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abstract

Multiresolution approximation (MRA) of the vector fields on $\mathbf{T}^3$ is studied. We introduced in the Fourier space a triad of vector fields called helical vectors which derived from the spherical coordinate system basis. Utilizing the helical vectors, we proved the orthogonal decomposition of $L^2(\mathbf{T}^3)$ which is a synthesis of the Hodge decomposition of the differential 1- or 2-form on $\mathbf{T}^3$ and the Beltrami decomposition that decompose the space of solenoidal vector fields into the eigenspaces of curl operator. In the course of proof, a general construction procedure of the divergence-free orthonormal complete basis from the basis of scalar function space is presented. Applying this procedure to MRA of $L^2(\mathbf{T}^3)$, we discussed the MRA of vector fields on $\mathbf{T}^3$ and the analyticity and regularity of vector wavelets. It is conjectured that the orthonormal solenoidal wavelet basis must break r-regular condition, i.e. some wavelet functions cannot be rapidly decreasing function because of the inevitable singularities of helical vectors. The localization property and spatial structure of solenoidal wavelets derived from the Littlewood-Paley type MRA (Meyer's wavelet) are also investigated numerically.

Keywords: orthonormal divergence-free vector wavelet. Hodge-Beltrami decomposition.

[1] Introduction

Divergence free vector fields with coherent structures are ubiquitous in a lot of natural phenomena, for example. coronal flare of the Sun, dipolar magnetic field of the Earth, or the coherent vortices in sufficiently subsonic motions of fluid, for example Great Red Spot in Jupiter. Wavelet analysis has been regarded one of the promising tools for surveying such coherent structures. Because they are remarkably localized not only in physical space but also in Fourier space within the limit of the uncertainty principle. Using wavelet transformation, one can obtain the information of scale and location simultaneously.

Theory of discrete wavelet transformation is well known as multiresolution approximation (MRA) of function spaces. The wavelet bases are remarkably useful because they are the orthonormal complete, for some cases unconditional, basis not only of L^2 space but also of many function spaces such as Sobolev spaces, Hölder spaces, Hardy space, Besov spaces,

etc [1]. Dyadic dilation property of the wavelet basis seems quite akin to the idea of scaling laws, which appears in many fields of physics such as quantum field theory [2], critical phenomena [3] or fully developed turbulence [4].

The wavelets, however, are scalar function so that application to divergence-free vector field contains a problem. Surely one can apply the scalar-valued wavelet transform to each component of a divergence-free vector field, say $u(x) = (\ u_x(x,y,z),\ u_y(x,y,z),\ u_z(x,y,z)\ )$. and obtain the wavelet spectrum of the field,

$$u(x) = \sum_\lambda \Big(\langle \psi_\lambda, u_x \rangle \psi_\lambda(x),$$
$$\langle \psi_\lambda, u_y \rangle \psi_\lambda(x), \langle \psi_\lambda, u_z \rangle \psi_\lambda(x) \Big), \quad (1)$$

where ψ_λ is scalar wavelet and $\langle *, * \rangle$ denotes inner product. Each term of the spectrum

$$u_\lambda(x) = \Big(\langle \psi_\lambda, u_x \rangle \psi_\lambda(x),$$

$$\langle\psi_\lambda, u_y\rangle\psi_\lambda(x), \langle\psi_\lambda, u_z\rangle\psi_\lambda(x)\Big) \quad (2)$$

is not a divergence-free vector field in general. This discrepancy has its root in the fact that only two of three components are independent, but on the whole they are dependent each other. (Furthermore they depends also on boundary conditions.) Thus the divergence-free vector-valued wavelet function is required for practical purposes.

Divergence-free vector wavelet bases have been proposed by Battle and Federbush[5], and Frick and Zimin[6]. Battle and Federbush adopted such a constructing way that minimizes the value of integral $\int(\nabla \times A)^2 dx$ under the two constraints, the divergence-free condition and an appropriate boundary condition. On the other hand, Frick and Zimin proposed such a wavelet that is, roughly speaking, given by the curl of the function given by Fourier integral of a step function supported on a spherical shell in the Fourier space. The former approach requires the variational calculations when the wavelet transformation is carried out. The latter approach, on the other hand, has no such steps because it is based on the sharp decomposition of Fourier space into spherical shells. The obtained wavelet functions, however, are not orthogonal each other in general, and not localized well, in other words, breaks r-regular condition. Because of these discrepancies, the approaches they proposed do not seem popular in practical applications.

In the present work, we propose a general construction procedure of the orthonormal complete divergence-free vector wavelet basis of $L^2(\mathbf{T}^3)$, in which only two popular algorithms, fast Fourier transform (FFT) and fast wavelet transform (FWT), and no additional novel one is required.

The procedure we will describe here is based on a quite different idea from the previous two. There are two key ideas for the construction. One is that any function expansion using an orthonormal complete basis, say $\{f_\lambda; \lambda \in \Lambda\}$, is a unitary transform from $L^2(\mathbf{T}^3)$ to $l^2(\Lambda)$, where Λ is an appropriate set of indices. Therefore the Fourier coefficients of the base functions $\widehat{f}_\lambda(k)$ are regarded as components of an infinite dimensional unitary matrix which acts on l^2 space which maps Fourier coefficients to $\{f_\lambda\}$-expansion coefficients. The other one is that all the orthonormal complete basis of the function space of solenoidal vector fields on $\mathbf{T}^3$, which is denoted by $L^2_\Sigma(\mathbf{T}^3)$ hereafter, is given by a certain unitary transform of the complex helical wave basis [7, 8]. Thus functions to which complex helical waves are unitary transformed by the matrix $\{\mathcal{F}\{f_\lambda\}\}$ constitutes an another orthonormal basis of $L^2_\Sigma(\mathbf{T}^3)$. As an orthonormal complete basis of $L^2(\mathbf{T}^3)$, wavelet basis is adopted here.

