B_0/q_0R_0$ and $B_0^\zeta = B_0/R_0$ are constants representing the values at the magnetic axis $s = 0$.

Since $g^{ss}(s=0) = 1$ and $g_{\zeta\zeta}(s=0) = (R_0)^2$, we can write g^{ss} and $g_{\zeta\zeta}$ near the magnetic axis as

$$\begin{aligned} g^{ss} &= 1 + \mathcal{O}(s), \\ g_{\zeta\zeta} &= (R_0)^2 + s g_{\zeta\zeta}^{(1)}(\theta) + \mathcal{O}(s^2). \end{aligned} \tag{44}$$

Noting that $|\partial\mathbf{x}/\partial\theta| = \mathcal{O}(s)$, we can put $g_{\theta\theta} = \mathcal{O}(s^2)$ and $g_{\theta\zeta} = s g_{\theta\zeta}^{(1)} + \mathcal{O}(s^2)$. Consider the expansion of the relation $B_\theta = g_{\theta\theta}B^\theta + g_{\theta\zeta}B^\zeta$ with respect to s and recall that, in the Boozer coordinates, the covariant poloidal and toroidal magnetic field components $B_\theta(s)$ and $B_\zeta(s)$ are independent of the angle coordinates θ and ζ. Then, we get $g_{\theta\zeta}^{(1)} = \text{const}$ and finally obtain $g_{\theta\zeta}^{(1)} = 0$ and $g_{\theta\zeta} = \mathcal{O}(s^2)$ because $\oint g_{\theta\zeta} d\theta = \oint d\theta\, \partial_\theta \mathbf{x}(s,\theta,\zeta) \cdot \partial_\zeta \mathbf{x}(s=0,\zeta) + \mathcal{O}(s^2)$ and $\oint d\theta\ \partial_\theta \mathbf{x}(s,\theta,\zeta) = 0$ where the θ integral is taken with s and ζ fixed. Consequently, $g_{\theta\zeta} = \mathcal{O}(s^2)$ is derived. From Eqs. (42), (44), and the relation

$$g g^{ss} = g_{\theta\theta}g_{\zeta\zeta} - (g_{\theta\zeta})^2, \tag{45}$$

we find that $g_{\theta\theta}$ and $g_{\theta\zeta}$ are written as

$$\begin{aligned} g_{\theta\theta} &= s^2 + \mathcal{O}(s^3), \\ g_{\theta\zeta} &= s^2 g_{\theta\zeta}^{(2)} + \mathcal{O}(s^3). \end{aligned} \tag{46}$$

Here, using the relation $B_\theta = g_{\theta\theta}B^\theta + g_{\theta\zeta}B^\zeta$ again, we see that $B_\theta(s) = \mathcal{O}(s^2)$ and $g_{\theta\zeta}^{(2)} = \text{const}$ in Eq. (46).

It should be noted that the relations $\theta' = \theta - \theta_0(s)$ and $\zeta' = \zeta - \zeta_0(s)$ give another set of the Boozer coordinates (s, θ', ζ') from the original (s, θ, ζ) by using arbitrary functions $\theta_0(s)$ and $\zeta_0(s)$. Under this transformation, we see that $g_{s\theta'} \neq g_{s\theta}$ and $g_{s\zeta'} \neq g_{s\zeta}$ hold generally although $g_{\theta'\theta'} = g_{\theta\theta}$, $g_{\theta'\zeta'} = g_{\theta\zeta}$, and $g_{\zeta'\zeta'} = g_{\zeta\zeta}$. Here, we generally have $g_{s\theta} = \mathcal{O}(s)$ and $g_{s\zeta} = \mathcal{O}(s^0)$. However, we can choose $\theta_0(s)$ and $\zeta_0(s)$ to satisfy $g_{s\theta'} = \mathcal{O}(s^2)$ and $g_{s\zeta'} = \mathcal{O}(s)$, from which we find that, in the local region, the new Boozer coordinates (s, θ', ζ') are related to the local cylinder coordinates (r, ϑ, z) by $r = s + \mathcal{O}(s^2)$, $\theta' = \vartheta + \mathcal{O}(s)$, and $\zeta' = z/R_0 + \mathcal{O}(s)$ with the origins of both coordinates coinciding with each other. Hereafter, we use this new set of the Boozer coordinates (s, θ', ζ') although they are represented by (s, θ, ζ) with the prime omitted. Thus, we now write $g_{s\theta}$ and $g_{s\zeta}$ as

$$\begin{aligned} g_{s\theta} &= s^2 g_{s\theta}^{(2)}(\theta,\zeta) + \mathcal{O}(s^3), \\ g_{s\zeta} &= s g_{s\zeta}^{(1)}(\theta,\zeta) + \mathcal{O}(s^2) \end{aligned} \tag{47}$$

We find from Eqs. (44), (46), and (47) that

$$g_{ss} = 1 + \mathcal{O}(s), \tag{48}$$

and that the contravariant basis vectors $\partial\mathbf{x}/\partial s$, $\partial\mathbf{x}/\partial\theta$, and $\partial\mathbf{x}/\partial\zeta$ are orthogonal to each other to the lowest order in s and they are related to the contravariant basis vectors ∇s, $\nabla\theta$, and $\nabla\zeta$ by

$$\begin{aligned} \partial\mathbf{x}/\partial s &= \nabla s + \mathcal{O}(s), \\ \partial\mathbf{x}/\partial\theta &= s^2\nabla\theta + \mathcal{O}(s^2), \\ \partial\mathbf{x}/\partial\zeta &= R_0^2\nabla\zeta + \mathcal{O}(s). \end{aligned} \tag{49}$$

