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NAGOYA, JAPAN
Eigenfunction for Dissipative Dynamics Operator

and

Attractor of Dissipative Structure

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It is shown that the states with the minimum dissipation rate in general dissipative dynamics systems are expressed by the eigenfunctions for the dissipative dynamics operators. The eigenfunctions for the dissipative dynamics operators are shown to constitute the self-organized and self-similar decay phase as the attractor of the dissipative structure.

Keywords: self-organization, eigenfunction, dissipative dynamics operator, self-similar decay, attractor, dissipative structure
"Dissipative structures" realized in nonlinear dynamical systems with dissipation have attracted much attention in many research fields, such as various self-organized structures in the thermodynamic system,\(^1,2\) the force-free fields in the cosmic magnetic fields,\(^3\) the self-organized relaxed state in the magnetized fusion plasmas,\(^4\) and the helical flow pattern after the turbulent puffs.\(^5\) We may find some common mathematical structure among the self-organized relaxed states of the dissipative structure or the proposed theories to explain those dissipative structures.\(^6,7\) It is interesting to investigate such a common universal mathematical structure which is embedded in dissipative nonlinear dynamics systems and leads to those dissipative structures. We propose here a theory which leads to the attractor of the dissipative structure.

We express here quantities with \(n\) elements in dynamics systems of interest as \(q(t, x) = \{q_1(t, x), q_2(t, x), \ldots, q_n(t, x)\}\). We first consider a nondissipative ideal dynamics system which may be described by

\[
\frac{\partial q_i}{\partial t} = L_i^N[q],
\]

where \(L_i^N[q]\) is nondissipative, linear or nonlinear dynamics operators, for example, \(q_i = u\) and \(L_i^N[q] = -\nabla p/\rho - \nabla u^2/2 + u \times \omega\) in the ideal incompressible fluid dynamics.

We sometimes treat steady state of \(\partial q_i/\partial t = 0\) and obtain the equilibrium equations from eq.(1) as follows,

\[
\text{equilibrium equations } \quad L_i^N[q] = 0.
\]

When no external input is applied, global auto-correlations \(W_{ii} = \int q_i q_i dV = \int q_i^2 dV\) over the space volume of the system, which represent usually the total energy of the system, are conserved in the steady state of eq.(2) because of no dissipation, and we obtain

\[
\frac{\partial W_{ii}}{\partial t} = 2 \int q_i \frac{\partial q_i}{\partial t} dV = 2 \int q_i L_i^N[q] dV = 0.
\]
We cannot find usually any peculiar or unique spatial profiles of \( q_i \) from the equilibrium equations, eq.(2), themselves.

We now proceed to a dissipative dynamics system which may be described by

\[
\frac{\partial q_i}{\partial t} = L_i^N[q] + L_i^P[q],
\]

(4)

where \( L_i^N[q] \) is the nondissipative dynamics operators shown above, and \( L_i^P[q] \) is dissipative, linear or nonlinear dynamics operators, for example, \( q_i = u \) and \( L_i^P[q] = (\nu/\rho) \nabla^2 u \) in the Navier-Stokes equation of the incompressible viscous fluid dynamics with the coefficient of viscosity \( \nu \). When no external input is applied, the steady state of \( \partial q_i/\partial t = 0 \) and \( \partial W_{ni}/\partial t = 0 \) is no longer realized because of dissipation. However, we can treat a quasi-steady state where the equilibrium equations, eq.(2), are applicable approximately. In this quasi-steady state with eq.(2), the dissipation rate of \( W_{ni} \) is written as follows, by using eqs.(2)-(4),

\[
\frac{\partial W_{ni}}{\partial t} = 2 \int q_i \frac{\partial q_i}{\partial t} \, dV = 2 \int q_i L_i^P[q] \, dV.
\]

(5)

We then investigate whether the quasi-steady state with the minimum dissipation rate of \( |\partial W_{ni}/\partial t| \) has some unique spatial profiles of \( q_i \) or not, as in the similar way in refs.3 and 6. This is a typical problem of the variational calculus with respect to the spatial variable \( x \) to find the spatial profiles of \( q_i \) such that satisfy the followings:

\[
\min |\frac{\partial W_{ni}}{\partial t}| \text{ for a given value of } W_{ni}.
\]

(6)

We use notations of \( q^*(W_{ni}, x) \) or simply \( q_i^* \) for the profiles of \( q_i \) such that satisfy eq.(6).