This study is an attempt to construct the multiresolution approximation of the vector fields. In the present study we restrict our interest on the vector fields on the three-torus $\mathbf{T}^3$ and the possibility

of construction of multiresolution approximation of them. The reason of choice of the manifold $\mathbf{T}^3$ is mainly due to the fact that the Hodge decomposition theorem is established on bounded manifolds. Thus we base our attempt on the MRA of $L^2(\mathbf{T}^3)$ though the theory of wavelets is firstly established on the unbounded Euclidean space $\mathbf{R}^N$.

This paper is organized as follows. In §2 notations in the paper are explained. The orthogonal decomposition of the vector field on $\mathbf{T}^3$ is proved in §3. Algorithm of helical wavelet decomposition is given in §4, which is an anthology of the properties of helical basis. Construction the theory of MRA of vector fields is tried in §5. The Riesz basis condition, which is one of the basic properties of MRA, is shown to be broken. Section 6 is devoted to the discussion on the *regularity*, i.e. the localization property of the helical wavelet. Finally some remarks are given in §7.

[2] Nomenclature

Before going into the details, some notations should be fixed.

Let us denote by $\mathbf{R}^3$ a linear space spanned by a Cartesian basis $\{e_x, e_y, e_z\}$, and $\mathbf{T}^3$ the quotient space $\mathbf{R}^3/\mathbf{Z}^3$. $\chi(\mathbf{T}^3)$ is a set of vector fields on $\mathbf{T}^3$ defined as

$$\chi(\mathbf{T}^3) := \Big\{ u \,;\, u(x) = \sum_{i=x,y,z} u_i(x)e_i,$$

$$u_x, u_y, u_z \in C^\infty(\mathbf{T}^3) \Big\},$$

where $\{e_x, e_y, e_z\}$ is the basis of the tangent space $T_x\mathbf{T}^3$ at the point $x \in \mathbf{T}^3$ obtained by the canonical identification. In other words, $\chi(\mathbf{T}^3)$ is the set of all the C^∞ sections of the tangent bundle $T\mathbf{T}^3$. We identify, if required, the space with the set of 1-forms $\Omega^1(\mathbf{T}^3)$, or that of 2-forms $\Omega^2(\mathbf{T}^3)$.

The completion of F with respect to the norm of the Banach space E is denoted by $\mathrm{clos}_E\{F\}$. In the following analysis, we study the function space given by the L^2-norm completion of $\chi(\mathbf{T}^3)$,

$$L^2_\chi(\mathbf{T}^3) := \mathrm{clos}_{L^2(\mathbf{T}^3)}\{\chi(\mathbf{T}^3)\}, \quad (3)$$

$$= \Big\{ u \,;\, u(x) = \sum_{i=x,y,z} u_i(x)e_i,$$

$$u_x, u_y, u_z \in L^2(\mathbf{T}^3) \Big\}, \quad (4)$$

which is a Hilbert space equipped with a *inner product*,

$$\langle u, v\rangle_\chi := \int_{\mathbf{T}^3} \overline{u}(x) \cdot v(x)\,dx \quad (5)$$

$$= \int_{\mathbf{T}^3} \Big(\overline{u_x(x)}v_x(x) + \overline{u_y(x)}v_y(x)$$

$$+ \overline{u_z(x)}v_z(x)\Big)\,dx,$$

where $\boldsymbol{u}$, $\boldsymbol{v} \in L^2_\chi(\mathbf{T}^3)$, $\cdot$ denotes the scalar product of two vectors, $*$ the complex conjugate, and $d\boldsymbol{x}$ the Lebesgue measure on $\mathbf{T}^3$. By definition, any Hilbert space is a Banach space with the norm naturally determined by its inner product. Concerning $L^2_\chi(\mathbf{T}^3)$, the norm is

$$\|\boldsymbol{u}|L^2_\chi(\mathbf{T}^3)\| = \sqrt{\langle \boldsymbol{u}, \boldsymbol{u} \rangle_\chi} \qquad (6)$$

$$= \left[\int_{\mathbf{T}^3} \left(|u_x(x,y,z)|^2 + |u_y(x,y,z)|^2 \right. \right.$$

$$\left. \left. + |u_z(x,y,z)|^2 \right) d\boldsymbol{x} \right]^{\frac{1}{2}},$$

where $\| * |E\|$ denotes the norm of a Banach space E. In the following, we drop the symbol $\mathbf{T}^3$ in definite integrals, and represent $\mathbf{T}^3$ by a periodic unit cube, i.e. $\mathbf{T}^3 = [0,1]^3$.

Fourier series representation of $f(\boldsymbol{x}) \in L^2_\chi(\mathbf{T}^3)$ is formally written as

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbf{Z}^3} \widehat{f}(\boldsymbol{k}) \exp(2\pi i \boldsymbol{k} \cdot \boldsymbol{x}), \qquad (7)$$

where $\widehat{f}(\boldsymbol{k})$'s ($\boldsymbol{k} \in \mathbf{Z}^3$) are Fourier coefficients. The calligraphic letter $\mathcal{F}$ is used to denote the sequence of Fourier coefficients, i.e. $\mathcal{F}f = \{\widehat{f}(\boldsymbol{k}); \boldsymbol{k} \in \mathbf{Z}^3\}$. It is also used for the Fourier transform of a set of function and a function space, for example $\mathcal{F}L^2(\mathbf{T}^3) = \{\mathcal{F}f; f \in L^2(\mathbf{T}^3)\}$.

Let us consider a trivial bundle $M := \mathbf{Z}^3 \times \boldsymbol{R}^3$. *Fourier transform of a vector field* is defined by the Fourier transform of the components with respect to the Cartesian basis as follows:

$$\widehat{\boldsymbol{u}}(\boldsymbol{k}) := \widehat{u}_x(\boldsymbol{k})\widehat{e}_x + \widehat{u}_y(\boldsymbol{k})\widehat{e}_y + \widehat{u}_z(\boldsymbol{k})\widehat{e}_z, \qquad (8)$$

where $\boldsymbol{k} \in \mathbf{Z}^3$ and $\{\widehat{e}_x, \widehat{e}_y, \widehat{e}_z\}$ is a Cartesian basis of the fiber $\boldsymbol{R}^3$ of M. Thus $(\boldsymbol{k}, \boldsymbol{u}(\boldsymbol{k})) \in M$ and $\mathcal{F}\boldsymbol{u}$ is a sequence of three dimensional vectors.