Using Eqs. (41) and (46)–(48), the covariant magnetic field components are shown to be written as

$$\begin{aligned} B_s &= s(B_0/R_0)g_{s\zeta}^{(1)}(\theta,\zeta) + \mathcal{O}(s^2), \\ B_\theta &= s^2(B_0/q_0R_0)(1 + q_0 g_{\theta\zeta}^{(2)}) + \mathcal{O}(s^3), \\ B_\zeta &= R_0B_0 + \mathcal{O}(s). \end{aligned} \tag{50}$$

From the relations $g^{s\theta} = (g_{s\zeta}g_{\theta\zeta} - g_{s\theta}g_{\zeta\zeta})/g$ and $g^{s\zeta} = (g_{s\theta}g_{\theta\zeta} - g_{s\zeta}g_{\theta\theta})/g$, we obtain

$$\begin{aligned} g^{s\theta} &= -g_{s\theta}^{(2)}(\theta,\zeta) + \mathcal{O}(s), \\ g^{s\zeta} &= -s g_{s\zeta}^{(1)}(\theta,\zeta)/R_0^2 + \mathcal{O}(s^2), \end{aligned} \tag{51}$$

where Eqs. (42), (44), (46), and (47) are used.

Now, Eq. (33) is rewritten by using Eqs. (34), (42), (44), and (46) as

$$\partial_\theta(\widetilde{g_{s\zeta}})_n = -i(ns)^{-1}(\partial_\theta^2 g_{\zeta\zeta}^{(1)})(\widetilde{g_{s\theta}})_n + \mathcal{O}(s)\cdot(\widetilde{g_{s\zeta}})_n. \tag{52}$$

Then, using Eqs. (35), (36), (42), (44), and (46), we obtain

$$(\partial_\theta^2 g_{\zeta\zeta}^{(1)})\partial_\theta(\widetilde{g_{s\theta}})_n = -(\partial_\theta^3 g_{\zeta\zeta}^{(1)})(\widetilde{g_{s\theta}})_n + \mathcal{O}(s^2)(\widetilde{g_{s\zeta}})_n. \tag{53}$$

The relation between $(\widetilde{g_{s\theta}})_n$ and $(\widetilde{g_{s\zeta}})_n$ is derived from Eqs. (34), (36), (38), (39), (42), (44), and (46) as

$$\begin{aligned}&\left(-1+\frac{4\pi\sum_a n_{a0}m_a}{B_0^2}R_0^2(V_0^\zeta)^2\right)(\widetilde{g_{s\zeta}})_n\\ &= (\partial_\theta^2 g_{\zeta\zeta}^{(1)})(\widetilde{g_{s\theta}})_n/(n^2 q_0 s),\end{aligned} \tag{54}$$

where n_{a0}, V_0^ζ, and q_0 represent the density, toroidal angular velocity, and safety factor at the magnetic axis $s=0$, respectively. Eliminating $(\widetilde{g_{s\zeta}})_n$ from Eq. (53) by using Eq. (54) gives

$$\partial_\theta(\widetilde{g_{s\theta}})_n = -\frac{\partial_\theta^3 g_{\zeta\zeta}^{(1)}}{\partial_\theta^2 g_{\zeta\zeta}^{(1)}}(\widetilde{g_{s\theta}})_n, \tag{55}$$

the solution of which is written as

$$(\widetilde{g_{s\theta}})_n = \frac{C_n(s)}{\partial_\theta^2 g_{\zeta\zeta}^{(1)}}, \tag{56}$$

where $C_n(s)$ is a function of s that still remains to be determined.

From Eqs. (54) and (56), we have

$$(\widetilde{g_{s\zeta}})_n = \frac{C_n(s)}{n^2 q_0 s}\left(-1+\frac{4\pi\sum_a n_{a0}m_a}{B_0^2}R_0^2(V_0^\zeta)^2\right)^{-1}, \tag{57}$$

where it is assumed that $-1+(4\pi\sum_a n_{a0}m_a/B_0^2)R_0^2(V_0^\zeta)^2 \neq 0$ holds generally. Then, substituting Eqs. (56) and (57) into Eq. (52), we see that, to the lowest order in s,

$$C_n(s)\partial_\theta^2 g_{\zeta\zeta}^{(1)} = 0 \tag{58}$$

which is found to reduce to $C_n(s)=0$ by noting that the component of the Riemann curvature tensor in Eq. (29) is proportional to the Gaussian curvature of the toroidal flux surface which is not identical to zero. Therefore, we finally obtain