Since \( \partial W_{ni}/\partial t \) has usually negative value, the mathematical expressions for eq.(6) are written as follows, by defining a functional \( F \) with use of a Lagrange multiplier \( \alpha \),

\[
F \equiv - \frac{\partial W_{ni}}{\partial t} - \alpha W_{ni},
\]

(7)
\( \delta F = 0 \), \\
\( \delta^2 F > 0; \)

where \( \delta F \) and \( \delta^2 F \) are the first and the second variations of \( F \) with respect to the variation \( \delta q(\mathbf{x}) \) only for the spatial variable \( \mathbf{x} \). Substituting eq.(5) into eqs.(7) and (8), we obtain

\[
\delta F = -2 \int \delta q_i (L^D_i[q] + \alpha q_i) + q_i \delta L^D_i[q] \, dV = 0. \tag{10}
\]

We now impose the following self-adjoint property upon the dissipative dynamics operators \( L^D_i[q] \),

\[
\int q_i \delta L^D_i[q] dV = \int \delta q_i L^D_i[q] dV + \oint P \cdot dS, \tag{11}
\]

where \( \oint P \cdot dS \) denotes the surface integral term which comes out such as from the Gauss theorem. The self-adjoint property of eq.(11) is satisfied actually by the dissipative dynamics operators such as \( (\nu/\rho) \nabla^2 \mathbf{u} \) in the Navier-Stokes equation, and the Ohm loss term of \( -\nabla \times (\eta \mathbf{J}) \) in the magnetic field equation of the resistive MHD plasma. The surface integral term of \( \oint P \cdot dS \) sometimes vanishes because of the boundary condition such as for the ideally conducting wall. Using the self-adjoint property of eq.(11), we obtain the followings from eq.(10),

\[
\delta F = -2 \int \delta q_i (2L^D_i[q] + \alpha q_i) \, dV - 2 \oint P \cdot dS = 0. \tag{12}
\]

We then obtain the Euler-Lagrange equations from the volume integral term in eq.(12) for arbitrary variations of \( \delta q_i \) as follows,

\[
L^D_i[q^*] = -\frac{\alpha}{2} q^*. \tag{13}
\]

We find from eq.(13) that the profiles of \( q^* \) are given by the eigenfunctions for the dissipative dynamics operators \( L^D_i[q^*] \), and therefore have peculiarity uniquely determined by the operator \( L^D_i[q^*] \) themselves. Substituting the eigenfunctions of eq.(13)
and the approximate equilibrium equation of eq.(2) for the quasi-steady state into eq.(4), we obtain the followings,

$$\frac{\partial q_i^*}{\partial t} \cong -\frac{\alpha}{2} q_i^*, \quad (14)$$

$$q_i^* \cong q_{R_i^*}(x) e^{-\frac{\alpha}{2} t}, \quad (15)$$

$$W_{ii}^* = \int (q_i^*)^2 dV \cong e^{-\alpha t} \int (q_{R_i}^*)^2 dV. \quad (16)$$