Being introduced the l^2_χ-norm of $\mathcal{F}L^2_\chi(\mathbf{T}^3)$ defined by

$$\|\mathcal{F}\boldsymbol{u}|l^2_\chi\| := \sqrt{\sum_{\boldsymbol{k} \in \mathbf{Z}^3} \left(|\widehat{u}_x(\boldsymbol{k})|^2 + |\widehat{u}_y(\boldsymbol{k})|^2 + |\widehat{u}_z(\boldsymbol{k})|^2 \right)}, \qquad (9)$$

$\mathcal{F}L^2_\chi(\mathbf{T}^3)$ becomes a Banach space. Applying Parseval identity to each Cartesian component u_x, u_y and u_z of a $L^2_\chi(\mathbf{T}^3)$ vector field $\boldsymbol{u}$, we conclude that the $L^2_\chi(\mathbf{T}^3)$ norm and $l^2_\chi(\mathbf{Z}^3)$ norm are equivalent. Due to this equivalence of two norms, the *fundamental sequence* of $\boldsymbol{u}(\boldsymbol{x})$ defined by

$$\boldsymbol{u}_N(\boldsymbol{x}) := \sum_{0 \le |\boldsymbol{k}| < N} \widehat{\boldsymbol{u}}(\boldsymbol{k}) \exp(2\pi i \boldsymbol{k} \cdot \boldsymbol{x}), \quad N \in \mathbf{N}. \quad (10)$$

is a Cauchy sequence of vector fields in the sense of L^2_χ-norm. In the following, the derivatives are formally defined by Fourier series,

$$\frac{\partial u_i}{\partial x_j} := \frac{1}{(2\pi)^3} \sum_{\boldsymbol{k} \in \mathbf{Z}^3 \backslash \{0\}} 2\pi i\, k_j\, \widehat{u}_i(\boldsymbol{k}) \exp(2\pi i \boldsymbol{k} \cdot \boldsymbol{x}),$$

$$(11)$$

where i, $j = x$, y and z.

We distinguish the terms *divergence-free* and *solenoidal* in the present work; the former is used for such vector fields $\boldsymbol{u}$ that satisfy $\nabla \cdot \boldsymbol{u} = 0$. The latter term, on the other hand, is used when a vector field $\boldsymbol{u}$ is given by curl of certain vector field $\boldsymbol{a}$, $\boldsymbol{u} = \nabla \times \boldsymbol{a}$. Difference of these two kinds of vectors persists in the fact that the harmonic vector fields, which are constant function for the case of $\mathbf{T}^3$, are also divergence-free. In terms of forms, divergence-free fields correspond to *closed 2-forms* and solenoidal fields *exact 2-forms*.

[3] orthogonal decomposition of the vector fields on $\mathbf{T}^3$

In order to construct complex helical waves, a triad of vector fields $\{e_r(\boldsymbol{k}), e_\vartheta(\boldsymbol{k}), e_\varphi(\boldsymbol{k})\}$ which is almost identical to a spherical coordinate system basis is introduced in the Fourier space. In the present study, they are defined in terms of the wavenumber vector $\boldsymbol{k}$ and Cartesian coordinate system basis $\{e_x, e_y, e_z\}$ as follows:

$$e_r(\boldsymbol{k}) := \frac{\boldsymbol{k}}{|\boldsymbol{k}|},$$

$$e_\varphi(\boldsymbol{k}) := \begin{cases} e_r(\boldsymbol{k}) \times e_x & (e_r(\boldsymbol{k}) /\!/ e_z) \\ \dfrac{e_z \times e_r(\boldsymbol{k})}{|e_z \times e_r(\boldsymbol{k})|} & (\text{otherwise}), \end{cases} \qquad (12)$$

$$e_\vartheta(\boldsymbol{k}) := e_\varphi(\boldsymbol{k}) \times e_r(\boldsymbol{k}).$$

The helical vectors $h_s(\boldsymbol{k})$, where the index s denotes *polarity* of them and is $+$, $-$ or 0, are a triad of complex valued vector fields on the Fourier space that are defined by

$$h_+(\boldsymbol{k}) := \frac{e_\vartheta(\boldsymbol{k}) + i e_\varphi(\boldsymbol{k})}{\sqrt{2}},$$

$$h_-(\boldsymbol{k}) := \frac{e_\vartheta(\boldsymbol{k}) - i e_\varphi(\boldsymbol{k})}{\sqrt{2}}, \qquad (13)$$

$$h_0(\boldsymbol{k}) := -i e_r(\boldsymbol{k}).$$

We also use the notations $h_{\Sigma+}(\boldsymbol{k})$, $h_{\Sigma-}(\boldsymbol{k})$ and $h_D(\boldsymbol{k})$ instead of $h_+(\boldsymbol{k})$, $h_-(\boldsymbol{k})$ and $h_0(\boldsymbol{k})$, respectively.

It should be noted here that we used definition of helical vectors which is used in Ref.[9] with slight modifications, introduction of $h_0(\boldsymbol{k})$ and normalization of $h_\pm(\boldsymbol{k})$ vectors. The helical vectors are defined on $\boldsymbol{R}^3 \backslash \{0\}$, and infinitely differentiable vector fields on $\boldsymbol{R}^3 \backslash \{0\}$ except on the line along the north pole and the south pole for x and y component of $h_\pm(\boldsymbol{k})$ and around $\boldsymbol{k} = 0$ for z components of $h_\pm(\boldsymbol{k})$ and all the components of $h_0(\boldsymbol{k})$. In this section, however, we treat the vector field on $\mathbf{T}^3$ and restrict the case for $\boldsymbol{k} \in \mathbf{Z}^3 \backslash \{0\}$.

