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Hamiltonian for the Toroidal Helical Magnetic Field Lines in the Vacuum

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Abstract

The hamiltonian is obtained using Lie perturbation expansion techniques and it is given up to the second perturbative order. The hamiltonian is given both for the cylindrical limit approximation of the field and for the toroidal case.

Keywords: Lie perturbation techniques, toroidal helical magnetic field, hamiltonian formulation

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1 Introduction

The toroidal helical magnetic field possesses magnetic surfaces in the proximity of the magnetic axis, while in regions far from the magnetic axis such surfaces do not exist. The domain in which magnetic surfaces do not exist is called magnetic chaos domain, and between the magnetic axis and the magnetic chaos domain lies the outermost magnetic surface, whose position is an important information from the point of view of magnetic confinement.

Since the divergence of the magnetic field is zero, the magnetic field lines system can be treated as a hamiltonian system. For hamiltonian systems the phase space structure is subdivided into chaotic and non-chaotic regions, and this subdivision has a correspondece with the magnetic chaos domain and the magnetic surfaces domain discussed above.

The existing research in this field is based on numerical calculations about the structure of the magnetic field, but in these methods numerical errors are inherently present and this poses limitations on the accuracy by which the properties of the outermost magnetic surface are investigated. To enhance the accuracy it is necessary to track many times the magnetic field lines around the torus, but there are indications that this procedure cause s shift of the surface towards the magnetic axis. A possible explanation of this fact is that the divergence-free nature of the magnetic field is not fully respected in these numerical methods. With this background in mind, we think that in order to further advance this kind of research it is necessary to approach the problem from the point of view of hamiltonian theory, in which the divergence-free property is rigorously respected. Besides, for hamiltonian systems it is possible to use symplectic integration techniques to solve the equation of motion, techniques which are free from secular, or dissipative errors. This fact make the application of symplectic techniques very attractive.

The research approaching the toroidal helical magnetic field problem from the hamiltonian point of view ends with works done more than ten years ago (ref. [1], [2] and [3]), partly because of the strong push towards research on tokamaks. However, we have two good reasons to revive the hamiltonian approach to the toroidal helical magnetic field problem:

1). The Japanese Ministry of Education is pushing the reseach concerning the helical systems approach to magnetic confinement.

2). The last ten years have seen an impetuous development of chaos physics, and times are ripen for reseach about the chaotic properties of the toroidal helical magnetic field lines system.

Recently T. Hatori and T. Watanabe (ref. [4]) have obtained an explicit form for the Boozer's magnetic coordinates in the first order toroidal correction to the cylindrical helical magnetic configuration. In this paper a more systematic procedure (Lie perturbation expansion technique) is used to proceed to higher orders.

2 General Observations

In this section we will give a very concise description of the Lie perturbation technique on which our work is based. This description is simply meant to serve as a reference for the reader and it is not meant to be a throughout discussion of the technique itself (the interested reader can consult ref. [5]). The equations for the field lines flows can be obtained from the variational principle:

$$\delta \int d\lambda A_\mu(x) dx^\mu / d\lambda = 0$$

where A_μ is the magnetic potential and λ is an arbitrary parameter. With a redefinition of the coordinates involved, the preceding formula can be written as: $\delta \int \gamma_\mu dz^\mu = 0$. The vector γ_μ will be called 1-form (more precisely the 1-form is the quantity $\gamma_\mu dz^\mu$). A gauge transformation on the potential A_μ is: $A_\mu \to A_\mu + \partial_\mu S$, where S is the gauge function. This gauge transformation induces a gauge transformation on the 1-form γ_μ as follows: $\gamma_\mu \to \gamma_\mu + \partial_\mu S$.

