ISSN 0915-633X

NATIONAL INSTITUTE FOR FUSION SCIENCE

Reduced form of MHD Lagrangian for Ballooning Modes

R.L. Dewar

(Received - June 10, 1997)

NIFS-507 Sep. 1997

RESEARCH REPORT
NIFS Series

NAGOYA, JAPAN

Reduced form of MHD Lagrangian for ballooning modes

R.L. Dewar*

National Institute for Fusion Science

322-6 Oroshi-cho, Toki

Gifu 509-52, Japan

Abstract

This mini-review sets out the derivation of the short-perpendicular-scale-length approximate reduction of the ideal magnetohydrodynamic Lagrangian to a scalar form in the case of incompressible MHD, or to a two-field form in the compressible case. No eikonal approximation is made. The nature of the field-line bending and field-line compression terms in the energy principle is clarified. Self adjointness and the role of the "kink term" in the energy principle are also discussed.

For submission to J. Plasma & Fusion Res. May 20, 1997

Keywords: Stability, Ideal Magnetohydrodynamics, Energy Principle, Three-dimensional Toroidal Confinement

*Permanent address: Research School of Physical Sciences and Engineering, The Australian National University, Canberra ACT 0200

1 Introduction

In the pioneering paper of Furth, Killeen, Rosenbluth and Coppi (FKRC) [1] a reduced form of the ideal magnetohydrodynamic (MHD) energy principle [2], involving only a single scalar field, was given for perturbations with short scale lengths perpendicular to the magnetic lines of force. A form of the energy principle that is more physically transparent than that of Bernstein *et al.* was introduced by FKRC as a starting point for this reduction.

As written, their starting expression for the potential energy, δW, is correct only if the component of the plasma displacement vector $\boldsymbol{\xi}$ parallel to the equilibrium magnetic field, $\mathbf{B}$, vanishes. The correct expression for the case of finite $\xi_{\|}$ was given by Greene and Johnson [3], and a heuristic physical interpretation of the terms making up the energy expression was given.

In this note we derive a reduced form of the energy principle, similar to that of FKRC, using a systematic asymptotic approach in which the only expansion parameter is the ratio of the perpendicular scale length, $1/k_{\perp}$, of $\boldsymbol{\xi}$ to a typical equilibrium scale length, L. The scale length of variation of $\boldsymbol{\xi}$ along the magnetic field lines, $1/k_{\|}$, is taken to be the same as the equilibrium scale length. The variational principle we work with is actually Hamilton's principle, or equivalently the Rayleigh-Ritz variational principle for mode frequency (or growth rate when the frequency is imaginary), so we also discuss models for the kinetic energy term in the Lagrangian.

Our derivation avoids the use of an eikonal approximation, and thus our variational principle (and associated Euler–Lagrange equations) should provide a convenient starting point for the discussion of cases for which the simple eikonal ansatz breaks down. The use of this ansatz was discussed by Dewar and Glasser [4], who used WKB ray tracing to investigate the spectrum of global ballooning eigenmodes in nonaxisymmetric geometries such as stellarators. One place our reduced Lagrangian could simplify their discussion is in calculating the reflection condition at caustics.

A more fundamental problem where it is hoped to apply the reduced Lagrangian is in the analysis of the case when the contours of constant growth rate in the three-dimensional reduced phase space, used by Dewar and Glasser for the ray tracing analysis, are topologically spherical. Although their WKB analysis indicated that these "broad continuum" modes are highly singular, the rays being attracted to fixed points in the phase space, it could not be used to analyse them in detail because the WKB assumption of a separation in scale length between the exponential eikonal term and the amplitude factor broke down.

Nakajima [5] has recently found interchange-stable cases for the Large Helical Device (LHD) that have topologically spherical marginal ballooning stability contours. Thus the only unstable ideal MHD modes in such LHD cases are the highly singular broad continuum modes. It is of considerable physical importance to determine whether kinetic effects stabilize, or greatly slow down the growth, of these singular modes. As a first step, one needs to revisit the ideal MHD analysis to derive an equation for these singular ballooning modes. It is hoped that the present paper will provide the basis for such an analysis.

In Sec. 2 we briefly review the ideal MHD equilibrium equations, straight-field-line magnetic coordinates and curvature identities, and in Sec. 3 we review variational principles for ideal MHD mode growth rates. In Secs. 4 and

5 we motivate the basic orderings by a study of the local plane wave dispersion relation and introduce a representation for the plasma displacement that enforces these orderings.

The reduced Lagrangians for the one- and two-field, incompressible and compressible, wave equations are derived in Sec. 6. A model wave equation is derived in the single-field case, and the relation of the two-field Lagrangian to the ballooning-mode Lagrangian of Dewar and Glasser [4] is discussed. In Appendix A we discuss the geometric interpretation of two important terms, the "field-line bending" and "field-line compression" terms, of the potential energy introduced in Sec. 3 and conclude that this terminology can only be rigorously justified within the short-wavelength ordering. In Appendix B we give a detailed discussion of self adjointness and the relation between the "kink term" and the geodesic part of the "curvature term" in the energy principle.

The approach in this paper is very standard. It is not claimed that the material is new. Indeed most of it is well known to experts in the field, but the author has not been able to find a clear discussion in the literature. Partly this is because modern text books tend to be specialized to axisymmetric systems because of the dominance of tokamaks in the last two decades. With large stellarators such as LHD coming into operation it is timely to review the general, fully three-dimensional formalism. Therefore we have set out the details and fine points of the derivation, in the hope that newcomers to the field will be helped in getting started and that established workers will find it a useful summary for reference. We have also tried to give an indication of the connection between this classic analysis and modern numerical MHD stability codes.

2 Equilibrium

In this paper we have tried to keep the discussion as coordinate free as possible, but, in discussing the choice of kinetic energy normalization, we shall have need of a generic curvilinear straight-field-line magnetic coordinate system. Assuming the equilibrium magnetic field lines all lie within nested invariant magnetic surfaces we label them with the enclosed poloidal flux $2\pi\psi$. Introducing poloidal and toroidal angles θ and ζ, we write the equilibrium magnetic field as

$$\mathbf{B} = \nabla\zeta\times\nabla\psi + q\nabla\psi\times\nabla\theta \; , \tag{1}$$

where $q(\psi)$ is the safety factor (inverse of the rotational transform, $\iota\!\!\!-$).

We assume there to be no flow in the unperturbed state of the plasma, so that the condition for a stationary state is the equilibrium condition

$$\nabla p = \mathbf{j}\times\mathbf{B} \; , \tag{2}$$

where $\mathbf{j} = \nabla\times\mathbf{B}/\mu_0$ is the equilibrium current and $p(\psi)$ is the equilibrium pressure, with μ_0 being the permeability of free space (SI units).

Following Greene and Johnson [3] we also define the quantities $\sigma \equiv \mathbf{j}\cdot\mathbf{B}/B^2$, proportional to the parallel component of the current, and $\boldsymbol{\kappa} \equiv \mathbf{e}_{\parallel}\cdot\nabla\mathbf{e}_{\parallel}$, the curvature of the magnetic field lines.

Using $\nabla\times\mathbf{B} = \mu_0\mathbf{j}$ and the equilibrium relation Eq. (2), we can prove the useful identity [3] relating the field-line curvature to the perpendicular component of the gradient of the total (field plus kinetic) pressure

$$\boldsymbol{\kappa} = \frac{\nabla_\perp(B^2 + 2\mu_0 p)}{2B^2} \,, \tag{3}$$

where subscript $\perp$ on any vector $\mathbf{f}$ denotes its projection $\mathsf{P}_\perp\cdot\mathbf{f}$ onto the plane locally perpendicular to $\mathbf{B}$, $\mathsf{P}_\perp$ being the *perpendicular projection operator*

$$\mathsf{I} - \mathbf{e}_\parallel \mathbf{e}_\parallel \,. \tag{4}$$

The curvature $\boldsymbol{\kappa}$ is orthogonal to $\mathbf{B}$. It can be decomposed into a component, $\kappa_\mathrm{n} \equiv \boldsymbol{\kappa}\cdot\nabla\psi/|\nabla\psi|$, in the direction of the normal to the magnetic surface on which it is being evaluated, and a *geodesic* component [3] $\kappa_\mathrm{g} \equiv \boldsymbol{\kappa}\cdot\nabla\psi\times\mathbf{B}/|B\nabla\psi|$ in the tangent plane of the magnetic surface.

Decomposing the current into its parallel and perpendicular components, $\mathbf{j} \equiv \sigma\mathbf{B} + \mathbf{j}_\perp$, and using $\nabla\cdot\mathbf{j} = 0$ we see that $\mathbf{B}\cdot\nabla\sigma = -\nabla\cdot\mathbf{j}_\perp$. From the equilibrium condition Eq. (2), it is easily seen that $\mathbf{j}_\perp = \mathbf{B}\times\nabla p/B^2$, which is called the *diamagnetic current.* Taking its divergence and using Eq. (3) gives an identity relating the parallel current to the geodesic curvature

$$\mathbf{B}\cdot\nabla\sigma = \frac{2\boldsymbol{\kappa}\cdot\mathbf{B}\times\nabla p}{B^2} \,. \tag{5}$$

3 Lagrangian

The ideal MHD equations for a linearized displacement $\boldsymbol{\xi}$ with time dependence cos or $\sin\omega t$ are the Euler–Lagrange equations that extremize the time-averaged Lagrangian

$$L = \omega^2 K - \delta W \,, \tag{6}$$

where $\omega^2 K$ is the kinetic energy and δW is the potential energy of the plasma. As this variational principle can also be arrived at by minimizing δW under the normalization constraint $K = \text{const}$, ω^2 then being a Lagrange multiplier, the choice of K is often referred to as the "normalization".