Utilizing the helical vectors, we define the *complex helical waves* that are complex valued vector fields by

$$V(\boldsymbol{k}, s; \boldsymbol{x}) := h_s(\boldsymbol{k}) \exp(2\pi i \boldsymbol{k} \cdot \boldsymbol{x}), \qquad (14)$$

for $k \in \mathbf{Z}^3 \backslash \{0\}$ and $s = +, -, 0$. According to the polarity of the helical vectors, we will call each helical wave $\Sigma+$-, $\Sigma--$ and D-mode, respectively. It should be remarked that we introduced D-mode of complex helical waves that is not defined by Lesieur[8] or Waleffe[9]. We will see in the following that this mode is curl-free.

The helical vectors are defined to satisfy the orthonormal relation

$$\overline{h_s(k)} \cdot h_{s'}(k) = \delta(s|s'), \qquad (15)$$

for s, $s' = +, -, 0$ at each k. Hereafter $\delta(A|B)$ denotes Kronecker's delta whose value is one only when the two arguments A and B, which are not only numbers but also vectors, symbols, etc., coincide, otherwise it gives zero. Therefore the complex helical waves are such vector fields that are orthonormal in $L^2_\chi(\mathbf{T}^3)$:

$$\left\langle V(k, s; x), V(k', s'; x) \right\rangle_\chi = \delta\left(k, s \,\middle|\, k', s'\right). \quad (16)$$

Scalar and vector products of the wavenumber vector and the helical vectors are

$$ik \cdot h_\pm(k) = 0, \quad ik \cdot h_0(k) = |k|. \qquad (17)$$

$$ik \times h_\pm(k) = \pm|k|h_\pm(k), \quad ik \times h_0(k) = 0, \, (18)$$

Corresponding to these products, divergence and curl of complex helical waves are

$$\nabla \cdot V(k, \pm; x) = 0,$$
$$\nabla \cdot V(k, 0; x) = 2\pi|k| \exp(2\pi ik \cdot x), \qquad (19)$$

$$\nabla \times V(k, \pm; x) = \pm 2\pi|k|V(k, \pm; x),$$
$$\nabla \times V(k, 0; x) = 0. \qquad (20)$$

The $\Sigma\pm$-modes are eigenfunctions of the curl operator which are associated with the eigenvalues $\pm 2\pi|k|$ and the D-mode belongs to the kernel of the curl operator. Thus it is obvious now that the indices of helical vectors $+$, $-$ and 0 correspond to the signs of eigenvalues, and that the implication of the alternative indices $\Sigma+$, $\Sigma-$ and D are *solenoidal with positive helicity, solenoidal with negative helicity* and *dilatational*, respectively.

Since $e_r(-k) = -e_r(k)$, $e_\vartheta(-k) = e_\vartheta(k)$, $e_\varphi(-k) = -e_\varphi(k)$, the complex conjugates of helical vectors at k are equal to ones at $-k$:

$$\overline{h_s(k)} = h_s(-k). \qquad (21)$$

Therefore, complex conjugates of the complex helical waves satisfy

$$\overline{V(k, s; x)} = V(-k, s; x), \qquad (22)$$

for $k \in \mathbf{Z}^3 \backslash \{0\}$, $s = +, -, 0$. The $\Sigma+$ and $\Sigma-$ vectors are complex conjugate each other,

$$\overline{h_\pm(k)} = h_\mp(k), \qquad (23)$$

so that the $\Sigma\pm$-modes of the complex helical waves are complex conjugate each other:

$$\overline{V(k, \pm; x)} = V(k, \mp; x). \qquad (24)$$

Using these helical vectors, we define the *helical decomposition of Fourier coefficients (or l^p_χ-sequence)*, say $\{\widehat{u}(k)\}$, by

$$\widehat{u}(k) = \widehat{u}_+(k)h_+(k) + \widehat{u}_-(k)h_-(k) + \widehat{u}_0(k)h_0(k), \qquad (25)$$

where $\widehat{u}_s(k)$'s are the *s-mode helical Fourier coefficients* defined by the scalar product

$$\widehat{u}_s(k) := \widehat{u}(k) \cdot \overline{h_s(k)}, \qquad (26)$$

for each $k \in \mathbf{Z}^3 \backslash \{0\}$ and $s = +, -, 0$. Each helical Fourier coefficients $\{\widehat{u}_s(k)\}$ is a sequence of scalars.

Conversely, by multiplying helical vector on each terms of a scalar $(l^p$-)sequence $\{\widehat{u}(k); k \in \mathbf{Z}^3 \backslash \{0\}\}$, one can make a vector sequence $\{\widehat{u}(k)h_s(k); k \in \mathbf{Z}^3 \backslash \{0\}\}$ which we call the *helical pull up of a sequence* $\{\widehat{u}(k)\}$ *to s-mode* hereafter.

It should be remarked here that the helical decomposition of solenoidal vector field on $\mathbf{T}^3$ has also been discussed by Constantin and Majda[10]. In their paper, the decomposition is given by $\widehat{u}_\pm(k) = \widehat{u}(k) \pm ie_r(k) \times \widehat{u}(k)$ for each $k \in \mathbf{Z}^3 \backslash \{0\}$ under the assumption that the vector field is solenoidal $k \cdot \widehat{u}(k) = 0$. It is easy to see that, under the same assumption, the identity $\widehat{u}_\pm(k) = \sqrt{2}\widehat{u}_\pm(k)h_\pm(k)$ is satisfied for each $k \in \mathbf{Z}^3 \backslash \{0\}$.