The equations of motion, that is the equations for the magnetic field lines flow, are obtained from the above equation carrying out the variation of the integral, and are not the hamilton equations if the variables are not canonical. However, if the variables are canonical, that is if a hamiltonian exists, carrying out the variation we obtain just the hamilton equations for the magnetic field lines flow. Our purpose in this paper is to transform the "noncanonical" expression for the magnetic potential into a canonical one. This will be done using Lie transformations, which are a special kind of coordinate transformation, specified by a generator g^μ, which relates the old and the new coordinates. The starting point is a 1-form of the type:

$$\gamma = \gamma^0 + \epsilon\gamma^1 + \epsilon^2\gamma^2 + \ldots \qquad (1)$$

where ϵ is a smallness parameter. Using the Lie transormation we change the coordinates from z^μ to Z^μ. If we define the operator L_g by $L_g = g^\mu \partial_\mu$, then the new and the old coordinates are related by $z^\mu = \exp(-\epsilon L_g) Z^\mu$. In the new coordinates the transformed 1-form

$$\Gamma = \Gamma^0 + \epsilon\Gamma^1 + \epsilon^2\Gamma^2 + ... \tag{2}$$

is given by:

$$\Gamma^0 = \gamma^0 \tag{3}$$

$$\Gamma^1 = dS_1 - L_1\gamma^0 + \gamma^1 \tag{4}$$

$$\Gamma^2 = dS_2 - L_2\gamma^0 + \gamma^2 - L_1\gamma^1 + \frac{1}{2}L_1^2\gamma^0 \tag{5}$$

where:

$$(L_n\gamma^0)_\mu = g_n^\nu \omega^0_{\nu\mu} \tag{6}$$

The tensor ω^0 is called Lagrange tensor, and it is defined by:

$$\omega^0_{\mu\nu} = \partial_\mu\gamma^0_\nu - \partial_\nu\gamma^0_\mu \tag{7}$$

In general, we define the Lagrange tensor with respect to an arbitrary 1-form ς as follows:

$$\omega^\varsigma_{\mu\nu} = \partial_\mu\varsigma_\nu - \partial_\nu\varsigma_\mu \tag{8}$$

We will proceed evaluating every single term in the expressions for Γ^0, Γ^1, Γ^2, and so on. The general form of Γ is:

$$\Gamma^n = dS_n - L_n\gamma^0 + C_n \tag{9}$$

where C_n is a 1-form calculated from γ^n and the results of the preceding lower order calculations. The generator is contained in the term $L_n\gamma_0$. Since we will be concerned only with expressions up to the second order, we give the form of C_1 and C_2:

$$C_1 = \gamma^1, \qquad C_2 = \gamma^2 - L_1\gamma^1 + \tfrac{1}{2}L_1^2\gamma^0 \tag{10}$$

It is possible to choose $g_n^0 = 0$ to all orders. The 2N components g_n^i and the scalar S_n can be chosen as to bring the 2N + 1 components of Γ^n, where N is the number of degrees of freedom, into some desired form. We want a form in which only the temporal component of Γ, which will be the φ-component in our case, is not zero, and this can be done choosing:

$$g_n^j = (\partial_i S_n + C_{ni}) J_0^{ij} \tag{11}$$

where J_0^{ij} is the inverse of the spatial part of the Lagrange tensor. With this choice of the generators the temporal component of the 1-form becomes:

$$\Gamma_t^n = V_0^\mu \partial_\mu S_n + C_{n\mu} V_0^\mu \tag{12}$$

where V_0^μ is the Poisson vector, defined as $V_0^i = J_0^{ij}\omega_{0j}^0; V_0^0 = 1$. We stress that with the choice (11) of the generators, the temporal component of the 1- form is the only one which survives after the transformation. Finally, in order to avoid secularities we must take the temporal component of the 1-form as the average over the unperturbed orbits of the quantity $C_{n\mu}V_0^\mu$.