Note that, to derive the first term in Eq. (6) from the physical kinetic energy involving the square of the linearized velocity, $\partial\boldsymbol{\xi}/\partial t$, we must assume that ω is real. The case of instability is often treated by considering ω to be imaginary, so that $\omega^2 < 0$, but we must regard this as an analytic continuation to be performed *after* Eq. (6) is derived.

3.1 Kinetic energy models

The kinetic energy factor K is defined by

$$K = \tfrac{1}{2} \int_\Omega d^3x \, \boldsymbol{\xi}\cdot\boldsymbol{\rho_M}\cdot\boldsymbol{\xi} \, , \tag{7}$$

where Ω is the region occupied by the plasma. Here we have allowed for the use of a model kinetic energy with an anisotropic inertia by use of a dyadic mass density, $\boldsymbol{\rho_M}$, since, as will become apparent in Sec. 6.1, it is often convenient to remove $\xi_\parallel$ from K in order to make the extremizing displacements purely incompressible (since $\xi_\parallel$ then enters the Lagrangian through the term in $|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}|^2$). This does not change points of marginal stability in parameter space, but can have a profound effect on the eigenvalues.

In this section we consider several choices for K:

1. **Physical normalization**

   Here we use an isotropic density

$$\boldsymbol{\rho_M} = \rho(\psi)\mathsf{I} \, . \tag{8}$$

2. **PEST2 normalization**

An inertia purely normal to the magnetic surfaces is used in the PEST2 code [6]

$$\boldsymbol{\rho}_M = \boldsymbol{\nabla}\psi\boldsymbol{\nabla}\psi \; . \tag{9}$$

Clearly $(\xi_{\|}\mathbf{e}_{\|})\cdot\boldsymbol{\rho}_M = \boldsymbol{\rho}_M\cdot(\xi_{\|}\mathbf{e}_{\|}) \equiv 0$, as desired. However this model also annihilates the perpendicular component of $\boldsymbol{\xi}$ within a surface, and thus gives rise to a very unphysical spectrum. Essentially the same normalization is used in the CAS3D1 code [7].

3. **TERPSICHORE normalization**: Following Cooper *et al.* [8], we can also use the dyadic mass density currently used in the global stability code TERPSICHORE

$$\boldsymbol{\rho}_M = (2q)^2\boldsymbol{\nabla}\psi\,\boldsymbol{\nabla}\psi + [\psi'(s)]^2\,\mathbf{e}^{\perp}\,\mathbf{e}^{\perp} \; , \tag{10}$$

where s is a dimensionless surface label and the basis vector

$$\begin{aligned} \mathbf{e}^{\perp} &\equiv \boldsymbol{\nabla}\zeta - q\boldsymbol{\nabla}\theta \; , \\ &= -\frac{\mathbf{B}\times(\boldsymbol{\nabla}\theta\times\boldsymbol{\nabla}\zeta)}{\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\theta\times\boldsymbol{\nabla}\zeta} \end{aligned} \tag{11}$$

so that it is orthogonal to $\mathbf{B}$ (thus eliminating $\xi_{\|}$) and to $\boldsymbol{\nabla}\theta\times\boldsymbol{\nabla}\zeta$, where θ and ζ are as in Sec. 2. For the surface label we follow the convention in the VMEC code [9] and label the magnetic surfaces using the toroidal magnetic flux, $2\pi\Phi$, so that $s \equiv \Phi(\psi)/\Phi(\psi_a)$ runs from zero at the magnetic axis to unity at the plasma edge, $\psi = \psi_a$.

This normalization is also not very physical and is not invariant under choice of toroidal and poloidal angle.

4. Incompressible physical model

If we take

$$\rho_M = \rho(\psi)\left(\frac{\nabla\psi\nabla\psi}{|\nabla\psi|^2} + \frac{\nabla\psi\times\mathbf{B}\,\nabla\psi\times\mathbf{B}}{|\nabla\psi|^2|\mathbf{B}|^2}\right) \tag{12}$$

we have a compromise choice that is physical in the plane perpendicular to $\mathbf{B}$, but still annihilates the parallel component of $\boldsymbol{\xi}$ from K, so $\nabla\cdot\boldsymbol{\xi}$ vanishes identically (see Sec. 6.1). From Eq. (18) we see that the Alfvén wave has no component in the parallel direction, so its spectrum will be unaffected by the use of this normalization. However from Eq. (19) we see that the polarization of the slow magnetosonic mode is primarily in the parallel direction, so how much this normalization distorts the eigenvalue depends on how much slow magnetosonic component the mode in question has.

One can also make the argument that the parallel dynamics in a collisionless plasma cannot be described by ideal MHD and that non-ideal effects (parallel viscosity) will damp parallel motions, so that pressure fluctations along the magnetic field line cannot be supported. The latter fact can be captured by assuming incompressibility as a constraint from the outset, and the former by only using the perpendicular part of the momentum equation, leading to a version of MHD called "collisionless MHD" by Freidberg [10, pp. 32–38 and p. 260]. As we shall see in Sec. 6.1.1, incompressibility is a consequence of Eq. (12), so the normalization of Eq. (12) is completely equivalent to "collisionless MHD".

3.2 Potential energy

For the potential energy we use the form due to Furth *et al.* [1], as given in full generality by Greene and Johnson [3]

$$\delta W = \tfrac{1}{2}\int_{\Omega} d^3x \left[\frac{\mathbf{Q}_\perp^2}{\mu_0} + \frac{(\mathbf{Q}\cdot\mathbf{B} - \mu_0\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p)^2}{\mu_0 B^2} + \gamma p(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2 - \sigma\boldsymbol{\xi}_\perp\times\mathbf{B}\cdot\mathbf{Q}_\perp - 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}\,\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}p\right] + \delta W_{\rm vac}\,, \qquad (13)$$

where $\mathbf{Q} \equiv \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\mathbf{B})$ is the perturbation in the magnetic field and the equilibrium quantities are as defined as in Sec. 2. The vacuum energy, $\delta W_{\rm vac}$, is the energy in the perturbed magnetic field associated with the vacuum region surrounding the plasma (if any). We also include in this any surface energy [2].

It is convenient to refer to the five terms making up the plasma contribution to δW by reference to their heuristic interpretations as discussed by Greene and Johnson [3] (see also Freidberg [10, p. 259]): We call

- the term, $\mathbf{Q}_\perp^2/\mu_0$, the *field-line bending term*;
- the second term, involving the parallel component of $\mathbf{Q}$, the *field-line compression term*;
- the third term, proportional to γp, the *fluid compression term*;
- the fourth term, proportional to the parallel equilibrium current, the *kink term*;
- and the final term, proportional to the pressure gradient and the curvature vector, the *curvature term*. (Which can also be called the *interchange term* [11, p. 250].)

This terminology is discussed further in Appendix A.

It is an extremely important property of ideal MHD that the linearized force operator is "self-adjoint", or Hermitian [10, pp. 242–243]. Because of this it is often useful to write δW in the form $\delta W(\boldsymbol{\xi},\boldsymbol{\xi})$, where $\delta W(\boldsymbol{\eta},\boldsymbol{\xi})$ is the *bilinear form* obtained by replacing the first $\boldsymbol{\xi}$ in each term of Eq. (13) (including those implicit in $\mathbf{Q}$) with the independent vector field $\boldsymbol{\eta}$. [1] That is, $\delta W(\boldsymbol{\eta},\boldsymbol{\xi})$ is symmetric under interchange of $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$.

For example, this allows us to write the first variation of δW as $2\delta W(\delta\boldsymbol{\xi},\boldsymbol{\xi})$. This means we can treat $\boldsymbol{\xi}$ as a complex variable, replacing $L(\boldsymbol{\xi},\boldsymbol{\xi})$ with the bilinear form $L(\boldsymbol{\xi}^*,\boldsymbol{\xi})$, where $\boldsymbol{\xi}^*$ denotes the complex conjugate of $\boldsymbol{\xi}$. The Euler–Lagrange equations for extremizing $L(\boldsymbol{\xi}^*,\boldsymbol{\xi})$ under *independent* variations of $\boldsymbol{\xi}^*$ and $\boldsymbol{\xi}$ are both physically valid because the equilibrium quantities and ω^2 are real and hence $\boldsymbol{\xi}$ and $\boldsymbol{\xi}^*$ obey the same equations. This trick is useful as it is often convenient to use complex exponential notation when dealing with wavelike perturbations.