$$(\widetilde{g_{s\theta}})_n = (\widetilde{g_{s\zeta}})_n = 0. \tag{59}$$

Now, we see that all components $g_{\alpha\beta}$ and accordingly $g^{\alpha\beta}$ $(\alpha,\beta=s,\theta,\zeta)$ of the metric tensor are independent of ζ. For such a case, an arbitrary closed ζ-curve defined by $\mathbf{x}=\mathbf{x}(s,\theta,\zeta)$ $(0\le\zeta\le 2\pi)$ with s and θ fixed is a circle because the curvature and the torsion of the ζ-curve, which can be derived from $g_{\alpha\beta}$, need to be independent of ζ and accordingly constants along the closed curve. Furthermore, two circular closed ζ-curves, $\mathbf{x}=\mathbf{x}(s,\theta,\zeta)$ and $\mathbf{x}=\mathbf{x}(s,\theta+d\theta,\zeta)$, has a constant interval given by $d\theta\sqrt{g_{\theta\theta}-(g_{\theta\zeta})^2/g_{\zeta\zeta}}$ between each other. Thus, toroidal surfaces, which the ζ-curves lie on, are axisymmetric in the local region near the magnetic axis and the magnetic axis itself is a circle of the radius R_0 on a flat plane. Note that $ds/\sqrt{g^{ss}}(=ds/|\nabla s|)$ represents the local distance between neighboring flux surfaces with the radial coordinates s and $s+ds$. Since $\partial g^{ss}/\partial\zeta=0$ is assumed to be satisfied globally in the whole volume region, all flux surfaces, which encircle the magnetic axis of the circular shape, should be axisymmetric. Now, it is concluded that quasi-axisymmetric rotating plasmas considered here need to be axisymmetric. The same result is obtained for general quasisymmetric rotating plasmas in a similar way to the above one.

Here, it is instructive to rewrite the component $R_{\theta\zeta\theta\zeta}$ of the Riemann curvature tensor in Eq. (29) by using Eqs. (42) and (44) as

$$R_{\theta\zeta\theta\zeta} = -\frac{s}{2}\frac{\partial^2 g_{\zeta\zeta}^{(1)}(\theta)}{\partial\theta^2} + \mathcal{O}(s^2), \tag{60}$$

and the components L, M, and N of the second fundamental form are calculated from $g_{\alpha\beta}$ $(\alpha,\beta=s,\theta,\zeta)$ as

$$\begin{aligned}L &= -s+\mathcal{O}(s^2),\\ M &= s\left(\frac{1}{2}\frac{\partial g_{s\zeta}^{(1)}}{\partial\theta} - g_{\theta\zeta}^{(2)}\right)+\mathcal{O}(s^2),\\ N &= -\frac{1}{2}g_{\zeta\zeta}^{(1)}+\mathcal{O}(s).\end{aligned} \tag{61}$$

From Eq. (61) and (C11), we have $LN-M^2=R_{\theta\zeta\theta\zeta}=sg_{\zeta\zeta}^{(1)}/2+\mathcal{O}(s^2)$ which is combined with Eqs. (60) to obtain $\partial_\theta^2 g_{\zeta\zeta}^{(1)} = -g_{\zeta\zeta}^{(1)}$. Thus, we obtain $g_{\zeta\zeta}^{(1)}\propto R_{\theta\zeta\theta\zeta}\propto\cos\theta$, where the origin of the poloidal angle θ is chosen such that the Gaussian curvature ($\propto LN-M^2$) of toroidal flux surfaces near the magnetic axis vanishes for $\theta=\pm\pi/2$. This form of $g_{\zeta\zeta}^{(1)}$ is a well-known result for large-aspect-ratio axisymmetric toroidal surfaces, where the Gaussian curvature is positive (negative) for $|\theta|<\pi/2$ ($|\theta|>\pi/2$).

V. CONCLUSIONS

In this work, the conditions for quasisymmetric toroidal systems to allow large flows on the order of the ion thermal velocity are investigated. Taking the case of rotating quasi-axisymmetric plasmas, in which the field strength B does not depend on the toroidal angle ζ, the equilibrium momentum balance equations are used to show that the component $g_{\zeta\zeta}$ of the metric tensor is shown to be independent of ζ in the systems with no local radial current, where the Boozer coordinates exist. It is also shown that, unless $\partial g_{\zeta\zeta}/\partial\zeta=0$, the toroidal flow velocity cannot take any value other than a very limited class of eigenvalues corresponding to very rapid rotation especially for low beta plasmas. In the Boozer coordinates, where $\partial g_{\zeta\zeta}/\partial\zeta=0$, the metric tensor components $g_{\alpha\beta}$ $(\alpha,\beta=\theta,\zeta)$ associated with the angle coordinates for each flux surface are shown to be independent of ζ. We finally find that, in order to globally satisfy the equilibrium momentum balance, all metric tensor components $g_{\alpha\beta}$ $(\alpha,\beta=s,\theta,\zeta)$ need to be independent of ζ, and therefore toroidal flux surfaces are axisymmetric. General quasisymmetric cases can be treated similarly. Thus, quasisymmetric toroidal equilibrium plasmas with large mean flows should be axisymmetric or they have local radial currents which do not allow existence of the Boozer coordinates and impose strong constraints on the toroidal flow velocity. Since, in the present work, quasisymmetry is assumed to be rigorously satisfied with the magnetic field strength B as an analytical function of the flux coordinates, we have not definitely answered yet whether large mean flows on the order of the ion thermal velocity can be compatible with nonaxisymmetric toroidal plasma equilibria such as quasi-omnigenous configurations. This remains as a future task.