After this quite concise and abstract introduction to the calculation strategy we will use, we discuss now the actual form of our magnetic potential, and some manipulation we will make before starting to apply the technique described. We begin by writing the magnetic potential in the form:

$$\gamma = (\epsilon A_\xi^{lm} + \partial_\xi S)d\xi + (A_\eta^T + \epsilon A_\eta^{lm} + \partial_\eta S)d\eta + (\epsilon' A_\varphi^{l0} + \partial_\varphi S)d\varphi \tag{13}$$

where ϵ and ϵ' are smallness parameters. The explicit form of the terms A_ξ^{lm}, A_η^T, A_η^{lm} and A_φ^{l0} can be found in the appendix. The coordinates ξ, η and φ are toroidal coordinates, in term of which the cartesian coordinates are $z = \xi \sin\eta/(1-\xi\cos\eta)$, $x = (1-\xi^2)^{1/2}\cos\psi/(1-\xi\cos\eta)$, and $y = (1-\xi^2)^{1/2}\sin\varphi/(1-\xi\cos\eta)$. Exploiting the gauge freedom, we choose the gauge function in order to make the ξ component of the 1-form to vanish, so we choose the gauge function to be:

$$S = -\epsilon \int d\xi A_\xi^{lm} \tag{14}$$

and therefore the 1-form becomes:

$$\gamma = (A_\eta^T + \epsilon A_\eta^{lm} - \epsilon \int d\xi \partial_\eta A_\xi^{lm})d\eta + (\epsilon' A_\varphi^{l0} - \epsilon \int d\xi \partial_\varphi A_\xi^{lm})d\varphi \tag{15}$$

We introduce the new variable $\theta = \eta + (m/l)\varphi$, where l is the poloidal multipolarity and m is the number of field periods, so that we get:

$$
\begin{aligned}
\gamma &= [A_\eta^T + \epsilon A_\eta^{lm} - \epsilon \int d\xi \partial_\eta A_\eta^{lm}]d\theta \\
&+[\epsilon' A_\varphi^{l0} - (m/2)(A_\eta^T + \epsilon A_\eta^{lm}) \\
&-\epsilon \int d\xi (\partial_\varphi A_\xi^{lm} - (m/l)\partial_\eta A_\xi^{lm})]d\varphi
\end{aligned} \tag{16}
$$

The 1-form is of the type: $\gamma = \gamma_\theta d\theta + \gamma_\varphi d\varphi$, and we will proceed to a perturbation expansion of the two components. This will be done first in the cylindrical limit approximation, with the purpose of illustrating the technique on a relatively simple form of magnetic potential, and then on the toroidal expression of the magnetic potential.

The analisys is based on an expansion of the magnetic potential in the cylindrical limit approximation. The expansion is a perturbative one and the perturbation parameters are the radial coordinate ξ and the parameters ϵ and ϵ'.

Thrughout this work we will choose $l = 2$.

3 Cylindrical limit approximation

The starting point is the expansion (see the Appendix):

$$
\begin{array}{ll}
\gamma_\theta^0 = \psi & \gamma_\varphi^0 = -(m/2)\psi \\
\gamma_\theta^1 = 0 & \gamma_\varphi^1 = \epsilon\psi(m^2/4)\sin(2\theta) \\
\gamma_\theta^2 = -\epsilon(m^3/8)\psi^2\sin(2\theta) & \gamma_\varphi^2 = \epsilon(m^4/12)\psi^2\sin(2\theta)
\end{array} \tag{17}
$$

Where the new coordinate $\psi = (1/2)\xi^2$ has been introduced. As discussed in the previous section, we now proceed to a perturbative expansion of the components of the 1-form such that the resulting expression is in the canonical form. The unperturbed part of the potential corresponds to the zeroth order and is expressed by:

$$
\gamma^0 = \psi d\theta - (m/2)\psi d\varphi \tag{18}
$$

In this expression θ and φ play the role of canonical conjugate variables and the unperturbed hamiltonian is $H = (m/2)\psi$. As discussed before the zeroth component of the transformed 1-form is equal to the untrasformed one, that is:

$$
\Gamma^0 = \psi d\theta - (m/2)\psi d\varphi \tag{19}
$$

We proceed now to the calculations for the first order. The expression for Γ^1 is given by:

$$\Gamma^1_\varphi = \partial_\varphi S_1 + V_0^\psi \partial_\psi S_1 + V_0^\theta \partial_\theta S_1 + C_{1\varphi} + V_0^\psi C_{1\psi} + V_0^\theta C_{1\theta} \qquad (20)$$