Self-adjointness can also be demonstrated when there is a vacuum region [2], but for simplicity we consider only internal modes, taking $\boldsymbol{\xi}$ to vanish at the plasma boundary and dropping the vacuum energy. Then the only terms for which the symmetry is not manifestly obvious are the kink and curvature terms. To demonstrate it for these terms we need to integrate by parts and use the identity, Eq. (5), to obtain a cancellation between the $\mathbf{B}\cdot\nabla\sigma$ term arising from the integration by parts of the kink term, and the

[1] Note that we have written the term involving $\boldsymbol{\kappa}$ in the reverse order to that used by Greene and Johnson [3] so as to define the bilinear form correctly. Alternatively one can explicitly symmetrize before writing the bilinear form.

geodesic correction arising from interchanging $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ in the curvature term (see Appendix B).

Hamilton's principle requires δL to be stationary under arbitrary variations of $\boldsymbol{\xi}$, the Euler–Lagrange equation giving the equations of motion. Equivalently, the Rayleigh–Ritz variational principle requires that

$$\omega^2 = \frac{\delta W}{K} \tag{14}$$

be stationary under arbitrary variations of $\boldsymbol{\xi}$.

4 Orderings

We shall denote our formal asymptotic expansion parameter by ϵ and adopt the following scalings as $\epsilon \to 0$:

$$L \sim k_\parallel = O(1) , \tag{15}$$

$$k_\perp = O(\epsilon^{-1}) , \tag{16}$$

$$\mathbf{B} \sim \xi_\perp \sim \xi_\parallel = O(1) . \tag{17}$$

Note that we do *not* assume $\mathbf{k}\cdot\mathbf{B} \equiv k_\parallel B$ to be small, but, rather, take $\mathbf{k}\times\mathbf{B}$ to be large. That is, the wavelength perpendicular to the magnetic field is short, but the wavelength along the field lines is comparable with the system size. By an architectural analogy, such perturbations are often called *flute-like*, as the magnetic surfaces of a plasma supporting such perturbations are deformed into a shape resembling a fluted column whose grooves are aligned approximately with the magnetic field lines. (Since we assume the field to have finite rotational transform, these helically fluted columns are more baroque than Grecian.)

Since there is no separation between the equilibrium and perturbation scale lengths along the magnetic field lines, $k_\parallel$ is not precisely defined as a wavelength and should be interpreted as denoting $\mathbf{e}_\parallel \cdot \boldsymbol{\nabla} \log f$, where $\mathbf{e}_\parallel \equiv \mathbf{B}/B$ is the unit vector in the direction of $\mathbf{B}$ and f is any perturbed quantity. However, in order to motivate these orderings it is nevertheless instructive to examine the local plane-wave dispersion relation for MHD waves, assuming *temporarily* that the concept of wavelength is well defined in all directions.

As discussed by Dewar and Glasser [4], the theory of the stability of static MHD equilibria can be based on finding relatively small geometric corrections to the local plane wave dispersion relations (see e.g. Freidberg [10, pp. 234–238]) for the two MHD wave branches whose frequencies are finite in the limit $\epsilon \to 0$. For $k_\parallel \ll k_\perp$ the dispersion relations and polarization vectors for the Alfvén wave ($\omega_{\rm A}$), the slow magnetosonic wave ($\omega_{\rm S}$) and the fast magnetosonic wave ($\omega_{\rm F}$) are given by

$$\rho\omega_{\rm A}^2 = (\mathbf{k}\cdot\mathbf{B})^2 \quad , \quad \boldsymbol{\xi}_{\rm A} = \mathbf{k}\times\mathbf{B}/k \; , \tag{18}$$

$$\rho\omega_{\rm S}^2 = \frac{\gamma p(\mathbf{k}\cdot\mathbf{B})^2}{B^2+\gamma p} \quad , \quad \boldsymbol{\xi}_{\rm S} = (B^2+\gamma p)\mathbf{B} - \gamma p \mathbf{k}\cdot\mathbf{B}\mathbf{k}_\perp/k_\perp^2 \; , \tag{19}$$

$$\rho\omega_{\rm F}^2 = (B^2+\gamma p)k^2 \quad , \quad \boldsymbol{\xi}_{\rm F} = \gamma p \mathbf{k}\cdot\mathbf{B}\mathbf{B}/k_\perp + (B^2+\gamma p)B^2\mathbf{k}_\perp/k_\perp \; , \tag{20}$$

where ρ is the mass density. (Note that this is for the physical, isotropic mass density case. The dispersion relations would be different for the dyadic mass density model discussed in Sec. 3.1.)

Since the frequencies of the Alfvén and the slow magnetosonic wave modes are proportional to $k_\parallel$, the orderings in Eq. (15) are necessary in order that ω be finite in the limit. (The actual value of ω cannot be found from the plane-wave dispersion relations of course, but its ordering must be consistent with these relations.) On the other hand, there is no direction of propagation for which ω_F is finite in the short-wavelength limit, so the fast magnetosonic branch must be excluded from the space in which we choose the lowest order approximation to the solution.

The ordering of the components of $\boldsymbol{\xi}$ can also be seen from these considerations: from Eq. (18) and Eq. (19) we see that we must allow both parallel and perpendicular components in $\boldsymbol{\xi}$. However, because we are excluding the fast magnetosonic wave, the lowest order approximation to $\boldsymbol{\xi}$ is constrained to be orthogonal to $\boldsymbol{\xi}_F$. As this mode is primarily longitudinal (i.e. $\boldsymbol{\xi}_F$ is parallel to $\mathbf{k}$ to lowest order), the constraint on $\boldsymbol{\xi}$ is *that, to lowest order in* ϵ, *it be transverse to* $\mathbf{k}$. As a consequence it is easily verified that the following orderings hold

$$\nabla\cdot\boldsymbol{\xi} \sim \nabla\cdot\boldsymbol{\xi}_\perp = O(1) , \tag{21}$$

$$\mathbf{Q}_\perp \sim Q_\parallel = O(1) . \tag{22}$$

For the fast magnetosonic mode, on the other hand, the ordering of these quantitities would be $O(\epsilon^{-1})$.

As a final check, it is readily verified that the orderings Eqs. (15–17), (21) and (22) are consistent with the Rayleigh-Ritz form of the variational principle, Eq. (14), in that they make $\omega^2 = O(1)$ as desired.

5 Representation of $\boldsymbol{\xi}$ and Q

¿From Eq. (18) and Eq. (19), the motion perpendicular to $\mathbf{B}$ is dominated by the Alfvén branch and is primarily transverse to $\mathbf{k}$, but the slow magnetosonic wave does provide a small longitudinal component. Thus we are led to the representation

$$\boldsymbol{\xi}_\perp = \frac{\mathbf{B}\times\boldsymbol{\nabla}\varphi}{B^2} - [\boldsymbol{\nabla}_\perp\chi]\,, \tag{23}$$

where $\varphi = O(\epsilon)$ is a stream function for the perpendicular displacement and $\chi = O(\epsilon^2)$ is a potential to provide the small longitudinal part. As the parallel component of $\boldsymbol{\xi}$ is a scalar, $\xi_\parallel$, we do not need any special representation for it at this stage.

In this section we indicate that terms are $O(\epsilon)$ by enclosing them in square brackets. Thus, in Eq. (23), the first term is $O(1)$ but the second is $O(\epsilon)$.

Since the curl of a gradient vanishes identically, the divergence

$$\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp = \boldsymbol{\nabla}\times\left(\frac{\mathbf{B}}{B^2}\right)\cdot\boldsymbol{\nabla}\varphi - \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}_\perp\chi \tag{24}$$

is $O(1)$ as required by Eq. (21). Also, $\boldsymbol{\nabla}\cdot(\xi_\parallel\mathbf{e}_\parallel) \equiv \mathbf{B}\cdot\boldsymbol{\nabla}(\xi_\parallel/B) = O(1)$, by Eq. (15). Thus $\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = O(1)$ also, as assumed in Eq. (21).

In order to calculate the magnetic field perturbation $\mathbf{Q} \equiv \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\mathbf{B})$, first note from Eq. (23) that

$$\boldsymbol{\xi}\times\mathbf{B} = \boldsymbol{\nabla}_\perp\varphi + [\mathbf{B}\times\boldsymbol{\nabla}_\perp\chi]\,. \tag{25}$$

Thus

$$\begin{aligned}\mathbf{Q} &= \boldsymbol{\nabla}\times(\boldsymbol{\nabla}_\perp\varphi) + \mathbf{B}\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}_\perp\chi \\ &\quad + [\boldsymbol{\nabla}_\perp\chi\cdot\boldsymbol{\nabla}\mathbf{B} - \mathbf{B}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}_\perp\chi]\,.\end{aligned} \tag{26}$$

It is immediately seen that the parallel component of $\mathbf{Q}$ is $O(1)$, as required by Eq. (22), but at first sight it appears that the perpendicular component is large, $O(\epsilon^{-1})$, because it contains two $\boldsymbol{\nabla}$ operators. However, by noting $\boldsymbol{\nabla}_\perp\varphi \equiv \boldsymbol{\nabla}\varphi - (\mathbf{B}/B^2)\mathbf{B}\cdot\boldsymbol{\nabla}\varphi$ and observing that $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\varphi \equiv 0$ we can prove the identity

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}_\perp\varphi = \mathbf{B}\times\boldsymbol{\nabla}\left(\frac{\mathbf{B}\cdot\boldsymbol{\nabla}\varphi}{B^2}\right) - \left[\mu_0\mathbf{j}\frac{\mathbf{B}\cdot\boldsymbol{\nabla}\varphi}{B^2}\right] , \tag{27}$$

whence we see that the perpendicular component of $\mathbf{Q}$ is also $O(1)$, so the ordering in Eq. (22) is now completely verified.