The helical decomposition of Fourier coefficients is unitary in the sense that the identities

$$\overline{\widehat{u}(k)} \cdot \widehat{v}(k) = \overline{\widehat{u}_+(k)}\widehat{v}_+(k) + \overline{\widehat{u}_-(k)}\widehat{v}_-(k) + \overline{\widehat{u}_0(k)}\widehat{v}_0(k) \qquad (27)$$

are satisfied for each $k \in \mathbf{Z}^3 \backslash \{0\}$ if $\{\widehat{u}(k)\}$ and $\{\widehat{v}(k)\}$ belongs to $l^\infty_\chi(\mathbf{Z}^3 \backslash \{0\})$. This property leads to the identities

$$|\widehat{u}_x(k)|^2 + |\widehat{u}_y(k)|^2 + |\widehat{u}_z(k)|^2$$
$$= |\widehat{u}_+(k)|^2 + |\widehat{u}_-(k)|^2 + |\widehat{u}_0(k)|^2 \qquad (28)$$

for every $k \in \mathbf{Z}^3 \backslash \{0\}$, and to the following consequences.

Proposition 1 *If a three-vector sequence $\{\widehat{u}(k)\}$ belongs to $l^2_\chi(\mathbf{Z}^3)$, each of its helical projections $\{\widehat{u}_s(k)\}$ is $l^2(\mathbf{Z}^3 \backslash \{0\})$-sequence for $s = +, -$ and 0. Conversely, if an sequence $\{\widehat{u}(k)\}$ belongs to $l^2(\mathbf{Z}^3 \backslash \{0\})$, each of its helical pull up $\{\widehat{u}(k)h_s(k)\}$ is $l^2_\chi(\mathbf{Z}^3 \backslash \{0\})$-sequence for $s = +, -$ and 0.*

PROOF: equation (28) leads to the inequalities

$$|\widehat{u}_s(k)|^2 \le |\widehat{u}_x(k)|^2 + |\widehat{u}_y(k)|^2 + |\widehat{u}_z(k)|^2,$$

for every $k \in \mathbf{Z}^3 \backslash \{0\}$ and $s = +, -$ and 0. Thus the l^2-norm of $\{\widehat{u}_s(k)\}$ satisfies

$$\sum_{k \in \mathbf{Z}^3 \backslash \{0\}} |\widehat{u}_s(k)|^2$$

$$\leq \sum_{k \in \mathbf{Z}^3 \setminus \{0\}} \left(|\hat{u}_x(k)|^2 + |\hat{u}_y(k)|^2 + |\hat{u}_z(k)|^2 \right)$$

$$\leq \|\hat{u}\|_{\chi}^2 (\mathbf{Z}^3)\|$$

$$< \infty$$

The latter part is proven by two steps. First, the absolute values of the helical vectors are one, i.e. $|h_s(k)| = 1$, so that all the Cartesian components of them are equal to or less than one, i.e. $|(h_s(k))_j| \leq 1$, for every $k \in \mathbf{Z}^3 \setminus \{0\}$, $s = +, -, 0$, and $j = x, y, z$. All the Cartesian component of helical pull up satisfies

$$|(\hat{u}(k)h_s(k))_j| = |\hat{u}(k)||(h_s(k))_j| \leq |\hat{u}(k)|$$

so that they are $l^2(\mathbf{Z}^3 \setminus \{0\})$-sequence. Thus the helical pull up is an $l_\chi^2(\mathbf{Z}^3 \setminus \{0\})$-sequence by definition. □

Thus the helical transform is unitary transform acts on $l_\chi^2(\mathbf{Z}^3 \setminus \{0\})$.

Theorem 2 *If a scalar sequence $\{u(k)\}$ belongs to $l^2(\mathbf{Z}^3 \setminus \{0\})$, the vector field that is given by*

$$u_s(x) := \sum_{k \in \mathbf{Z}^3 \setminus \{0\}} \hat{u}(k)h_s(k)\exp(2\pi i k \cdot x) \quad (29)$$

converges in the sense of $L_\chi^2(\mathbf{T}^3)$-norm for each polarity $s = +, -$ and 0.

PROOF: Applying Riesz-Fisher theorem to each component, they are proved to belong to $L^2(\mathbf{T}^3)$. Thus the vector field belongs to $L_\chi^2(\mathbf{T}^3)$ by definition. □

We call the vector field expressed by Eq.(29) *s-mode helical vector field*. This theorem allows us to define the function subspaces of $L_\chi^2(\mathbf{T}^3)$ by

$$L_s^2(\mathbf{T}^3) := \Big\{ u_s(k) \, ; \, \{\hat{u}(k)\} \in l^2(\mathbf{Z}^3 \setminus \{0\}),$$

$$u_s(x) = \sum_{k \in \mathbf{Z}^3 \setminus \{0\}} \hat{u}(k)h_s(k)\exp\left(2\pi i k \cdot x\right) \Big\}, \quad (30)$$

and the projection operators $\mathcal{P}_s$ that are linear maps from $L_\chi^2(\mathbf{T}^3)$ to $L_s^2(\mathbf{T}^3)$ by

$$\mathcal{P}_s u(x) := \sum_{k \in \mathbf{Z}^3 \setminus \{0\}} \left(\hat{u}(k) \cdot \overline{h_s(k)} \right) h_s(k)\exp(2\pi i k \cdot x),$$

$$(31)$$

for $s = \Sigma+, \Sigma-, D$. We call the subspaces *helical vector spaces* in general, and call each one $\Sigma+$-, $\Sigma-$-, D-mode space, respectively.