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APPENDIX A: SOLUTIONS OF EQ. (19)

In this Appendix, we investigate solutions of Eq. (19). As a function of ζ, $\widetilde{g_{\zeta\zeta}}$ is expanded by the Fourier series,

$$\widetilde{g_{\zeta\zeta}} = \sum_n (\widetilde{g_{\zeta\zeta}})_n e^{-in\zeta}, \tag{A1}$$

where $(\widetilde{g_{\zeta\zeta}})_0 = 0$. Then, Eq. (19) is represented in terms of $(\widetilde{g_{\zeta\zeta}})_n$ by

$$\begin{aligned} &\frac{\partial}{\partial\theta}\left(\frac{1}{\sqrt{g}}\frac{\partial[e^{-inq\theta}(\widetilde{g_{\zeta\zeta}})_n]}{\partial\theta}\right) \\ &= -4\pi\sqrt{g}\left(\sum_a n_a m_a\right)\left(\frac{nV^\zeta}{\chi'}\right)^2 e^{-inq\theta}(\widetilde{g_{\zeta\zeta}})_n. \end{aligned} \tag{A2}$$

Introducing new variables $\xi_n \equiv e^{-inq\theta}(\widetilde{g_{\zeta\zeta}})_n$ and η_n, Eq. (A2) is given in the form of the fist-order ordinary differential equations,

$$\frac{d}{d\theta}\begin{bmatrix}\xi_n\\ \eta_n\end{bmatrix} = \begin{bmatrix} 0 & \sqrt{g} \\ -4\pi\sqrt{g}\left(\sum_a n_a m_a\right)(nV^\zeta/\chi')^2 & 0\end{bmatrix}\begin{bmatrix}\xi_n\\ \eta_n\end{bmatrix}, \tag{A3}$$

where the radial coordinate s is included as a parameter in ξ_n, η_n, $\sqrt{g}$, n_a, V^ζ, and χ' although it is not shown explicitly. The solution of Eq. (A3) is written as

$$\begin{bmatrix}\xi_n(\theta)\\ \eta_n(\theta)\end{bmatrix} = \Theta\exp\left[\int_0^\theta d\theta'\, A_n(\theta')\right]\begin{bmatrix}\xi_n(0)\\ \eta_n(0)\end{bmatrix} \tag{A4}$$

where

$$A_n(\theta') \equiv \begin{bmatrix} 0 & \sqrt{g(\theta')} \\ -4\pi\sqrt{g(\theta')}\left(\sum_a m_a n_a(\theta')\right)(nV^\zeta/\chi')^2 & 0\end{bmatrix}. \tag{A5}$$

On the right-hand side of Eq. (A4), Θ represents the θ-ordered product of matrix functions of θ and $\Theta\exp\left[\int_0^\theta d\theta'\, A_n(\theta')\right]$ is defined by

$$\begin{aligned} &\Theta\exp\left[\int_0^\theta d\theta'\, A_n(\theta')\right] \\ &\equiv 1 + \sum_{l=1}^{\infty}\int_0^\theta d\theta_1\int_0^{\theta_1} d\theta_2\cdots\int_0^{\theta_{l-1}} d\theta_l \\ &\quad\times A_n(\theta_1)A_n(\theta_2)\cdots A_n(\theta_l). \end{aligned} \tag{A6}$$

Generally, $A_n(\theta)$ and $A_n(\theta')$ are noncommutative, $A_n(\theta)A_n(\theta') \neq A_n(\theta')A_n(\theta)$, for $\theta \neq \theta'$. If $A_n(\theta)$ and $A_n(\theta')$ are commutative for arbitrary values of θ and θ', Eq. (A6) reduces to the well-known exponential form,

$$\begin{aligned} \Theta\exp\left[\int_0^\theta d\theta'\, A_n(\theta')\right] &= \sum_{l=0}^{\infty}\frac{1}{l!}\left(\int_0^\theta d\theta' A_n(\theta')\right)^l \\ &\equiv \exp\left[\int_0^\theta d\theta'\, A_n(\theta')\right]. \end{aligned} \tag{A7}$$

Since $(\widetilde{g_{\zeta\zeta}})_n$ and $d(\widetilde{g_{\zeta\zeta}})_n/d\theta$ are periodic functions of θ with the period 2π, $e^{inq\theta}\xi_n$ and $e^{inq\theta}\eta_n$ are periodic, too. These periodic conditions give

$$e^{2\pi inq}\Theta\exp\left[\int_0^\theta d\theta'\, A_n(\theta')\right]\begin{bmatrix}\xi_n(0)\\ \eta_n(0)\end{bmatrix} = \begin{bmatrix}\xi_n(0)\\ \eta_n(0)\end{bmatrix}. \tag{A8}$$

Then, the condition for $[\xi_n(0), \eta_n(0)] \neq [0, 0]$ to satisfy Eq. (A8) is written as

$$\det\left(e^{2\pi inq}\Theta\exp\left[\int_0^\theta d\theta'\, A_n(\theta')\right] - I\right) = 0, \tag{A9}$$

where I denotes the 2×2 unit matrix. Thus, for a given n and a given radial coordinate s, V^ζ included in the matrix $A_n(\theta)$ needs to take one of eigenvalues determined by the condition shown in Eq. (A9).