Note that, although χ does not contribute to $\boldsymbol{\xi}$ at leading order, it *does* contribute at leading order to its divergence [see Eq. (24)], and hence to both the fluid compression term of δW *and* to the field-line compression term through the following identity [3], derived using Eq. (3)

$$\mathbf{Q}\cdot\mathbf{B} - \mu_0\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p = -B^2(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}) . \tag{28}$$

6 Reduction of the Lagrangian

Using the above results we find, to leading order in ϵ,

$$\begin{aligned} 2\delta W^{(0)} = \int_\Omega d^3x \Bigg\{ & \frac{B^2}{\mu_0}\left|\boldsymbol{\nabla}_\perp\left(\frac{\mathbf{B}\cdot\boldsymbol{\nabla}\varphi}{B^2}\right)\right|^2 + \frac{|B^2\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\kappa}\times\mathbf{B}\cdot\boldsymbol{\nabla}_\perp\varphi|^2}{\mu_0 B^2} \\ & + \gamma p\left|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{\xi_\parallel}{B}\right)\right|^2 \\ & + \frac{(\mathbf{B}\times\boldsymbol{\nabla}\sigma\cdot\boldsymbol{\nabla}_\perp\varphi^*)(\mathbf{B}\cdot\boldsymbol{\nabla}\varphi)}{2B^2} + \frac{(\mathbf{B}\cdot\boldsymbol{\nabla}\varphi^*)(\mathbf{B}\times\boldsymbol{\nabla}\sigma\cdot\boldsymbol{\nabla}_\perp\varphi)}{2B^2} \\ & - \frac{(\boldsymbol{\kappa}\times\mathbf{B}\cdot\boldsymbol{\nabla}_\perp\varphi^*)(\boldsymbol{\nabla}p\times\mathbf{B}\cdot\boldsymbol{\nabla}_\perp\varphi)}{B^4} \\ & - \frac{(\boldsymbol{\nabla}p\times\mathbf{B}\cdot\boldsymbol{\nabla}_\perp\varphi^*)(\boldsymbol{\kappa}\times\mathbf{B}\cdot\boldsymbol{\nabla}_\perp\varphi)}{B^4}\Bigg\} , \end{aligned} \tag{29}$$

and

$$2K^{(0)} = \int_\Omega d^3x \left(\frac{\mathbf{B}\times\nabla_\perp\varphi^*}{B^2} + \xi_\|^*\mathbf{e}_\|\right)\cdot\boldsymbol{\rho}_M\cdot\left(\frac{\mathbf{B}\times\nabla_\perp\varphi}{B^2} + \xi_\|\mathbf{e}_\|\right) . \tag{30}$$

where the superscript (0) denotes the part that is $O(\epsilon^0)$ and we have used the "complexified" bilinear form discussed in Sec. 3.2 in order to make sure the self adjointness is explicit (see Appendix B).

Note that the term in $\nabla\sigma$, arising from the kink term after integration by parts (again see Appendix B), is formally one order in ϵ smaller than the other terms. It has been retained here for several reasons:

- It is closely related to the geodesic part of the curvature term (see Appendix B).
- It is of opposite parity in $\mathbf{B}\cdot\nabla$ to the other terms so it could produce physically interesting qualitative changes to "normal" ballooning mode behavior (e.g. it reduces to the term in $j'(r)$ that drives tearing modes in axisymmetric, large-aspect-ratio machines [12] so it could be important for resistive ballooning stability).
- For machines close to marginal stability against ballooning, where the leading order terms balance $[\omega^2 = O(\epsilon)]$, it could be an important stabilizing or destabilizing term.
- It could be important in systems with steep current density gradients $[\nabla\sigma = O(\epsilon^{-1})]$, for instance near rational surfaces in stellarators when there are large Pfirsch–Schlüter currents [13, 14, 15].

However, if one is really to take this term seriously one should also check that we have not omitted other $O(\epsilon)$ terms that should be retained for consistency.

6.1 One-field, incompressible models

If we use the kinetic energy models 2–4 in Sec. 3, which annihilate the parallel component of $\boldsymbol{\xi}$, then the kinetic energy norm reduces to

$$2K^{(0)} = \int_\Omega d^3x \, \frac{\mathbf{B}\times\boldsymbol{\nabla}_\perp\varphi^*\boldsymbol{\cdot}\boldsymbol{\rho}_M\boldsymbol{\cdot}\mathbf{B}\times\boldsymbol{\nabla}_\perp\varphi}{B^4} \,. \tag{31}$$

Now the only place $\xi_\|$ appears in the Lagrangian is in the fluid compression term. Requiring stationarity of the Lagrangian under arbitrary variations of $\xi_\|^*$ gives, after integrating by parts, the Euler–Lagrange equation

$$\mathbf{B}\boldsymbol{\cdot}\boldsymbol{\nabla}\left[\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\xi}_\perp + \mathbf{B}\boldsymbol{\cdot}\boldsymbol{\nabla}\left(\frac{\xi_\|}{B}\right)\right] \equiv \mathbf{B}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\xi} = 0 \,. \tag{32}$$

6.1.1 Fluid incompressibility

On a surface of irrational rotational transform, the only solution of this equation is the *incompressibility condition*

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\xi}_\perp + \mathbf{B}\boldsymbol{\cdot}\boldsymbol{\nabla}\left(\frac{\xi_\|}{B}\right) \equiv \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\xi} = 0 \,. \tag{33}$$

On a rational surface we could in principle have $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\xi}$ different on the different closed field lines making up the surface. However, if we restrict the solution space for $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\xi}$ to continuous functions of ψ then incompressibility must apply also on rational surfaces because they are nested arbitrarily closely by irrational surfaces. However, we must still ask if it is possible to find $\xi_\|$ such that $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\xi} = 0$ everywhere. This requires the solution of the magnetic differential equation [10, p. 262]

$$\mathbf{B}\boldsymbol{\cdot}\boldsymbol{\nabla}\left(\frac{\xi_\|}{B}\right) = -\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\xi}_\perp \,. \tag{34}$$

This can be solved by Fourier analysis [11, p. 59]

$$\xi_{\parallel,m,n} = \frac{i(\mathcal{J}\nabla\cdot\boldsymbol{\xi}_{\perp})_{m,n}}{m - nq} , \tag{35}$$

where the Jacobian factor $\mathcal{J} \equiv 1/\nabla\psi\cdot\nabla\theta\times\nabla\zeta$ and m and n are the poloidal and toroidal mode numbers, respectively.

As $\nabla\cdot\boldsymbol{\xi}_{\perp}$ is completely independent of $\xi_{\parallel}$, there is no reason to suppose that the resonant coefficients $(\mathcal{J}\nabla\cdot\boldsymbol{\xi}_{\perp})_{m,n}$, $m = nq$, will in general vanish on rational surfaces, so that this model has the fault that it implies divergent $(\xi_{\parallel})_{m,n}$ on rational surfaces.[2] However, it can be shown [10, p. 290–291] that, by regularizing the singularity in Eq. (35) and taking the limit as the regularizing parameter tends to zero, the contribution of the fluid compression term to δW can indeed be made to vanish, so that the incompressibility constraint does not affect marginal stability boundaries. The contribution to the kinetic energy would however tend to infinity as the regularizing parameter tends to zero if we were to use the full isotropic density model of Sec. 3.1, which implies from Eq. (14) that the incompressibility constraint can have a significant effect on growth rates (especially in low-magnetic shear systems with rotational transform close to a low-order rational).

(**Aside:** It might be thought that the ballooning formalism, in which Eq. (34) is solved on a covering space in which the periodicity condition is relaxed [4] somehow magically gets around this problem. This is not the case, because $\int d\theta\, \mathcal{J}\nabla\cdot\boldsymbol{\xi}_{\perp}$ over a field line does not in general vanish: if we impose $\xi_{\parallel} \to 0$ as $\theta \to -\infty$, then $\xi_{\parallel}$ cannot also tend to zero as $\theta \to +\infty$. Then

[2] In fact there is a problem on *all* surfaces for $m = n = 0$, but this part of Fourier space is excluded asymptotically by the large-$\mathbf{k}_{\perp}$ ordering. It can also be rigorously avoided by restricting $\boldsymbol{\xi}$ to a "mode family"[7] not including $n = 0$.

the infinite sum in the ballooning representation for $\xi_\parallel$ diverges on rational surfaces as $\theta \to +\infty$.)

6.1.2 Field-line incompressibility

We now observe that, to this order, χ appears only in the compressive terms and only through the term $\nabla\cdot\nabla_\perp\chi$ in $\nabla\cdot\boldsymbol{\xi}_\perp$. (Since we have now set $\nabla\cdot\boldsymbol{\xi} = 0$, in this section $\nabla\cdot\boldsymbol{\xi}_\perp$ in fact appears only in the field-line compression term.) Thus varying χ^* is equivalent to varying $\nabla\cdot\boldsymbol{\xi}_\perp^*$, and doing this we find the simple Euler–Lagrange equation

$$B^2\nabla\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\kappa}\times\mathbf{B}\cdot\nabla_\perp\varphi \equiv (\mathbf{Q}\cdot\mathbf{B} - \mu_0\boldsymbol{\xi}\cdot\nabla p)^{(0)} = 0 \,. \tag{36}$$

Using Eq. (3) and Eq. (24), Eq. (36) can be written as

$$\nabla\cdot\nabla_\perp\chi = -\frac{\mu_0\mathbf{j}\cdot\nabla_\perp\varphi}{B^2} \,. \tag{37}$$

We do not actually need to solve this equation, but it is worthwhile to convince ourselves that it *can* be solved without having to restrict the right hand side.