This expression results if we choose the generators of the Lie transformation in order to make the components Γ^1_θ and Γ^1_ψ vanish. Proceeding to the calculation of the various terms appearing in the equation, we obtain:

$$C_{1\varphi} = \gamma^1_\varphi, \quad C_{1\theta} = \gamma^1_\theta, \quad C_{1\varphi} = 0 \qquad (21)$$

The components of the Lagrange tensor are calculated directly from the definition, and we obtain:

$$\omega^0_{\varphi\theta} = \partial_\varphi \gamma^0_\theta - \partial_\theta \gamma^0_\varphi = 0 \qquad (22)$$

$$\omega^0_{\varphi\psi} = \partial_\varphi \gamma^0_\psi - \partial_\psi \gamma^0_\varphi = m/2 \qquad (23)$$

$$\omega^0_{\theta\psi} = \partial_\theta \gamma^0_\psi - \partial_\psi \gamma^0_\theta = -1 \qquad (24)$$

The Poisson vector is obtained from the formula $V_0^j = J_0^{ij}\omega^0_{0j}$, where J^{ij} is the inverse of the spatial part of the Lagrange tensor. We therefore obtain:

$$V_0^\psi = 0, \qquad V_0^\theta = m/2 \qquad (25)$$

and we remind the reader that by definiton $V_0^\varphi = 1$. The expression for Γ^1_φ becames:

$$\Gamma^1_\varphi = \partial_\varphi S_1 + (m/2)\partial_\theta S_1 + \gamma^1_\varphi + (m/2)\gamma^1_\theta \qquad (26)$$

Now, we have to take Γ^1_φ as the average over θ of the quantity $V_0^\mu C_{1\mu} = \gamma^1_\varphi + (m/2)\gamma^1_\theta$, which vanishes, since:

$$\gamma^1_\varphi + (m/2)\gamma^1_\theta = \epsilon(m^2/4)\psi \sin 2\theta \qquad (27)$$

We have now to calculate the first order gauge function and the first order Lie generators, which will be necessary in order to calculate the second order component of the transformed 1-form. To calculate the first order gauge function, we have to solve the equation:

$$\partial_\varphi S_1 + (m/2)\partial_\theta S_1 = -\gamma^1_\varphi - (m/2)\gamma^1_\theta \qquad (28)$$

A solution of this equation is a gauge function which does not depend on φ, that is:

$$S_1 = (1/4)\epsilon\psi\cos 2\theta \tag{29}$$

The generators are obtained from the formula

$$g_1^j = (\partial_i S_1 + C_{1i})J_0^{ij} \tag{30}$$

which explicitly reads:

$$g_1^\theta = -\partial_\psi S_1 = -(1/4)\epsilon m\cos 2\theta \tag{31}$$

$$g_1^\psi = \partial_\theta S_1 + \gamma_\theta^1 = -(1/2)\epsilon m\psi\sin 2\theta \tag{32}$$

Now, the second order contribution to the 1-form is:

$$\Gamma_\varphi^2 = \partial_\varphi S_2 + (m/2)\partial_\theta S_2 + C_{2\varphi} + (m/2)C_{2\theta} \tag{33}$$

and therefore we now have to evaluate the θ and φ components of the quantity:

$$C_2 = \gamma^2 - L_1\gamma^1 + (1/2)L_1^2\gamma^0 \tag{34}$$

We have:

$$\begin{aligned}(L_1\gamma^1)_\theta &= g_1^\psi\omega_{\psi\theta}^1\\ &= g_1^\psi\partial_\psi\gamma_\theta^1 = 0\end{aligned} \tag{35}$$

$$\begin{aligned}(L_1\gamma^1)_\varphi &= g_1^\theta\omega_{\theta\varphi}^1 + g_1^\psi\omega_{\psi\varphi}^1\\ &= g_1^\psi\partial_\psi\gamma_\varphi^1 + g_1^\theta\partial_\theta\gamma_\varphi^1\\ &= -(1/8)\epsilon^2 m^3\psi\end{aligned} \tag{36}$$