$$\frac{\partial W_{ii}^*}{\partial t} \cong -\alpha W_{ii}^*, \quad (17)$$

where $q_{R_i^*}(x)$ denotes the eigensolution for eq.(13) which is supposed to be realized at the state with the minimum dissipation rate during the time evolution of the dynamical system of interest. We find from eq.(15) that the eigenfunctions $q_i^*$ for the dissipative dynamics operators $L^D_i[q^*]$ constitute the self-organized and self-similar decay phase during the time evolution of the present dynamic system. We see from eq.(17) that the factor $\alpha$ of eq.(13) being the Lagrange multiplier is equal to the decay const. of $W_{ii}$ at the self-organized and self-similar decay phase. We should bear in mind, however, that since the present dynamic system evolves basically by eq.(4), the dissipation and being open of the system with respect to $W_{ii}$ will still lead to some gradual deviation from the self-similar decay. When some external input is applied in order to recover the dissipation of $W_{ii}$, the present dynamic system will come back close to the self-organized and self-similar decay phase repeatedly. When we observe the time evolution of the system of interest during long time interval, we would come to find that the system behaves as if it is repeatedly attracted to and trapped in the self-organized and self-similar decay phase of eq.(15) where the system stays longest time during one cycle of the time evolution. In this meaning, the eigenfunctions $q_i^*$ of eq.(13) for the dissipative dynamics operators $L^D_i[q^*]$ are "the attractors of
the dissipative structure" introduced by Prigogine.\textsuperscript{1,2} When the dynamic system of interest is nonlinear and dissipative, the system would come back close to the self-organized state of eq.(13) repeatedly, not exactly to the same point but with some chaos-like deviations. The eigenfunctions $q_i^*$ of eq.(13) for the dissipative dynamics operators $L^D_i[q^*]$ is considered to become the core of "the attractor of the chaos-like behavior" in the longer time observations. It has been reported in ref.6 that eq.(13) leads to $\nabla \times \nabla \times u^* = \alpha^2 u^*$ in the case of incompressible viscous fluids, and also to $\nabla \times (\eta u^*) = \alpha B^*/2$ in the case of resistive MHD plasmas, and both of them are proved to be followed by the self-similar decay phase.

Using eq.(9), we next discuss the mode transition point or the bifurcation point of the self-organized dissipative structure. Substituting eq.(5) into eqs.(7) and (9), we obtain

$$\delta^2 F = -2 \int \delta q_i (\delta L^D_i[q] + \frac{\alpha}{2} \delta q_i) \, dV > 0 . \quad (18)$$

We consider here the following associated eigenvalue problem for critical perturbations $\delta q_i$ that make $\delta^2 F$ in eq.(18) vanish,

$$\left( \delta L^D_i[q] \right)_k + \frac{\alpha_k}{2} \delta q_{ik} = 0 , \quad (19)$$

with boundary conditions given for $\delta q_i$, for example $\delta q_i = 0$ at the boundary wall. Here, $\alpha_k$ is the eigenvalue, and $\left( \delta L^D_i[q] \right)_k$ and $\delta q_{ik}$ denote the eigensolution. Substituting the eigensolution $\delta q_{ik}$ into eq.(18) and using eq.(19), we obtain the following,

$$\delta^2 F = (\alpha_k - \alpha) \int \delta q_{ik}^2 \, dV > 0 . \quad (20)$$

Since eq.(20) is required for all eigenvalues, we obtain the following condition for the state with the minimum dissipation rate,

$$0 < \alpha < \alpha_1 , \quad (21)$$
where $\alpha_1$ is the smallest positive eigenvalue, and $\alpha$ is assumed to be positive. When the value of $\alpha$ goes out of the condition of eq. (21), like as $\alpha_1 < \alpha$, then the mixed mode, which consists of the basic mode by the solution of eq. (13) with $\alpha = \alpha_1$ and the lowest eigenmode of eq. (19), becomes the self-organized dissipative structure with the minimum dissipation rate. This result for the the bifurcation point of the self-organized dissipative structure has the same mathematical structure with that for the self-organized relaxed state of the resistive MHD plasmas.\(^6,7\)

In conclusion, the states with the minimum dissipation rate in the dissipative dynamics system of eq. (4) have been shown to be expressed by the eigenfunctions $q^*_i$ of eq. (13) for the dissipative dynamics operators $L^D_i[q^*]$. The eigenfunctions $q^*_i$ have been shown to constitute the self-organized and self-similar decay phase of eq. (15). The factor $\alpha$ of eq. (13) being the Lagrange multiplier is equal to the decay const. of $W_r$ at the self-organized and self-similar decay phase. The eigenfunctions $q^*_i$ for the dissipative dynamics operators $L^D_i[q^*]$ are considered to be "the attractor of the dissipative structure", and are expected to become the core of "the attractor of the chaos-like behavior" in the longer time observations.

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