If it were not for the fact that $\nabla\chi$ must be projected at each point onto the plane perpendicular to $\mathbf{B}$ there would be no problem — we would have Poisson's equation. Applying the Dirichlet boundary condition $\chi = 0$ on the plasma boundary $\partial\Omega$ (appropriate for internal modes), the equation could be solved using, for instance, the standard Green's function for the Laplace operator on the left-hand side.

If we can convince ourselves that the operator $\nabla\cdot\nabla_\perp$ is an *elliptic* differential operator, then Eq. (37) is a simple generalization of Poisson's equation and can still in principle be solved by standard techniques. It is here that there appears to be a problem: The elementary pointwise definition of an elliptic operator requires the form $\mathbf{k}\cdot\mathsf{P}_\perp\cdot\mathbf{k}$ to be nonzero for *all* nonzero $\mathbf{k}$. Here $\mathbf{k} \equiv \boldsymbol{\nabla} S$, with S being the *eikonal*, defined through the representation $\chi = \hat{\chi}\exp(iS/\epsilon)$, where S and $\hat{\chi}$ are slowly varying as $\epsilon \to 0$. However $\mathbf{k}\cdot\mathsf{P}_\perp\cdot\mathbf{k}$ clearly *is* zero for $\mathbf{k}$ parallel to $\mathbf{B}$.

We can argue on the basis of our large $\mathbf{k}_\perp/\epsilon$ ordering that the case $\mathbf{k}_\perp = 0$ is excluded, and that is perhaps sufficient for our purposes. However, by adopting a more general "weak sense" viewpoint of ellipticity, requiring $\int_\Omega |\boldsymbol{\nabla}_\perp\chi|^2\, d^3x > 0$ for all χ such that $\int_\Omega |\chi|^2\, d^3x > 0$. we can also make the stronger claim that the problem does indeed appear to be a standard elliptic one irrespective of asymptotic arguments.

We argue by *reductio ad absurdum*: If the weak-sense ellipticity condition were to be violated, then we would have to have $\boldsymbol{\nabla}_\perp\chi = 0$ throughout Ω, but with $\chi = \text{const} \neq 0$ in some finite region, $\Omega' \subset \Omega$.

We can solve $\boldsymbol{\nabla}_\perp\chi = 0$ at each point by integrating back along any curve Γ made up of line elements everywhere perpendicular to $\mathbf{B}$ (so $\chi = \text{const}$ on Γ) to a surface on which we know the value of χ. In particular, if Γ connects with $\partial\Omega$, then $\chi = 0$. Thus, for χ to be nonzero in Ω', all integral curves Γ starting within Ω' must remain in Ω'. As a consequence, any such curve touching its boundary, $\partial\Omega'$, must be *tangential* to $\partial\Omega'$. In other words, $\mathbf{B}$ must be everywhere *normal* to $\partial\Omega'$.

But in a magnetic plasma confinement device we can assume that $\mathbf{B} \neq 0$, and thus the normal component of $\mathbf{B}$ on $\partial\Omega'$ must be of constant sign. However, by Gauss' theorem this is impossible. Thus the hypothesized region

Ω', isolated from the plasma edge, cannot exist. We conclude that Eq. (37) is a generalized Poisson's equation and so, unlike the fluid incompressibility condition, the field-line incompressibility condition should be easy to satisfy.

We note that field-line incompressibility has been found to be very accurately satisfied, even for low-n modes, in both stellarators and tokamaks by using the CAS3D code [7].

6.1.3 One-field δW and wave equation

Accepting that both the kinetic and field-line compression terms drop out of Eq. (29), we are left with a potential energy involving the single field φ

$$
\begin{aligned}
2\delta W^{(0)} &= \int_\Omega d^3x \left\{ \frac{B^2}{\mu_0} \left| \nabla_\perp \left(\frac{\mathbf{B}\cdot\nabla\varphi}{B^2} \right) \right|^2 - \frac{(\nabla p \times \mathbf{B}\cdot\nabla_\perp\varphi^*)(\boldsymbol{\kappa}\times\mathbf{B}\cdot\nabla_\perp\varphi)}{B^4} \right. \\
&\qquad \left. - \frac{(\boldsymbol{\kappa}\times\mathbf{B}\cdot\nabla_\perp\varphi^*)(\nabla p\times\mathbf{B}\cdot\nabla_\perp\varphi)}{B^4} \right\} , \qquad (38)
\end{aligned}
$$

where we have dropped the kink term because it is $O(\epsilon)$.

The wave equation, obtained from Eq. (38) by extremizing the Lagrangian with respect to φ^*, is

$$
\begin{aligned}
&\mathbf{B}\cdot\nabla \left\{ \frac{1}{B^2} \nabla\cdot \left[\frac{B^2}{\mu_0} \mathsf{P}_\perp \cdot \nabla \left(\frac{\mathbf{B}\cdot\nabla\varphi}{B^2} \right) \right] \right\} + \nabla\cdot \left(\frac{\nabla p \times \mathbf{B}\, \boldsymbol{\kappa}\times\mathbf{B}\cdot\nabla\varphi}{B^4} \right) \\
&+ \nabla\cdot \left(\frac{\boldsymbol{\kappa}\times\mathbf{B}\, \nabla p\times\mathbf{B}\cdot\nabla\varphi}{B^4} \right) + \omega^2 \nabla\cdot \left(\frac{\rho}{B^2} \mathsf{P}_\perp \cdot \nabla\varphi \right) = 0 . \qquad (39)
\end{aligned}
$$

6.2 Two-field, compressible model

Although, as Freidberg cautions [10, pp. 262–263], the predictions of ideal MHD regarding the effect of finite compressibility are likely to be unreliable, it is nevertheless worthwhile to calculate growth rates using both models 1 and 4 in Sec. 3.1. If they differ greatly, then that is a good indication that the parallel dynamics is important and a more sophisticated description is needed, since "collisionless MHD" is not reliable either.

For model 1 of Sec. 3.1, the kinetic energy norm is

$$2K^{(0)} = \int_\Omega d^3x\ \rho(\psi) \left[\frac{|\nabla_\perp \varphi|^2}{B^2} + \left|\xi_\|\right|^2 \right] . \tag{40}$$

As in Sec. 6.1.2 we vary $\nabla\cdot\boldsymbol{\xi}_\perp^*$, obtaining from Eq. (29) the Euler–Lagrange equation

$$\left(\frac{B^2}{\mu_0} + \gamma p\right) \nabla\cdot\boldsymbol{\xi}_\perp = -\frac{2\boldsymbol{\kappa}\times\mathbf{B}\cdot\nabla_\perp\varphi}{\mu_0} - \gamma p \mathbf{B}\cdot\nabla\left(\frac{\xi_\|}{B}\right) . \tag{41}$$

Using this to eliminate $\nabla\cdot\boldsymbol{\xi}_\perp$ from $\delta W^{(0)}$ we obtain from Eq. (29) (again dropping the kink term)

$$\begin{aligned} 2\delta W^{(0)} \ &= \ \int_\Omega d^3x \left\{ \frac{B^2}{\mu_0} \left| \nabla_\perp \left(\frac{\mathbf{B}\cdot\nabla\varphi}{B^2} \right) \right|^2 \right. \\ &\quad + \frac{\gamma p B^2}{(B^2 + \mu_0 \gamma p)} \left| \mathbf{B}\cdot\nabla\left(\frac{\xi_\|}{B}\right) - \frac{2\boldsymbol{\kappa}\times\mathbf{B}\cdot\nabla_\perp\varphi}{B^2} \right|^2 \\ &\quad - \frac{(\nabla p\times\mathbf{B}\cdot\nabla_\perp\varphi^*)(\boldsymbol{\kappa}\times\mathbf{B}\cdot\nabla_\perp\varphi)}{B^4} \\ &\quad \left. - \frac{(\boldsymbol{\kappa}\times\mathbf{B}\cdot\nabla_\perp\varphi^*)(\nabla p\times\mathbf{B}\cdot\nabla_\perp\varphi)}{B^4} \right\} . \end{aligned} \tag{42}$$

By adding Eq. (41) divided by $(B^2/\mu_0 + \gamma p)$ and the identity $\nabla\cdot(\xi_\|\mathbf{e}_\|) = \mathbf{B}\cdot\nabla(\xi_\|/B)$ we readily obtain the result

$$\frac{B^2}{(B^2 + \mu_0\gamma p)} \left[\mathbf{B}\cdot\nabla\left(\frac{\xi_\|}{B}\right) - \frac{2\boldsymbol{\kappa}\times\mathbf{B}\cdot\nabla_\perp\varphi}{B^2} \right] = (\nabla\cdot\boldsymbol{\xi})^{(0)} , \tag{43}$$

which makes the identification of the compressibility term in Eq. (42) transparent.