The helical vector spaces $L_{\Sigma+}^2(\mathbf{T}^3)$. $L_{\Sigma-}^2(\mathbf{T}^3)$ and $L_D^2(\mathbf{T}^3)$ are Hilbert spaces w.r.t. the inner product $\langle * \rangle_\chi$ and orthogonal each other, that is, the inner product of arbitrary two vectors $u \in L_s^2(\mathbf{T}^3)$ and $v \in L_{s'}^2(\mathbf{T}^3)$ is zero if the helical mode indices do not coincide $s \neq s'$. Therefore the following relation holds;

$$L_{\Sigma+}^2(\mathbf{T}^3) \oplus L_{\Sigma-}^2(\mathbf{T}^3) \ominus L_D^2(\mathbf{T}^3) \subset L_\chi^2(\mathbf{T}^3). \quad (32)$$

The orthogonal complement of the direct sum in $L_\chi^2(\mathbf{T}^3)$ is the space of the harmonic functions on $\mathbf{T}^3$. We will discuss it later. This orthogonal relations are expressed in terms of the projection operators by

$$\mathcal{P}_s \mathcal{P}_{s'} = \delta(s|s')\mathcal{P}_s, \quad (33)$$

for $s, s' = \Sigma+, \Sigma-. D$

The theorem also enable us to define the *pull up operator* $\mathcal{P}_s^\dagger$ which is a map from the space of scalar functions $L^2(\mathbf{T}^3)$ to the space of s-mode helical vector fields $L_s^2(\mathbf{T}^3)$ by

$$\mathcal{P}_s^\dagger u(x) := \sum_{k \in \mathbf{Z}^3 \setminus \{0\}} \hat{u}(k)h_s(k)\exp(2\pi i k \cdot x), \quad (34)$$

and their adjoint operators $\mathcal{P}^{\dagger*}$ by

$$\mathcal{P}_s^{\dagger*} u(x) := \sum_{k \in \mathbf{Z}^3 \setminus \{0\}} \left(\hat{u}(k) \cdot \overline{h_s(k)} \right) \exp(2\pi i k \cdot x),$$

$$(35)$$

which satisfy the identity

$$\langle \mathcal{P}_s^\dagger f, g \rangle_\chi = \langle f, \mathcal{P}_s^{\dagger*} g \rangle, \quad (36)$$

for every $f \in L^2(\mathbf{T}^3)$ and $g \in L_\chi^2(\mathbf{T}^3)$. Each pull up operator is unitary in the sense that the identity

$$\langle f, g \rangle = \overline{\hat{f}(0)}\hat{g}(0) + \langle \mathcal{P}_s^\dagger f, \mathcal{P}_s^\dagger g \rangle_\chi \quad (37)$$

holds for every $f, g \in L^2(\mathbf{T}^3)$ and $s = +, -, 0$. Let us define *the zero-mean function space* $L_0^2(\mathbf{T}^3)$ by

$$L_0^2(\mathbf{T}^3) := \left\{ u(x) \, ; \, u(x) \in L^2(\mathbf{T}^3), \, \int u(x)dx = 0 \right\}.$$

$$(38)$$

The space is Hilbert with respect to the inner product Eq.(5), and the pull up operators become the unitary operators from $L_0^2(\mathbf{T}^3)$ in the strict sense. This unitary nature of the pull up operators allows us to construct an orthonormal complete basis of s-mode space from a scalar function basis.

Theorem 3 *If $\{f_\lambda \, \lambda \in \Lambda\}$ is an orthonormal complete basis of $L_0^2(\mathbf{T}^3)$ where Λ is appropriate set of indices, then the pull up of the basis to s-mode space $\mathcal{P}_s^\dagger \{f_\lambda\}$ is an orthonormal complete basis of $L_s^2(\mathbf{T}^3)$ where $s = \Sigma+, \Sigma-, D$*

PROOF: Orthonormality is obvious because of the unitary relation Eq.(37). Completeness is proved as follows: if $\langle \mathcal{P}_s^\dagger f_\lambda, \mathcal{P}_s^\dagger g \rangle_\chi = 0$ for all $\lambda \in \Lambda$, then $g = 0$. Because the unitary relation $\langle \mathcal{P}_s^\dagger f_\lambda, \mathcal{P}_s^\dagger g \rangle_\chi = \langle f_\lambda, g \rangle$ holds and $\{f_\lambda\}$ is complete. Therefore $\mathcal{P}_s^\dagger g = 0$ is concluded. □

The theorem leads to an important corollary.

Corollary 4 *Under the same conditions as the previous theorem, the union of the pull up of the basis to $\Sigma+$- and $\Sigma-$-mode spaces, $\mathcal{P}_{\Sigma+}^\dagger \{f_\lambda\} \cup \mathcal{P}_{\Sigma-}^\dagger \{f_\lambda\}$, is an orthonormal complete basis of the function space of the square integrable solenoidal vector fields.*

5

Finally, we shall discuss the "residual" of the projection operators in order to salvage the modes with wavenumber 0, which are left behind the helical modes. It is easy to see that the function defined by

$$u_h(x) := u(x) - \left(\mathcal{P}_{\Sigma+} + \mathcal{P}_{\Sigma-} + \mathcal{P}_D\right)u(x) \quad (39)$$

is a uniform vector field on $\mathbf{T}^3$ for every $u(x) \in L^2_\chi(\mathbf{T}^3)$. Because

$$u_N(x) - \left(\mathcal{P}_{\Sigma+} + \mathcal{P}_{\Sigma-} + \mathcal{P}_D\right)u_N(x) \\ = \hat{u}_x(0)e_x + \hat{u}_y(0)e_y + \hat{u}_z(0)e_z \quad (40)$$

for every elements of the fundamental sequence of $u(x)$. We will denote the space of uniform vector fields by L^2_H and define the projection operator $\mathcal{P}_H$ from L^2_χ to L^2_H by

$$\mathcal{P}_H u(x) := \hat{u}_x(0)e_x + \hat{u}_y(0)e_y + \hat{u}_z(0)e_z. \quad (41)$$

By definition, $L^2_H(\mathbf{T}^3) \cap L^2_s(\mathbf{T}^3) = \{0\}$, for $s = \Sigma+$, $\Sigma-$, D. Every elements of $L^2_H(\mathbf{T}^3)$ is a harmonic function on $\mathbf{T}^3$, i.e. satisfies $\nabla \cdot h = 0$, $\nabla \times h = 0$. The degree of freedom of $L^2_H(\mathbf{T}^3)$ is three. It is known that the number coincides with *the first Betti number* of the manifold $\mathbf{T}^3$[11].

At this point, the orthogonal decomposition of $L^2_\chi(\mathbf{T}^3)$ is completed.