In the neighborhood of the magnetic axis $s = 0$ as discussed in Sec. IV, $A_n(\theta)$ is approximated as,

$$\begin{aligned} A_n(\theta) \simeq A_{n0} &\equiv \begin{bmatrix} 0 & sR_0 \\ -n^2\lambda^2/sR_0 & 0\end{bmatrix} \\ &= -n\lambda\begin{bmatrix}1 & 0\\ 0 & n\lambda/sR_0\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & sR_0/n\lambda\end{bmatrix}, \end{aligned} \tag{A10}$$

where the radial coordinate s is defined by $\psi \equiv \frac{1}{2}B_0 s^2$ in the same way as in Sec. IV. Here, λ is a constant defined by

$$\lambda \equiv (4\pi \sum_a n_{a0} m_a)^{1/2} q_0 R_0 V_0^\zeta / B_0, \tag{A11}$$

where n_{a0}, q_0, and V_0^ζ are the density, the safety factor, and the toroidal angular velocity on the magnetic axis $s = 0$, respectively. From Eqs. (A9) and (A10), we obtain

$$\begin{aligned} & \det \left(e^{2\pi i n q_0} \exp(2\pi A_{n0}) - I \right) \\ = & \det \begin{bmatrix} e^{2\pi i n q_0} \cos(2\pi n\lambda) - 1 & e^{2\pi i n q_0} \sin(2\pi n\lambda) \\ -e^{2\pi i n q_0} \sin(2\pi n\lambda) & e^{2\pi i n q_0} \cos(2\pi n\lambda) - 1 \end{bmatrix} \\ = & (e^{2\pi i n (q_0 + \lambda)} - 1)(e^{2\pi i n (q_0 - \lambda)} - 1) = 0. \end{aligned} \tag{A12}$$

Equations (A11) and (A12) yield

$$\begin{aligned} \lambda \pm q_0 &= q_0 (\sqrt{\beta_0} R_0 V_0^\zeta / v_{T0} \pm 1) \\ &= \frac{l}{n} \quad (l = 0, \pm 1, \pm 2, \cdots), \end{aligned} \tag{A13}$$

where $\beta_0 \equiv 8\pi (\sum_a n_{a0} T_{a0}) / B_0^2$ is the central beta and $v_{T0} \equiv [2(\sum_a n_{a0} T_{a0}) / (\sum_a n_{a0} m_a)]^{1/2}$ represents a characteristic value of the central temperature. Eigenvalues which V_0^ζ should take are given by Eq. (A13), from which we obtain

$$R_0 V_0^\zeta / v_{T0} = (\pm 1 + l / n q_0) / \sqrt{\beta_0} \quad (l = 0, \pm 1, \pm 2, \cdots). \tag{A14}$$

For the regions other than the neighborhood of the magnetic axis, eigenvalues of V^ζ need to be determined by Eq. (A9). Equation (A14) shows that, unless $g_{\zeta\zeta}$ is independent of ζ, the toroidal flow velocity of $\mathcal{O}(v_{T0})$ cannot take any value other than a very limited class of eigenvalues corresponding to very rapid rotation (which could be supersonic) especially for low beta plasmas. Therefore, the case of $\partial g_{\zeta\zeta} / \partial \zeta = 0$ is considered in the main text of the present paper.

APPENDIX B: TRANSFORMATION OF FLUX COORDINATES

In terms of arbitrary flux coordinates (s, θ, ζ), the magnetic field, which forms nested toroidal surfaces, is represented by

$$\mathbf{B} = \psi'(s) \nabla s \times \nabla \theta + \chi'(s) \nabla \zeta \times \nabla s, \tag{B1}$$

where θ and ζ are the poloidal and toroidal angles, respectively, and s is a radial coordinate or a label to specify a flux surface. As explained after Eq. (1), the toroidal and poloidal fluxes within the volume inside the surface with the label s are given by $2\pi\psi(s)$ and $2\pi\chi(s)$, respectively, and $' \equiv \partial/\partial s$ denotes the derivative with respect to s.

Consider two sets of flux coordinates (s, θ, ζ) and (s, θ_A, ζ_A), where the same radial coordinate s is used. Then, there exists a generating function $G_A(s, \theta, \zeta)$ by which the coordinate transformation is written as [35]

$$\begin{aligned} \theta_A &= \theta + \chi'(s) G_A(s, \theta, \zeta), \\ \zeta_A &= \zeta + \psi'(s) G_A(s, \theta, \zeta). \end{aligned} \tag{B2}$$

We find from Eqs. (B1) and (B2) that the generating function G_A satisfies the magnetic differential equation,

$$\mathbf{B} \cdot \nabla G_A = \frac{1}{\sqrt{g_A}} - \frac{1}{\sqrt{g}} \tag{B3}$$

where $\sqrt{g} \equiv [\nabla s \cdot (\nabla \theta \times \nabla \zeta)]^{-1}$ and $\sqrt{g_A} \equiv [\nabla s \cdot (\nabla \theta_A \times \nabla \zeta_A)]^{-1}$ are the Jacobians associated with the flux coordinates (s, θ, ζ) and (s, θ_A, ζ_A), respectively. Using Eqs. (B1) and (B2), we also obtain

$$\begin{aligned} \frac{\partial \mathbf{x}}{\partial \theta} &= \frac{\partial \mathbf{x}}{\partial \theta_A} + \frac{\partial G}{\partial \theta} \sqrt{g_A} \mathbf{B}, \\ \frac{\partial \mathbf{x}}{\partial \zeta} &= \frac{\partial \mathbf{x}}{\partial \zeta_A} + \frac{\partial G}{\partial \zeta} \sqrt{g_A} \mathbf{B}. \end{aligned} \tag{B4}$$