Besides, we have:

$$\begin{aligned}(L_1^2\gamma^0)_\theta &= g_1^\psi\omega_{\psi\theta}^{L_1\gamma^0}\\ &= g_1^\psi[\partial_\psi(L_1\gamma^0)_\theta - \partial_\theta(L_1\gamma^0)_\psi]\end{aligned} \tag{37}$$

$$
\begin{aligned}
(L_1^2\gamma^0)_\varphi &= g_1^\theta \omega_{\theta\varphi}^{L_1\gamma^0} + g_1^\psi \omega_{\psi\varphi}^{L_1\gamma^0} \\
&= g_1^\theta[\partial_\theta(L_1\gamma^0)_\varphi - \partial_\psi(L_1\gamma^0)_\theta] \\
&\quad + g_1^\psi[\partial_\psi(L_1\gamma^0)_\varphi - \partial_\varphi(L_1\gamma^0)_\psi]
\end{aligned} \tag{38}
$$

and therefore we have to calculate first the components of the operator $L_1\gamma^0$. We obtain:

$$(L_1\gamma^0)_\theta = g_1^\psi \omega_{\psi\theta}^0 = -(1/2)\epsilon m\psi \sin 2\theta \tag{39}$$

$$(L_1\gamma^0)_\psi = g_1^\theta \omega_{\theta\psi}^0 = (1/4)\epsilon m \cos 2\theta \tag{40}$$

$$(L_1\gamma^0)_\varphi = g_1^\theta \omega_{\theta\varphi}^0 + g_1^\psi \omega_{\psi\theta}^0 = (1/4)m^2\epsilon\psi \sin 2\theta \tag{41}$$

Substituting, we get

$$(1/2)(L_1^2\gamma^0)_\theta = 0 \tag{42}$$

$$(1/2)(L_1^2\gamma^0)_\varphi = -(1/16)\epsilon^2 m^3\psi \tag{43}$$

Using these results we obtain:

$$C_{2\theta} = \gamma_\theta^2 - (L_1\gamma^1)_\theta + (1/2)(L_1^2\gamma^0)_\theta = -(1/8)\epsilon m^3\psi^2 \sin 2\theta \tag{44}$$

$$
\begin{aligned}
C_{2\varphi} &= \gamma_\varphi^2 - (L_1\gamma^1)_\psi + (1/2)(L_1^2\gamma^0)_\varphi \\
&= (1/12)\epsilon m^4\psi^2 \sin 2\theta + (1/8)\epsilon^2 m^3\psi - (1/16)\epsilon^2 m^3\psi \\
&= (1/12)\epsilon m^4\psi^2 \sin 2\theta + (1/16)\epsilon^2 m^3\psi
\end{aligned} \tag{45}
$$

Averaging over θ the quantity

$$C_{2\varphi} + (m/2)C_{2\theta} \tag{46}$$

we get the second order contribution to the transformed 1-form:

$$\Gamma_\varphi^2 = (1/16)\epsilon^2 m^3\psi \tag{47}$$

Accordingly, the expression of the transformed 1-form up to the second order is:

$$\Gamma = \psi d\theta + [-(m/2)\psi + (1/16)\epsilon^2 m^3 \psi] d\varphi \tag{48}$$

In this expression the magnetic potential is expressed in canonical form, with ψ and θ playing the role of action angle variables. The hamiltonian up to the second order is

$$H = (m/2)\psi - (1/16)\epsilon^2 m^3 \psi \tag{49}$$

The equations of the magnetic field lines flow are obtained directly from the above expression. This result is consistent with the result obtained in the ref. [2] .