To make correspondence with the ballooning-mode Lagrangian of Dewar and Glasser [4], set $\varphi = -i(\epsilon/k_\alpha)\exp{(iS/\epsilon)}\,\xi$ and $\xi_\parallel = B\exp{(iS/\epsilon)}\,\eta$, S is the eikonal and ξ and η are the components of the ballooning eigenfunction defined in Eq. (49) of Dewar and Glasser. We drop lower order terms, so $\boldsymbol{\nabla}\_\varphi$ is replaced by $i\mathbf{k}\varphi/\epsilon$. Then the only operators remaining in the Lagrangian are of the form $\mathbf{B}\cdot\boldsymbol{\nabla}$, so that the volume integration can be restricted to an arbitrarily narrow flux tube, thus explaining the field-line integral form of the ballooning mode Lagrangian. From Eq. (43) it is clear that the term $(\dot{\eta} - 2\boldsymbol{\kappa}\cdot\mathbf{B}\times\mathbf{k}\,\xi/B^2k_\alpha)$ occurring in Dewar and Glasser is proportional to $(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^{(0)}$ and thus vanishes for incompressible plasmas.

7 Conclusions

We have reviewed some very old concepts in a modern perspective and have found a model wave equation for ballooning modes whose analysis should shed light on the nature of the unstable continuum solutions in three-dimensional toroidal plasma confinement geometries. By taking care to explain some obscure points, it is hoped that this mini-review will help make some of the early literature on ideal MHD stability more accessible to a modern audience.

Acknowledgments

The author wishes to thank Mr J. L. V. Lewandowski for raising some of the issues addressed in this brief review, Drs J. M. Greene and J. L. Johnson for their comments on a draft manuscript, and Dr Z. Yoshida for useful discussions regarding the solution of Eq. (37). He also takes pleasure in acknowledging the kind hospitality of Professor A. Iiyoshi during his visit to the National Institute for Fusion Science, Nagoya.

A Interpretation of terms of δW

In view of the fact that there is current interest in alternative forms of δW [16, 17, 18] exhibiting MHD-stability-related geometric properties (in particular local magnetic shear) it is timely to revisit the geometric interpretation of the original FKRC form to enquire whether, and in what sense, the interpretations of some of the terms in the energy principle given in Sec. 3.2 are rigorously geometric. (Though in this paper we do not examine local magnetic shear.)

We consider, in particular, the field-line bending and compression terms and seek to make precise the definition of a displacement field $\boldsymbol{\xi}$ that (a) bends but does not compress and (b) compresses but does not bend the magnetic field lines. If the geometric identifications of "field-line bending term" and "field-line compression term" are precise, the second term of δW should vanish under displacements of type (a), while the first term should vanish under displacements of type (b).

First we need to introduce the concept of the *Lagrangian variation*, Δ, in a physical quantity as the plasma fluid is displaced from its background (in our case, equilibrium) position to its perturbed state as $\boldsymbol{\xi}$ is "turned on" at fixed t. Symbolically, the Lagrangian variation is defined by [19]

$$\Delta \equiv \delta + \boldsymbol{\xi} \cdot \boldsymbol{\nabla} \ , \tag{44}$$

where δ is the *Eulerian variation*, i.e. the variation of a quantity at a fixed point in space. The Lagrangian variation is the change as seen by a fluid element as it is perturbed from its background to its final position. It is a property [19] of ideal MHD that the Lagrangian variation of the physical quantities density, pressure and magnetic field can all be expressed in terms of the strain dyadic $\boldsymbol{\nabla}\boldsymbol{\xi}$ and are thus precisely of the same order, $O(\epsilon^{-1})$, in the short-wavelength approximation [allowing $k_{\parallel} = O(\epsilon^{-1})$ for the purposes of this discussion]. Lagrangian variations also have the advantage that they vanish for uniform translations of the system, which should not change any energy terms.

A.1 Type (a) variations: field-line incompressibility

To say that field lines are not compressed must mean that their separation is preserved under the perturbation of the system by the displacement field $\boldsymbol{\xi}$. Since the field-line density is measured by $|\mathbf{B}|$, we define *field-line incompressibility* by the requirement

$$\Delta\left(\frac{B^2}{2}\right) \equiv \mathbf{B}\cdot\mathbf{Q} + \boldsymbol{\xi}\cdot\boldsymbol{\nabla}\left(\frac{B^2}{2}\right) \equiv \mathbf{B}\cdot(\boldsymbol{\nabla}\boldsymbol{\xi})\cdot\mathbf{B} - B^2\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = 0 \ , \tag{45}$$

where the Eulerian variation in $\mathbf{B}$ is $\delta\mathbf{B} \equiv \mathbf{Q} \equiv \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\mathbf{B})$.

The first form in Eq. (45) is reminiscent of the expression $(\mathbf{Q}\cdot\mathbf{B} - \mu_0\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p)$ occurring in the second term of Eq. (13). However, the equilibrium condition

Eq. (2) implies

$$\boldsymbol{\xi}\cdot\nabla\left(\frac{B^2}{2}\right) = \mathbf{B}\cdot(\nabla\mathbf{B})\cdot\boldsymbol{\xi} - \mu_0\boldsymbol{\xi}\cdot\nabla p\,. \tag{46}$$

Thus, unless we restrict $\boldsymbol{\xi}$ so that $\mathbf{B}\cdot(\nabla\mathbf{B})\cdot\boldsymbol{\xi} = 0$, requiring the "field-line compressibility term" to vanish is *not* precisely equivalent to the field-line incompressibility criterion defined by Eq. (45).

The difference, however, is $O(\epsilon^0)$ in the short-wavelength limit, whereas $\mathbf{Q}\cdot\mathbf{B} = O(\epsilon^{-1})$, so that the two versions of field-line incompressibility *are* equivalent to leading order.

A.2 Type (b): curvature-preserving variations

To say that field lines are not bent must mean that the magnitude of their curvature, $\boldsymbol{\kappa} \equiv \nabla\mathbf{e}_\parallel\cdot\mathbf{e}_\parallel$, is preserved under Lagrangian variations. Thus we define *curvature-preserving variations* by the condition

$$\boldsymbol{\kappa}\cdot\Delta\boldsymbol{\kappa} = 0\,, \tag{47}$$

From its definition we can readily show that the Eulerian variation of the unit vector along the field is $\delta\mathbf{e}_\parallel = \mathbf{Q}_\perp/B$, while the Lagrangian variation is $(\boldsymbol{\xi}\cdot\nabla\mathbf{B})_\perp/B$. The Eulerian variation of the curvature is $\delta\boldsymbol{\kappa} \equiv \delta\mathbf{e}_\parallel\cdot\nabla\mathbf{e}_\parallel + \mathbf{e}_\parallel\cdot\nabla\delta\mathbf{e}_\parallel$, while the Lagrangian variation has the same form, but with δ replaced by Δ. Thus we see that setting $\mathbf{Q}_\perp = 0$ to make the "field-line bending" term in δW vanish will make the *Eulerian* variation of $\boldsymbol{\kappa}$ vanish, but not the Lagrangian variation, so that curvature-preserving variations do not in general make the first term of δW vanish.

The difference between the Lagrangian and Eulerian variations of the curvatures, however, is $\boldsymbol{\xi}\cdot\nabla\boldsymbol{\kappa} = O(\epsilon^0)$. In comparison, $\Delta\boldsymbol{\kappa}$ is $O(\epsilon^{-2})$ if

$k_\parallel = O(\epsilon^{-1})$, or $O(\epsilon^{-1})$ if $k_\parallel = O(\epsilon^0)$. Either way, the two criteria are equivalent to leading order in the short-wavelength limit.

Thus. although the terminology in Sec. 3.2 is not rigorously correct for long-wavelength perturbations, it is a convenient and appropriate designation for the purposes of this paper.

B The kink term and self adjointness

As only the kink and curvature terms in Eq. (13) fail to manifest obvious Hermitian symmetry we restrict attention in this Appendix to these terms, giving some details regarding the relationship between them and showing that the antisymmetric part of the unsymmetrized version of the reduced curvature term cancels the antisymmetric part of the corresponding version of the kink term. This shows that the kink term plays a fundamental role and may be an argument for its retention despite being formally small in the short-perpendicular-wavelength expansion. We also show how to get the version of the kink term used in Eq. (29), where its formal smallness is manifest.

The main result we wish to prove is that

$$\int_\Omega d^3x \left[\sigma \boldsymbol{\nabla}_\perp \varphi^* \boldsymbol{\cdot} \boldsymbol{\nabla} \times (\boldsymbol{\nabla}_\perp \varphi) + \frac{2(\boldsymbol{\kappa} \times \mathbf{B} \boldsymbol{\cdot} \boldsymbol{\nabla}_\perp \varphi^*)(\boldsymbol{\nabla} p \times \mathbf{B} \boldsymbol{\cdot} \boldsymbol{\nabla}_\perp \varphi)}{B^4}\right] \tag{48}$$

is an Hermitian symmetric form. This is the "raw" form of the relevant part of Eq. (13) obtained after "complexifying" by changing the first occurrence of $\boldsymbol{\xi}$ in the two terms by its complex conjugate, as discussed in Sec. 3.2, then replacing $\boldsymbol{\xi}_\perp$ with the representation in Eq. (23) and dropping the small terms involving χ.