When the local radial current vanishes, we have

$$\frac{4\pi}{c} J^s = \sqrt{g} \left(\frac{\partial B_\zeta}{\partial \theta} - \frac{\partial B_\theta}{\partial \zeta} \right) = 0, \tag{B5}$$

where B_θ and B_ζ are the covariant poloidal and toroidal magnetic field components, respectively. If the conventional equilibrium condition $\mathbf{J} \times \mathbf{B}/c = \nabla p(s)$ is assumed, $J^s = 0$ is immediately derived. However, it is emphasized that, in the present paper, we consider the equilibrium force balance given by Eq. (17) where the inertia term associated with the large flow velocity $\mathbf{V}_0$ is included. Therefore, $J^s = 0$ is not trivial here but it imposes additional constraints on the magnetic geometry as discussed in Sec. II. It is shown from Eq. (B5) that B_θ and B_ζ are written in terms of a certain function $\eta(s, \theta, \zeta)$ as

$$\begin{aligned} B_\theta &= \frac{2}{c} I_t(s) + \frac{\partial \eta(s, \theta, \zeta)}{\partial \theta}, \\ B_\zeta &= \frac{2}{c} I_p^d(s) + \frac{\partial \eta(s, \theta, \zeta)}{\partial \zeta}, \end{aligned} \tag{B6}$$

where $I_t(s) \equiv (c/4\pi) \oint B_\theta d\theta$ [$I_p^d(s) \equiv (c/4\pi) \oint B_\zeta d\zeta$] represents the toroidal (poloidal) current flowing inside (outside) the flux surface with the label s, and η is periodic in θ and ζ. Then, the magnetic field is written as

$$\begin{aligned} \mathbf{B} &= B_s \nabla s + B_\theta \nabla \theta + B_\zeta \nabla \zeta \\ &= \left(B_s - \frac{\partial \eta}{\partial s} \right) \nabla s + \frac{2}{c} I_t(s) \nabla \theta + \frac{2}{c} I_p^d(s) \nabla \zeta + \nabla \eta. \end{aligned} \tag{B7}$$

Now, the Boozer coordinates (s, θ_B, ζ_B) are obtained from the coordinate transformation in Eq. (B2) with the generating function G_A replaced by

$$G_B(s, \theta, \zeta) \equiv \frac{c\eta}{2(\psi' I_p^d + \chi' I_t)}. \tag{B8}$$

The Boozer coordinates (s, θ_B, ζ_B) are characterized by the properties that the covariant poloidal and toroidal components of the magnetic field are independent of θ_B and ζ_B. In fact, using Eqs. (B2), (B7), and (B8), we see

$$B_{\theta_B} = \frac{2}{c} I_t(s), \qquad B_{\zeta_B} = \frac{2}{c} I_p^d(s). \tag{B9}$$

Thus, we have proved that there exist the Boozer coordinates (s, θ_B, ζ_B) for the case of $J^s = 0$ even though the conventional equilibrium condition $\mathbf{J} \times \mathbf{B}/c = \nabla p(s)$ is not assumed.

Using $B^{\theta_B} = \chi'/\sqrt{g_B}$, $B^{\zeta_B} = \psi'/\sqrt{g_B}$, and Eq. (B9), shows

$$B^2 = B^{\theta_B} B_{\theta_B} + B^{\zeta_B} B_{\zeta_B} = \frac{2(\chi' I_t + \psi' I_p^d)}{c\sqrt{g_B}}, \tag{B10}$$

from which we obtain $\langle B^2 \rangle \equiv \oint d\theta_B \oint d\zeta_B \sqrt{g_B} B^2/V' = 8\pi^2(\chi' I_t + \psi' I_p^d)/(cV')$ and

$$\sqrt{g_B} = \frac{V'(s)}{4\pi^2} \frac{\langle B^2 \rangle}{B^2}. \tag{B11}$$

Here, $V'(s) = \oint d\theta_B \oint d\zeta_B \sqrt{g_B}$, is the radial derivative of the volume $V(s)$ within the flux surface labeled s. Thus, in the Boozer coordinates, the Jacobian $\sqrt{g_B}$ depends on the angle coordinates (θ_B, ζ_B) only through $1/B^2$.

In the quasisymmetric toroidal plasma with large mean flows, the flow velocity $\mathbf{V}_0$ is parallel to the direction of quasisymmetry of the magnetic field strength B and the Jacobian $\sqrt{g}$ of the flux coordinates (s, θ, ζ) does not vary in the symmetry direction as explained in Sec. II. Let us take the case of quasi-axisymmetry for simplicity although we can treat general quasisymmetric cases in the same way as shown below. Then, $\partial B/\partial\zeta = 0$ and $\partial\sqrt{g}/\partial\zeta = 0$, where we should note that (s, θ, ζ) does not need to be the Boozer coordinates but they should be at least flux coordinates satisfying Eq. (B1). Now, in addition to the above quasisymmetry conditions, we consider again the case of $J^s = 0$ in which there exist the Boozer coordinates (s, θ_B, ζ_B) satisfying Eqs. (B9) and (B11) as we have shown.