4 Toroidal case

We now proceed to the calculation of the transformed 1-form in the toroidal case, along the same line followed for the cylindrical limit approximation. The starting point is again the expansion of the magnetic potential:

$$\begin{array}{ll} \gamma_\theta^0 = \psi & \gamma_\varphi^0 = -\frac{m}{2}\psi \\ \gamma_\theta^1 = (2\psi)^{3/2}\frac{\cos\eta}{3} & \gamma_\varphi^1 = -\frac{m}{2}(2\psi)^{3/2}\frac{\cos\eta}{3} \\ & +2\psi\epsilon U_{2m}^0 \sin 2\theta + \epsilon' U_{20}^0 l\psi \sin\eta \\ \gamma_\theta^2 = (1+\cos\eta^2)\psi^2 & \gamma_\varphi^2 = -\frac{m}{2}(1+\cos\eta^2)\psi^2 + \epsilon'(2\psi)^{3/2} \\ & (1/6)[2U_{20}^0 \sin 2\eta \cos\eta - U_{20}^0 \cos 2\eta \sin\eta] \\ & -(1/2m)\epsilon(2\psi)^{3/2} U_{2m}^0 \cos\eta \sin 2\theta \end{array} \tag{50}$$

The quantities U_{2m}^0 and U_{20}^0 are constants (see the appendix). We notice that the zeroth order term is identic to the zeroth order term of the cylindrical limit approximation. The zeroth order in the expansion is the unperturbed part of the potential:

$$\gamma^0 = \psi d\theta - \frac{m}{2}\psi d\varphi \tag{51}$$

and it is in canonical form, with θ and ψ playing the role of action angle variables and $-\gamma_\varphi^0$ playing the role of the hamiltonian for the umperturbed system. However this is true only for the zeroth order, as can be deduced observing the structure of the expansion.

We now proceed to the calculations of the transformed 1-form to the various perturbative orders. For the zeroth order we have that the 1-form is left unchanged, that is:

$$\Gamma^0 = \gamma^0 = \psi d\theta - (m/2)\psi d\varphi \tag{52}$$

We also notice that this is the same expression that we obtained in the cylindrical limit approximation. For the first order we have to calculate the expression:

$$\Gamma^1_\varphi = \partial_\varphi S_1 + V_0^\psi \partial_\psi S_1 V_0^\theta \partial_\theta S_1 + C_{1\varphi} + V_0^\psi C_{1\psi} + V_0^\theta C_{1\theta} \tag{53}$$

Since the Lagrange tensor is calculated on the non perturbed part of the 1-form, we obtain the same result obtained in the cylindrical limit approximation:

$$\omega^0_{\varphi\theta} = 0, \quad \omega^0_{\varphi\psi} = m/2, \quad \omega^0_{\theta\psi} = -1 \tag{54}$$

The same is true for the Poisson tensor, which is therefore:

$$V_0^\psi = 0, \quad V_0^\theta = m/2, \quad V_0^\varphi = 1 \tag{55}$$

The equation for Γ^1_φ then becomes:

$$\Gamma^1_\varphi = \partial_\varphi S_1 + (m/2)\partial_\theta S_1 + \gamma^1_\varphi + (m/2)\gamma^1_\theta \tag{56}$$

Again, in order to avoid secularities we take Γ^1 as the average over θ, with η fixed, of the quantity $V_0^\mu C_{1\mu} = \gamma^1_\varphi + (m/2)\gamma^1_\theta$, which is:

$$\Gamma^1_\varphi = 2\epsilon' U^0_{20}\psi \sin 2\eta \tag{57}$$

For the gauge function we have the equation:

$$(m/2)\partial_\theta S_1 = -2\psi\epsilon U^0_{2m} \sin 2\theta \tag{58}$$

which has the solution:

$$S_1 = U^0_{2m}\frac{2\psi\epsilon}{m}\cos 2\theta \tag{59}$$

For the first order generators we obtain:

$$g_1^\theta = -\partial_\psi S_1 = -U_{2m}^0 \frac{2\epsilon}{m} \cos 2\theta \quad (60)$$

$$g_1^\varphi = \partial_\theta S_1 + \gamma_\theta^1 = -U_{2m}^0 \frac{4\epsilon\psi}{m} \sin 2\theta + (2\psi)^{3/2} \frac{\cos\eta}{3} \quad (61)$$