Theorem 5

$$L^2_\chi(\mathbf{T}^3) = L^2_{\Sigma+}(\mathbf{T}^3) \oplus L^2_{\Sigma-}(\mathbf{T}^3) \oplus L^2_D(\mathbf{T}^3) \oplus L^2_H(\mathbf{T}^3). \quad (42)$$

This is the synthesis of a special case of Hodge decomposition and the Beltrami decomposition.

Let us finish this section by giving a theorem to construct the orthonormal complete basis of *divergence-free* vector fields. For this purpose, we introduce a triad of uniform vector fields { $h_{\Sigma+}(0)$, $h_{\Sigma-}(0)$, $h_D(0)$ }. Choice of the vector fields is arbitrary except for the requirement that orthonormal relations $\overline{h_s(0)} \cdot h_{s'}(0) = \delta(s|s')$ must be hold.

Theorem 6 *If $\{f_\lambda\}$ is an orthonormal complete basis of $L^2(\mathbf{T}^3)$, then*

$$\left\{ \hat{f}_\lambda(0)h_{\Sigma+}(0) + \mathcal{P}^\dagger_{\Sigma+} f_\lambda(x) \right\} \\ \cup \left\{ \hat{f}_\lambda(0)h_{\Sigma-}(0) + \mathcal{P}^\dagger_{\Sigma-} f_\lambda(x) \right\} \quad (43) \\ \cup \left\{ h_D(0) \right\}$$

is an orthonormal complete basis of divergence-free vector fields $L^2_{\Sigma+}(\mathbf{T}^3) \oplus L^2_{\Sigma-}(\mathbf{T}^3) \oplus L^2_H(\mathbf{T}^3)$.

[4] helical basis

Here we gives an anthology of the properties which are satisfied for every helical basis.

First, we describe the procedure to obtain the expansion coefficient of the helical basis. Consider the helical basis which is obtained by the helical pull up of an orthonormal basis of $L^2_0(\mathbf{T}^3)$, say wavelet basis $\{\psi_\lambda : \lambda \in \Lambda\}$ where Λ is an appropriate set of indices. The expansion coefficients of an L^2_χ-vector field $u(x)$ with respect to $\mathcal{P}^\dagger_s\{\psi_\lambda\}$, the helical pull up of $\{\psi_\lambda\}$ are given by the inner products

$$\begin{aligned} u_{\lambda,s} &:= \left\langle \mathcal{P}^\dagger_s \psi_\lambda(x), u(x) \right\rangle_\chi \\ &= \left\langle \psi_\lambda(x), \mathcal{P}^{\dagger*}_s u(x) \right\rangle \\ &= \sum_{k \in \mathbf{Z}^3 \setminus \{0\}} \overline{\hat{\psi}_\lambda(k)} \left(\hat{u}(k) \cdot \overline{h_s(k)} \right), \quad (44) \end{aligned}$$

for $\lambda \in \Lambda$. The last equation leads to the following procedures; (1) calculate $\mathcal{F}u$, the vector-valued Fourier transform of $u(x)$: (2) calculate the inner product $\hat{u} \cdot \overline{h_s}$, and we obtain s-mode helical Fourier coefficients $\{\hat{u}_s\}$: (3) calculate $\mathcal{F}^{-1}\{\hat{u}_s\}$, the scalar-valued inverse Fourier transform of the s-mode helical Fourier coefficients: (4) applying the procedure of the scalar-valued function expansion, for example, the fast wavelet transform algorithm to $\mathcal{F}^{-1}\{u_s\}$. Thus we obtain the expansion coefficients $\{u_{\lambda,s}\}$. It should be remarked here that the procedure given above requires no novel and specific algorithm, but only such tools as FFT and FWT, which are popular in the signal processing.

The helical pull up of a real valued scalar function $f(x) \in L^2(\mathbf{T}^3)$ is also a real valued vector field. Because the helical vector satisfies

$$h_s(k) = \overline{h_s(-k)} \quad (45)$$

for each $k \in \mathbf{Z}^3$ so that each component of the helical pull up of the Fourier coefficient $\hat{f}(k)$ satisfies

$$\left(\hat{f}(k)h_s(k) \right)_j = \overline{\left(\hat{f}(-k)h_s(-k) \right)_j} \quad (46)$$

for $j = 1, 2, 3$ and each $k \in \mathbf{Z}^3 \setminus \{0\}$. Thus each element of the helical basis pulled up from a real-valued function basis of $L^2(\mathbf{T}^3)$ is also real-valued.

The continuity and differentiability of each element of a helical basis depends on those of the original scalar function basis. These properties of the helical basis, however, may not be identical to those of original one. Since the multiplication of the polarity vector $h_s(k)$ lead to the 'twisting' of the values of the scalar function to each components. In order to evaluate the properties, here we go via such a path that goes through the Sobolev's imbedding theorem.

When the function $f(x)$ belongs to the Sobolev space $H^r(\mathbf{T}^3)$, i.e. $f(x)$ satisfies

$$\|f(x)|H^r(\mathbf{T}^3)\|^2 = \sum_{k \in \mathbf{Z}^3} (1 + |k|^2)^{\frac{r}{2}} |\hat{f}(k)|^2 < \infty, \quad (47)$$

each component of its helical pull up satisfies

$$\begin{aligned} \|(\mathcal{P}^\dagger_s f(x))_j | H^r(\mathbf{T}^3)\|^2 \\ = \sum_{k \in \mathbf{Z}^3} (1 + |k|^2)^{\frac{r}{2}} |\hat{f}(k)|^2 |(h_s(k))_j|^2 \\ < \|f(x)|H^r(\mathbf{T}^3)\|^2 \quad (48) \end{aligned}$$

for $j = x$, y and z. This convergence property and the Sobolev's embedding theorem guarantees the continuity of the helical pull up of $f(x)$ when $s > \frac{3}{2}$. For this case, the formula of the helical pull up $\mathcal{P}_j^\dagger f(x)$ converges pointwise. The differentiability of the helical pull up of $f(x)$ is guaranteed up to r-th order when $f(x) \in H^{s+\frac{3}{2}}(\mathbf{T}^3)$ where $s \geq r$.