Since $\partial B/\partial\zeta = 0$ and $\partial\sqrt{g}/\partial\zeta = 0$, the generating function G_B yielding the coordinate transformation from (s, θ, ζ) to (s, θ_B, ζ_B) can be obtained by directly integrating the magnetic differential equation in the form of Eq. (B3) as

$$\begin{aligned} G_B &= \frac{1}{\chi'(s)} \int_0^\theta d\theta' \left(\frac{4\pi^2}{V'(s)} \frac{\{B(s,\theta')\}^2}{\langle B^2 \rangle} \sqrt{g(s,\theta')} - 1 \right) \\ &\quad + K(s), \end{aligned} \tag{B12}$$

where $K(s)$ is independent of (θ, ζ) and appears as a constant of integration. Using Eq. (B4) and noting that G_B is independent of ζ, we find

$$\frac{\partial \mathbf{x}}{\partial \zeta_B} = \frac{\partial \mathbf{x}}{\partial \zeta}, \qquad \frac{\partial B}{\partial \zeta_B} = \frac{\partial B}{\partial \zeta} = 0, \tag{B13}$$

from which $\partial\sqrt{g_B}/\partial\zeta_B = 0$ is immediately derived. Equation (B13) implies that, if a rapid rotating toroidal plasma with no local radial current is quasisymmetric in terms of a certain set of flux coordinates (s, θ, ζ), then it is so in the Boozer coordinates (s, θ_B, ζ_B), too. Thus, as remarked after Eq. (21) in Sec. II, we can use the Boozer coordinates from the beginning to investigate quasisymmetric toroidal systems with large mean flows if $J^s = 0$ is satisfied.

In passing, for the present case of $\partial\sqrt{g}/\partial\zeta = 0$, the coordinate transformation in Eq. (B2) using another generation function defined by

$$G_A = \frac{1}{\chi'(s)} \int_0^\theta d\theta' \left(\frac{4\pi^2}{V'(s)} \sqrt{g(s,\theta')} - 1 \right), \tag{B14}$$

yields the new flux coordinates (s, θ_A, θ_B) for which the Jacobian is given by

$$\sqrt{g_A} = \frac{V'(s)}{4\pi^2}. \tag{B15}$$

Equation (B15) shows that the Jacobian $\sqrt{g_A}$ is a flux-surface function, which is a characteristic of the Hamada coordinates [34]. Under the conventional equilibrium condition $\mathbf{J} \times \mathbf{B}/c = \nabla p(s)$, we find that J^{θ_A} and J^{ζ_A} are flux-surface functions, where J^{θ_A} and J^{ζ_A} represent the contravariant poloidal and toroidal components of the current density vector $\mathbf{J}$, respectively. However, it should be noted that, since the large-flow inertia term is included in the equilibrium force balance here, this property of the Hamada coordinates is not generally satisfied in the flux coordinates (s, θ_A, ζ_A), where J^{θ_A} and J^{ζ_A} may not be functions of s alone while $\sqrt{g_A} = V'(s)/(4\pi^2)$.

APPENDIX C: DIFFERENTIAL GEOMETRY OF SURFACES

In this Appendix, we briefly review the definitions of basic quantities used in the differential geometry for surfaces [32]. Here, in order to easily derive these quantities in Sec. III for the case of quasisymmetric toroidal surfaces, we represent an arbitrary surface locally by $\mathbf{x} = \mathbf{x}(\theta, \zeta)$ with θ and ζ used as two parameters even though they do not need to be angle coordinates in this Appendix.

The first fundamental form is the metric tensor for the surface $\mathbf{x} = \mathbf{x}(\theta, \zeta)$, the components of which are defined by

$$\begin{aligned} E &\equiv g_{\theta\theta} \equiv |\partial_\theta \mathbf{x}|^2, \\ F &\equiv g_{\theta\zeta} \equiv (\partial_\theta \mathbf{x}) \cdot (\partial_\zeta \mathbf{x}), \\ G &\equiv g_{\zeta\zeta} \equiv |\partial_\zeta \mathbf{x}|^2, \end{aligned} \tag{C1}$$

where simplified notations $\partial_\theta = \partial/\partial\theta$ and $\partial_\zeta = \partial/\partial\zeta$ are used. Then, the Christoffel symbols $\Gamma_{\gamma\alpha\beta}$ and $\Gamma^\gamma_{\alpha\beta}$ $(\alpha, \beta, \gamma = \theta, \zeta)$ are defined by using the components of the first fundamental form in Eq. (C1) as

$$\Gamma_{\gamma\alpha\beta} \equiv \sum_{\delta=\theta,\zeta} g_{\gamma\delta}\Gamma^{\delta}_{\alpha\beta} \equiv \frac{1}{2}(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}). \quad \text{(C2)}$$