The equation for the second order contribution is:

$$\Gamma_\varphi^2 = \partial_\varphi S_2 + V_0^\theta \partial_\theta S_2 + V_0^\psi \partial_\psi S_2 + C_{2\varphi} + C_{2\theta} V_0^\theta + C_{2\psi} V_0^\psi \quad (62)$$

which becomes, after substituting the values of the Poisson vector:

$$\Gamma_\varphi^2 = \partial_\psi S_2 + (m/2)\partial_\theta S_2 + C_{2\varphi} + (m/2) C_{2\theta} \quad (63)$$

For the second order we need therefore to evaluate the quantity $C_2 = \gamma^2 - L_1\gamma^1 + (1/2)L_1^2\gamma^0$. The calculations of the explicit expression of the operators L's is straightforward but quite long, so we give only the results:

$$\begin{aligned}(L_1\gamma^1)_\varphi \;=\;& -(2/m)\epsilon U_{2m}^0 \cos 2\theta [4\psi\epsilon U_{2m}^0 \cos 2\theta + 4\epsilon' U_{20}^0 \psi \cos 2\eta] \\ & +[(1/3)(2\psi)^{3/2}\cos\eta - (4/m)\epsilon\psi U_{2m}^0 \sin 2\theta] \\ & [-(m/2)(2\psi)^{3/2}\cos\eta + 2\epsilon U_{2m}^0 \sin 2\theta + \epsilon' U_{20}^0 2\sin 2\eta] \quad (64)\end{aligned}$$

,

$$\begin{aligned}(L_1\gamma^1)_\theta \;=\;& (2\psi)^{(1/2)}\cos\eta[(1/3)(2\psi)^{3/2}\cos\eta \\ & -(4/m)\epsilon\psi U_{2m}^0 \sin 2\theta] \quad (65)\end{aligned}$$

$$\begin{aligned}(1/2)(L_1^2\gamma^0)_\theta \;=\;& (1/2)[(1/3)(2\psi)^{3/2}\cos\eta - (4/m)\epsilon\psi U_{2m}^0 \sin 2\theta] \\ & [(2\psi)^{1/2}\cos\eta - (4/m)\epsilon U_{2m}^0 \sin 2\theta \\ & -(4/m)\epsilon U_{2m}^0 \sin 2\theta] \quad (66)\end{aligned}$$

$$\begin{aligned}(1/2)(L_1^2\gamma^0)_\varphi \;=\;& -(1/2)\epsilon U_{2m}^0 \cos 2\theta[(8/m)\epsilon\psi U_{2m}^0 \cos 2\theta] \\ & +(1/4)m[(4/m)\epsilon\psi U_{2m}^0 \sin 2\theta + (1/3)(2\psi)^{3/2}\cos\eta] \\ & [(2\psi)^{1/2}\cos\eta - (4/m)\epsilon U_{2m}^0 \sin 2\theta] \quad (67)\end{aligned}$$

Again, taking the average over θ of the quantity $C_{2\varphi} + (m/2)C_{2\theta}$, we finally obtain:

$$\Gamma_{\varphi}^{2} = \epsilon'(2\psi)^{3/2}(1/6)U_{20}^{0}[-2\sin 2\eta\cos\eta - \cos 2\eta\sin\eta] + \frac{4\psi\epsilon^{2}}{m}{U_{2m}^{0}}^{2} \tag{68}$$

Therefore, up to the second order, we have the following expression for the hamiltonian:

$$-H = \Gamma_{\varphi} = \Gamma_{\varphi}^{0} + \Gamma_{\varphi}^{1} + \Gamma_{\varphi}^{2} \tag{69}$$

$$\begin{aligned} H &= +(m/2)\psi - \epsilon' U_{20}^{0} 2\psi\sin 2\eta - \epsilon'(2\psi)^{3/2} \\ &\quad (1/6)U_{20}^{0}[-2\sin 2\eta\cos\eta - \cos 2\eta\sin\eta] - \frac{4\psi\epsilon^{2}}{m}{U_{2m}^{0}}^{2} \end{aligned} \tag{70}$$

5 Conclusions

We have derived in this work an expression for the hamiltonian for the toroidal helical magnetic field lines system up to the second perturbative order, both in the cylindrical limit approximation and in the toroidal case. Going to higher orders will be a straightforward application of the same procedure used in this work, and the only difficulty to be expected is algebraic complication. The particular hamiltonian we have derived is not the only possible choice, since we can manipulate the Lie transformation in order to get different hamiltonians (in different canonical coordinates). Different expressions for the hamiltonian could be of interest when analyzing particular problems, or for the application of symplectic integration schemas to the solution of the equations deriving from the hamiltonian.