To prove Hermiticity we first decompose the curvature term into a symmetric and an antisymmetric part and note the following lemma about the antisymmetric part

$$\frac{(\boldsymbol{\kappa}\times\mathbf{B}\cdot\boldsymbol{\nabla}_\perp\varphi^*)(\boldsymbol{\nabla}p\times\mathbf{B}\cdot\boldsymbol{\nabla}_\perp\varphi)}{B^4} - \frac{(\boldsymbol{\nabla}p\times\mathbf{B}\cdot\boldsymbol{\nabla}_\perp\varphi^*)(\boldsymbol{\kappa}\times\mathbf{B}\cdot\boldsymbol{\nabla}_\perp\varphi)}{B^4}$$
$$= \frac{\mathbf{B}\cdot\boldsymbol{\nabla}\sigma}{2B^2}\boldsymbol{\nabla}_\perp\varphi^*\cdot\mathbf{B}\times\boldsymbol{\nabla}_\perp\varphi$$
$$= \frac{1}{2}\boldsymbol{\nabla}_\perp\varphi^*\cdot\boldsymbol{\nabla}\sigma\times\boldsymbol{\nabla}_\perp\varphi\,. \tag{49}$$

where the second form on the right-hand side follows from observing that the perpendicular part of $\boldsymbol{\nabla}\sigma$, i.e. $\boldsymbol{\nabla}_\perp\sigma$, does not contribute because the scalar triple product of three vectors lying in the perpendicular plane is zero (three vectors cannot be linearly independent in a two-dimensional space). Thus only the parallel part of $\boldsymbol{\nabla}\sigma$ contributes, which is just the first form.

The identity Eq. (49) can conveniently be proved by decomposing the curvature into normal and geodesic components [16] $\boldsymbol{\kappa} = \kappa_\psi\boldsymbol{\nabla}\psi + \kappa_s\mathbf{s}$, where $\mathbf{s} \equiv \boldsymbol{\nabla}\psi\times\mathbf{B}/|\boldsymbol{\nabla}\psi|^2$. Then $\kappa_s = -\mathbf{B}\cdot\boldsymbol{\nabla}\sigma/2p'(\psi)$ by Eq. (5). Writing out the left-hand side of Eq. (49) in terms of these components, we find that the normal component, κ_ψ cancels, while the geodesic component becomes $(\mathbf{B}\cdot\boldsymbol{\nabla}\sigma/2B^2)\boldsymbol{\nabla}_\perp\varphi^*\cdot[(\boldsymbol{\nabla}\psi)\mathbf{s} - \mathbf{s}(\boldsymbol{\nabla}\psi)]\cdot\boldsymbol{\nabla}_\perp\varphi$. It is then readily verified using $\mathbf{B} \equiv \mathbf{s}\times\boldsymbol{\nabla}\psi$ that this is the same as the first form on the right-hand side of Eq. (49).

We now do a similar symmetric/antisymmetric decomposition on the kink term in Eq. (48). Using integration by parts (ignoring boundary contributions, as always in this paper), the antisymmetric part is readily found to obey the identity

$$\frac{1}{2}\int_\Omega d^3x\,\sigma\left[\boldsymbol{\nabla}_\perp\varphi^*\cdot\boldsymbol{\nabla}\times(\boldsymbol{\nabla}_\perp\varphi) - \boldsymbol{\nabla}_\perp\varphi\cdot\boldsymbol{\nabla}\times(\boldsymbol{\nabla}_\perp\varphi^*)\right]$$

$$\equiv -\frac{1}{2}\int_\Omega d^3x\ \boldsymbol{\nabla}_\perp\varphi^*\boldsymbol{\cdot}\boldsymbol{\nabla}\sigma\times\boldsymbol{\nabla}_\perp\varphi\ . \tag{50}$$

Comparing with Eq. (49) we see that the antisymmetric parts of the terms in Eq. (48) cancel each other and thus only the Hermitian symmetric terms survive, as claimed.

If we try to apply the identity Eq. (27) directly, the kink term appears to be $O(1)$. To see that it is really $O(\epsilon)$ we need first to use the identity in the form $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}_\perp\varphi) \equiv -\boldsymbol{\nabla}\times[\mathbf{BB}\boldsymbol{\cdot}(\boldsymbol{\nabla}\varphi)/B^2]$, then integrate by parts and only *now* use Eq. (27), giving

$$\begin{aligned}\int_\Omega d^3x\ \sigma\boldsymbol{\nabla}_\perp\varphi^*\boldsymbol{\cdot}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}_\perp\varphi) &= -\int_\Omega d^3x\ \frac{(\mathbf{B}\times\boldsymbol{\nabla}\sigma\boldsymbol{\cdot}\boldsymbol{\nabla}_\perp\varphi^*)(\mathbf{B}\boldsymbol{\cdot}\boldsymbol{\nabla}\varphi)}{B^2} \\ &\quad + \left[\int_\Omega d^3x\ \frac{\mu_0\sigma^2}{B^2}\mathbf{B}\boldsymbol{\cdot}\boldsymbol{\nabla}\varphi^*\mathbf{B}\boldsymbol{\cdot}\boldsymbol{\nabla}\varphi\right]\ ,\end{aligned} \tag{51}$$

the leading term of which is $O(\epsilon)$, the second term being $O(\epsilon^2)$. The symmetrized version of this identity is used in Eq. (29).

References

[1] H. P. Furth, J. Killeen, M. N. Rosenbluth, and B. Coppi, in *Plasma Physics and Controlled Nuclear Fusion Research* (IAEA, Vienna, 1966), Vol. 1, pp. 103–126.

[2] I. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, Proc. R. Soc. London Ser. A **244**, 17 (1958).

[3] J. M. Greene and J. L. Johnson, Plasma Phys. **10**, 729 (1968).

[4] R. L. Dewar and A. H. Glasser, Phys. Fluids **26**, 3038 (1983).

[5] N. Nakajima, Phys. Plasmas **3**, 4556 (1996).

[6] R. C. Grimm, R. L. Dewar, and J. Manickam, J. Comput. Phys. **49**, 94 (1983).

[7] C. Schwab, Phys. Fluids B **5**, 3195 (1993).

[8] W. A. Cooper, D. B. Singleton, and R. L. Dewar, Phys. Plasmas **3**, 275 (1996), erratum: Phys. Plasmas **3**, 3520 (1996).

[9] S. P. Hirshman and O. Betancourt, J. Comput. Phys. **96**, 99 (1991).

[10] J. P. Freidberg, *Ideal Magnetohydrodynamics* (Plenum, New York, 1987).

[11] R. D. Hazeltine and J. D. Meiss, *Plasma Confinement*, No. 86 in *The Advanced Book Program* (Addison-Wesley, Redwood City, 1992).

[12] R. L. Dewar and M. P. Persson, Phys. Fluids B **5**, 4273 (1993).

[13] H. J. Gardner and B. D. Blackwell, Nucl. Fusion **32**, 2009 (1992).

[14] R. Moeckli and W. A. Cooper, Nucl. Fusion **33**, 1899 (1993).

[15] K. Ichiguchi, N. Nakajima, and H. J. Gardner, Nucl. Fusion **36**, 1157 (1996).

[16] R. L. Dewar, D. A. Monticello, and W. N.-C. Sy, Phys. Fluids **27**, 1723 (1984).

[17] J. M. Greene, Phys. Plasmas **3**, 8 (1996).

[18] W. A. Cooper, Phys. Plasmas **4**, 153 (1997).

[19] R. L. Dewar, Phys. Fluids **13**, 2710 (1970).

Recent Issues of NIFS Series

NIFS-461 Y. Hamada, A. Nishizawa, Y. Kawasumi, A. Fujisawa, K. Narihara, K. Ida, A. Ejiri, S. Ohdachi, K. Kawahata, K. Toi, K. Sato, T. Seki, H. Iguchi, K. Adachi, S. Hidekuma, S.Hirokura, K. Iwasaki, T. Ido, M. Kojima, J. Koong, R. Kumazawa, H. Kuramoto, T. Minami, I. Nomura, H. Sakakita, M. Sasao, K.N. Sato, T. Tsuzuki, J. Xu, I. Yamada and T. Watari,
Density Fluctuation in JIPP T-IIU Tokamak Plasmas Measured by a Heavy Ion Beam Probe, Nov. 1996

NIFS-462 N. Katsuragawa, H. Hojo and A. Mase,
Simulation Study on Cross Polarization Scattering of Ultrashort-Pulse Electromagnetic Waves; Nov. 1996

NIFS-463 V. Voitsenya, V. Konovalov, O. Motojima, K. Narihara, M. Becker and B. Schunke,
Evaluations of Different Metals for Manufacturing Mirrors of Thomson Scattering System for the LHD Divertor Plasma; Nov. 1996

NIFS-464 M. Pereyaslavets, M. Sato, T. Shimozuma, Y. Takita, H. Idei, S. Kubo, K. Ohkubo and K. Hayashi,
Development and Simulation of RF Components for High Power Millimeter Wave Gyrotrons; Nov. 1996

NIFS-465 V.S. Voitsenya, S. Masuzaki, O. Motojima, N. Noda and N. Ohyabu,
On the Use of CX Atom Analyzer for Study Characteristics of Ion Component in a LHD Divertor Plasma; Dec. 1996