We take the inverse of the 2×2 matrix consisting of $g_{\alpha\beta}$ $(\alpha, \beta = \theta, \zeta)$ when deriving $\Gamma^{\gamma}_{\alpha\beta}$ from $\Gamma_{\gamma\alpha\beta}$. This inverse matrix should not be identified with $g^{\alpha\beta}$ used in Secs. III–IV because the latter is obtained by taking the inverse of the 3×3 matrix which includes additional components g_{ss}, $g_{s\theta}$, and $g_{s\zeta}$. Using the Christoffel symbols, the components $R^{\gamma}_{\delta\alpha\beta}$ and $R_{\gamma\delta\alpha\beta}$ $(\delta, \gamma, \alpha, \beta = \theta, \zeta)$ of the Riemann curvature tensor are defined by

$$\begin{aligned} R^{\gamma}_{\delta\alpha\beta} &= \partial_\alpha \Gamma^{\gamma}_{\beta\delta} - \partial_\beta \Gamma^{\gamma}_{\alpha\delta} + \Gamma^{\gamma}_{\alpha\sigma}\Gamma^{\sigma}_{\beta\delta} - \Gamma^{\gamma}_{\beta\sigma}\Gamma^{\sigma}_{\alpha\delta}, \\ R_{\gamma\delta\alpha\beta} &= \sum_{\sigma=\theta,\zeta} g_{\gamma\sigma} R^{\sigma}_{\delta\alpha\beta}. \end{aligned} \quad \text{(C3)}$$

The components of the Riemann curvature tensor have the symmetry properties written as

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma}, \quad \text{(C4)}$$

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0. \quad \text{(C5)}$$

Then, it is found that nonzero covariant components of the Riemann tensor for the surface $\mathbf{x} = \mathbf{x}(\theta, \zeta)$ are only $R_{\theta\zeta\theta\zeta}$ and those obtained from $R_{\theta\zeta\theta\zeta}$ by using the symmetric properties in Eq. (C4).

The unit normal vector $\mathbf{e}$ to the surface is written as

$$\mathbf{e} \equiv \frac{\nabla s}{|\nabla s|} \equiv \frac{\partial \mathbf{x}/\partial\theta \times \partial \mathbf{x}/\partial\zeta}{|\partial \mathbf{x}/\partial\theta \times \partial \mathbf{x}/\partial\zeta|}. \quad \text{(C6)}$$

Then, the components L, M, and N of the second fundamental form for the surface are defined by using the normal vector $\mathbf{e}$ as

$$\begin{aligned} L &\equiv h_{\theta\theta} \equiv \frac{\partial^2 \mathbf{x}}{\partial\theta^2} \cdot \mathbf{e}, \\ M &\equiv h_{\theta\zeta} \equiv \frac{\partial^2 \mathbf{x}}{\partial\theta\partial\zeta} \cdot \mathbf{e}, \\ N &\equiv h_{\zeta\zeta} \equiv \frac{\partial^2 \mathbf{x}}{\partial\zeta^2} \cdot \mathbf{e}. \end{aligned} \quad \text{(C7)}$$

In terms of the components of the first and second fundamental forms in Eqs. (C1) and (C7), the Gaussian curvature of the surface is given by

$$K = \frac{LN - M^2}{EG - F^2}. \quad \text{(C8)}$$

The components (E, F, G) and (L, M, N) of the first and second fundamental forms must satisfy the Codazzi-Mainardi equations which are written as

$$\begin{aligned} \partial_\zeta L - \partial_\theta M &= L\Gamma^{\theta}_{\theta\zeta} + M(\Gamma^{\zeta}_{\theta\zeta} - \Gamma^{\theta}_{\theta\theta}) - N\Gamma^{\zeta}_{\theta\theta}, \\ \partial_\zeta M - \partial_\theta N &= L\Gamma^{\theta}_{\zeta\zeta} + M(\Gamma^{\zeta}_{\zeta\zeta} - \Gamma^{\theta}_{\theta\zeta}) - N\Gamma^{\zeta}_{\theta\zeta}, \end{aligned} \quad \text{(C9)}$$

where we should recall that the Christoffel symbols $\Gamma^{\gamma}_{\alpha\beta}$ contain the components of the first fundamental form as shown in Eq. (C2). Furthermore, (E, F, G) and (L, M, N) must satisfy the Gauss equation given by

$$K = \frac{R_{\theta\zeta\theta\zeta}}{EG - F^2}, \quad \text{(C10)}$$

which is rewritten by using Eq. (C8) as

$$LN - M^2 = R_{\theta\zeta\theta\zeta}. \quad \text{(C11)}$$

Here, note that $R_{\theta\zeta\theta\zeta}$ is calculated from the components of the first fundamental form or the metric tensor by using Eq. (C2) and (C3). The Gauss equation in Eq. (C10) [or (C11)] is the basis for Gauss's theorema egregium which states that the Gaussian curvature of a surface can be determined by residents confined onto the surface who measure the dependence of the metric tensor $(g_{\alpha\beta})$ on the parameters (θ, ζ) without even knowing about the three-dimensional space in which the surface is embeded. Bonnet's fundamental theorem of surface theory states that, if given functions (E, F, G) and (L, M, N) satisfy $E > 0$, $EG - F^2 > 0$, and the Gauss-Codazzi-Mainardi equations in Eqs. (C9) and (C10) [or (C11)], then there exists a surface for which the components of the first and second fundamental forms are given by (E, F, G) and (L, M, N), respectively, and that such a surface is determined uniquely up to congruence.

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