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A Appendix

The explicit form of the various terms appearing in the expression of the magnetic potential for the toroidal case is given by:

$$A_\eta^T = (1/2)[(1-\cos\eta)^{-1}\ln(\frac{1-\xi\cos\eta}{1-\xi}) + (1-\cos\eta)^{-1}\ln(\frac{1-\xi\cos\eta}{1+\xi})]$$

$$A_\xi^{lm} = \frac{-\imath}{m\xi(1-\xi\cos\eta)}U_{lm}(\xi)\partial_\eta(1-\xi\cos\eta)^{1/2}\exp^{\imath l\eta+\imath m\varphi}$$

$$A_\eta^{lm} = \frac{\imath}{m(1-\xi\cos\eta)}\xi(1-\xi^2)\partial_\xi(1-\xi\cos\eta)^{1/2}U_{lm}(\xi)\exp^{\imath l\eta+\imath m\varphi}$$

$$A_\varphi^{l0} = -\int_0^\xi dx[x(1-x\cos\eta)]^{-1}U_{l0}(\xi)\partial_\eta(1-x\cos\eta)^{1/2}\exp^{\imath l\eta}$$

The $U_{lm}(\xi)$satisfy the equation:

$$\partial_\xi[\xi(1-\xi^2)\partial_\xi U_{lm}(\xi)] = (\frac{3\xi}{4} + \frac{l^2}{\xi} + \frac{m^2\xi}{1-\xi^2})U_{lm}(\xi)$$

and can be expanded as:

$$U_{lm}(\xi) = \xi^l[U_{lm}^0 + \xi U_{lm}^1 + \xi^2 U_{lm}^2 + ...]$$

where U_{2m}^0, U_{2m}^1, etc. are constant. For $l=2$ we have $U_{2m}^0 = m^2/8$ and $U_{0m}^0 = 1/2$. These are the only terms we will need.

We now proceed to a Taylor expansion of the various terms, using as smallness parameters ξ, ϵ and ϵ'. Introducing also the new variable $\psi = (1/2)\xi^2$, we obtain, up to the second order:

$$\gamma_\theta^0 = \psi \qquad \gamma_\varphi^0 = -\frac{m}{l}\psi$$

$$\gamma_\theta^1 = (2\psi)^{3/2}\frac{\cos\eta}{3} \qquad \gamma_\varphi^1 = -\frac{m}{l}(2\psi)^{3/2}\frac{\cos\eta}{3} + 2\psi\epsilon U_{lm}^0\sin 2\theta + \epsilon' U_{l0}^0 l\psi\sin\eta$$

$$\gamma_\theta^2 = (1+\cos\eta^2)\psi^2 \qquad \gamma_\varphi^2 = -\frac{m}{l}(1+\cos\eta^2\psi^2 + \epsilon'(2\psi)^{3/2} (1/6)[U_{l0}^0 l\sin 2\eta\cos\eta - U_{l0}^0\cos 2\eta\sin\eta] -(1/6m)\epsilon(2\psi)^{3/2}U_{lm}^0(l^2-1)\cos\eta\sin 2\theta$$

Throughout this work we have chosen $l = 2$. In the case of the cylindrical limit approximation we obtain (for $l = 2$):

$$\begin{array}{ll} \gamma_\theta^0 = \psi & \gamma_\varphi^0 = -(m/2)\psi \\ \gamma_\theta^1 = 0 & \gamma_\varphi^1 = \epsilon\psi(m^2/4)\sin(2\theta) \\ \gamma_\theta^2 = -\epsilon(m^3/8)\psi^2\sin(2\theta) & \gamma_\varphi^2 = \epsilon(m^4/12)\psi^2\sin(2\theta) \end{array}$$

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