NIFS-466 H. Miura and S. Kida,
Identification of Tubular Vortices in Complex Flows; Dec. 1996

NIFS-467 Y. Takeiri, Y. Oka, M. Osakabe, K. Tsumori, O. Kaneko, T. Takanashi, E. Asano, T. Kawamoto, R. Akiyama and T. Kuroda,
Suppression of Accelerated Electrons in a High-current Large Negative Ion Source; Dec. 1996

NIFS-468 A. Sagara, Y. Hasegawa, K. Tsuzuki, N. Inoue, H. Suzuki, T. Morisaki, N. Noda, O. Motojima, S. Okamura, K. Matsuoka, R. Akiyama, K. Ida, H. Idei, K. Iwasaki, S. Kubo, T. Minami, S. Morita, K. Narihara, T. Ozaki, K. Sato, C. Takahashi, K. Tanaka, K. Toi and I. Yamada,
Real Time Boronization Experiments in CHS and Scaling for LHD; Dec. 1996

NIFS-469 V.L. Vdovin, T.Watari and A. Fukuyama,
3D Maxwell-Vlasov Boundary Value Problem Solution in Stellarator Geometry in Ion Cyclotron Frequency Range (final report); Dec. 1996

NIFS-470 N. Nakajima, M. Yokoyama, M. Okamoto and J. Nührenberg,
Optimization of M=2 Stellarator; Dec. 1996

NIFS-471 A. Fujisawa, H. Iguchi, S. Lee and Y. Hamada, *Effects of Horizontal Injection Angle Displacements on Energy Measurements with Parallel Plate Energy Analyzer*; Dec. 1996

NIFS-472 R. Kanno, N. Nakajima, H. Sugama, M. Okamoto and Y. Ogawa,
Effects of Finite-β and Radial Electric Fields on Neoclassical Transport in the Large Helical Device; Jan. 1997

NIFS-473 S. Murakami, N. Nakajima, U. Gasparino and M. Okamoto,
Simulation Study of Radial Electric Field in CHS and LHD; Jan. 1997

NIFS-474 K. Ohkubo, S. Kubo, H. Idei, M. Sato, T. Shimozuma and Y. Takita,
Coupling of Tilting Gaussian Beam with Hybrid Mode in the Corrugated Waveguide; Jan. 1997

NIFS-475 A. Fujisawa, H. Iguchi, S. Lee and Y. Hamada,
Consideration of Fluctuation in Secondary Beam Intensity of Heavy Ion Beam Prove Measurements; Jan. 1997

NIFS-476 Y. Takeiri, M. Osakabe, Y. Oka, K. Tsumori, O. Kaneko, T. Takanashi, E. Asano, T. Kawamoto, R. Akiyama and T. Kuroda,
Long-pulse Operation of a Cesium-Seeded High-Current Large Negative Ion Source; Jan. 1997

NIFS-477 H. Kuramoto, K. Toi, N. Haraki, K. Sato, J. Xu, A. Ejiri, K. Narihara, T. Seki, S. Ohdachi, K. Adati, R. Akiyama, Y. Hamada, S. Hirokura, K. Kawahata and M. Kojima,
Study of Toroidal Current Penetration during Current Ramp in JIPP T-IIU with Fast Response Zeeman Polarimeter; Jan., 1997

NIFS-478 H. Sugama and W. Horton,
Neoclassical Electron and Ion Transport in Toroidally Rotating Plasmas; Jan. 1997

NIFS-479 V.L. Vdovin and I.V. Kamenskij,
3D Electromagnetic Theory of ICRF Multi Port Multi Loop Antenna; Jan. 1997

NIFS-480 W.X. Wang, M. Okamoto, N. Nakajima, S Murakami and N. Ohyabu,
Cooling Effect of Secondary Electrons in the High Temperature Divertor Operation; Feb. 1997

NIFS-481 K. Itoh, S.-I. Itoh, H. Soltwisch and H.R. Koslowski,
Generation of Toroidal Current Sheet at Sawtooth Crash; Feb. 1997

NIFS-482 K. Ichiguchi,
Collisionality Dependence of Mercier Stability in LHD Equilibria with Bootstrap Currents; Feb. 1997

NIFS-483 S. Fujiwara and T. Sato,
Molecular Dynamics Simulations of Structural Formation of a Single

Polymer Chain: Bond-orientational Order and Conformational Defects; Feb. 1997

NIFS-484 T. Ohkawa,
Reduction of Turbulence by Sheared Toroidal Flow on a Flux Surface; Feb. 1997

NIFS-485 K. Narihara, K. Toi, Y. Hamada, K. Yamauchi, K. Adachi, I. Yamada, K. N. Sato, K. Kawahata, A. Nishizawa, S. Ohdachi, K. Sato, T. Seki, T. Watari, J. Xu, A. Ejiri, S. Hirokura, K. Ida, Y. Kawasumi, M. Kojima, H. Sakakita, T. Ido, K. Kitachi, J. Koog and H. Kuramoto,
Observation of Dusts by Laser Scattering Method in theJIPPT-IIU Tokamak Mar. 1997

NIFS-486 S. Bazdenkov, T. Sato and The Complexity Simulation Group,
Topological Transformations in Isolated Straight Magnetic Flux Tube; Mar. 1997

NIFS-487 M. Okamoto,
Configuration Studies of LHD Plasmas; Mar. 1997

NIFS-488 A. Fujisawa, H. Iguchi, H. Sanuki, K. Itoh, S. Lee, Y. Hamada, S. Kubo, H. Idei, R. Akiyama, K. Tanaka, T. Minami, K. Ida, S. Nishimura, S. Morita, M. Kojima, S. Hidekuma, S.-I. Itoh, C. Takahashi, N. Inoue, H. Suzuki, S. Okamura and K. Matsuoka,
Dynamic Behavior of Potential in the Plasma Core of the CHS Heliotron/Torsatron; Apr. 1997

NIFS-489 T. Ohkawa,
Pfirsch - Schlüter Diffusion with Anisotropic and Nonuniform Superthermal Ion Pressure; Apr. 1997

NIFS-490 S. Ishiguro and The Complexity Simulation Group,
Formation of Wave-front Pattern Accompanied by Current-driven Electrostatic Ion-cyclotron Instabilities; Apr. 1997

NIFS-491 A. Ejiri, K. Shinohara and K. Kawahata,
An Algorithm to Remove Fringe Jumps and its Application to Microwave Reflectometry; Apr. 1997

NIFS-492 K. Ichiguchi, N. Nakajima, M. Okamoto,
Bootstrap Current in the Large Helical Device with Unbalanced Helical Coil Currents; Apr. 1997

NIFS-493 S. Ishiguro, T. Sato, H. Takamaru and The Complexity Simulation Group,
V-shaped dc Potential Structure Caused by Current-driven Electrostatic Ion-cyclotron Instability; May 1997

NIFS-494 K. Nishimura, R. Horiuchi, T. Sato,
Tilt Stabilization by Energetic Ions Crossing Magnetic Separatrix in Field-Reversed Configuration; June 1997

NIFS-495 T. -H. Watanabe and T. Sato,
Magnetohydrodynamic Approach to the Feedback Instability; July 1997

NIFS-496 K. Itoh, T. Ohkawa, S. -I.Itoh, M. Yagi and A. Fukuyama
Suppression of Plasma Turbulence by Asymmetric Superthermal Ions; July 1997

NIFS-497 T. Takahashi, Y. Tomita, H. Momota and Nikita V. Shabrov,
Collisionless Pitch Angle Scattering of Plasma Ions at the Edge Region of an FRC; July 1997

NIFS-498 M. Tanaka, A.Yu Grosberg, V.S. Pande and T. Tanaka,
Molecular Dynamics and Structure Organization in Strongly-Coupled Chain of Charged Particles; July 1997

NIFS-499 S. Goto and S. Kida,
Direct-interaction Approximation and Reynolds-number Reversed Expansion for a Dynamical System; July 1997

NIFS-500 K. Tsuzuki, N. Inoue, A. Sagara, N. Noda, O. Motojima. T. Mochizuki, T. Hino and T. Yamashina,
Dynamic Behavior of Hydrogen Atoms with a Boronized Wall; July 1997

NIFS-501 I. Viniar and S. Sudo,
Multibarrel Repetitive Injector with a Porous Pellet Formation Unit; July 1997

NIFS-502 V. Vdovin, T. Watari and A. Fukuyama,
An Option of ICRF Ion Heating Scenario in Large Helical Device; July 1997

NIFS-503 E. Segre and S. Kida,
Late States of Incompressible 2D Decaying Vorticity Fields; Aug. 1997

NIFS-504 S. Fujiwara and T. Sato,
Molecular Dynamics Simulation of Structural Formation of Short Polymer Chains; Aug. 1997

NIFS-505 S. Bazdenkov and T. Sato
Low-Dimensional Model of Resistive Interchange Convection in Magnetized Plasmas; Sep. 1997

NIFS-506 H. Kitauchi and S. Kida,
Intensification of Magnetic Field by Concentrate-and-Stretch of Magnetic Flux Lines; Sep. 1997

NIFS-507 R.L. Dewar,
Reduced form of MHD Lagrangian for Ballooning Modes; Sep. 1997