Next we consider the relation between the Σ+-mode and the Σ--mode associated with exchange of parity, i.e. reversion of orientation of the coordinate system on $\mathbf{T}^3$. The parity exchange operator is denoted by Γ here. When Γ acts on a real-valued scalar function, say $u(x)$, the relation

$$\Gamma u(x) = \sum_{k \in \mathbf{Z}^3 \backslash \{0\}} \widehat{u}(k) \exp(2\pi i k \cdot (-x))$$
$$= \sum_{k \in \mathbf{Z}^3 \backslash \{0\}} \overline{\widehat{u}(k)} \exp(2\pi i k \cdot x) \qquad (49)$$

holds. Thus the mirror image of a real-valued scalar function u is given by the inverse Fourier transformation of the complex conjugate of the Fourier transform of u, that is

$$\Gamma u(x) = \mathcal{F}^{-1}\left(\overline{\mathcal{F}u}\right)(x). \qquad (50)$$

The mirror image of the helical pull up of a real-valued scalar function is given by

$$\Gamma \mathcal{P}_{\Sigma\pm}^\dagger u(x)$$
$$= \sum_{k \in \mathbf{Z}^3 \backslash \{0\}} \widehat{u}(k) h_{\Sigma\pm}(k) \exp(2\pi i k \cdot (-x)), \quad (51)$$
$$= \sum_{k \in \mathbf{Z}^3 \backslash \{0\}} \overline{\widehat{u}(k)} h_{\Sigma\mp}(k) \exp(2\pi i k \cdot x). \qquad (52)$$

This leads to the commutation relation between the parity exchange and the helical pull up for real-valued functions:

$$\Gamma \mathcal{P}_{\Sigma\pm}^\dagger = \mathcal{P}_{\Sigma\mp}^\dagger \Gamma. \qquad (53)$$

[5] multiresolution approximation of the vector fields

In this section, the properties of the MRA of $L_0^2(\mathbf{T}^3)$, which is defined by the restriction of the MRA of $L^2(\mathbf{T}^3)$ to $L_0^2(\mathbf{T}^3)$, and its helical pull up's are discussed. The omission of constant functions from the MRA causes the absence of such solenoidal basis that is homogeneous, i.e. given by the orbit of a function by the action of finite group. It is discussed that this "discrepancy" is consistent with the practically natural postulation for approximating the constant vector field on the finite number of grid points.

First, we briefly review the multiresolution approximation (MRA) of the spaces of the functions of period 1. For details, one should consult §3.11 of Ref.[1]. The MRA of the function spaces on $\mathbf{T}$ is obtained by the periodification of the wavelet functions.

Consider the r-regular MRA of $L^2(\mathbf{R})$, denoted by $\{V_j(\mathbf{R}); j \in \mathbf{Z}\}$. The completion of each subspaces $V_j(\mathbf{R})$ w.r.t. the L^∞-norm. $\mathrm{clos}_{L^\infty(\mathbf{R})}\{V_j(\mathbf{R})\}$, retains the relation $f(x) \in V_j \iff f(2x) \in V_{j+1}$, which is one of the MRA conditions. Let us define $V_j(\mathbf{T})$ by

$$V_j(\mathbf{T}) := \left\{ f(x), f(x) \in \mathrm{clos}_{L^\infty(\mathbf{R})}\{V_j(\mathbf{R})\}. \right.$$
$$\left. f(x+1) = f(x) \right\}. \qquad (54)$$

The nested sequence $\{V_j(\mathbf{T}); j \in \mathbf{Z}\}$ is called the r-regular multiresolution approximation of $L^2(\mathbf{T})$

According to the lemma 13 in §3.11 of Ref.[1], the spaces $V_j(\mathbf{T})$ have the following properties. If $j \leq 0$, they are identical. The space $V_0(\mathbf{T})$ consists of constant functions. The dimension of the space $V_j(\mathbf{T})$ is 2^j.

Let $W_j(\mathbf{T})$ be the orthogonal complement of $V_j(\mathbf{T})$ in $V_{j+1}(\mathbf{T})$, the space $L^2(\mathbf{T})$ is represented in terms of a direct sum of the subspaces as follows:

$$L^2(\mathbf{T}) = V_0(\mathbf{T}) \oplus W_0(\mathbf{T}) \oplus W_1(\mathbf{T})$$
$$\oplus W_2(\mathbf{T}) \oplus W_3(\mathbf{T}) \oplus .... \qquad (55)$$

In the present study, we will define the multiresolution approximation of $L^2(\mathbf{T}^3)$ by the tensor product of the one-dimensional ones:

$$V_j(\mathbf{T}^3) := V_j(\mathbf{T}) \otimes V_j(\mathbf{T}) \otimes V_j(\mathbf{T}) \qquad (56)$$

for $j = 0$, 1, 2,.... The corresponding spaces $W_j(\mathbf{T}^3)$ ($j \geq 0$) are defined by the orthogonal complement of $V_j(\mathbf{T}^3)$ in $V_{j+1}(\mathbf{T}^3)$. According to the construction procedures, $V_0(\mathbf{T}^3)$ consists of the constant functions. The dimension of $V_j(\mathbf{T}^3)$ and $W_j(\mathbf{T}^3)$ are 2^{3j} and $7 \cdot 2^{3j}$, respectively.

Now we will try to swim away from the shore of the established MRA theory in order to find out the hidden bank of the MRA theory of the three-dimensional vector field.

In section three, we introduced the zero-mean function space $L_0^2(\mathbf{T}^3)$ in order to construct the solenoidal function space by helical pull up. In terms of the MRA of $L^2(\mathbf{T}^3)$, one can obtain a nested sequence of the zero-mean subspaces of $L^2(\mathbf{T}^3)$, $\{\widetilde{V}_j(\mathbf{T}^3)\}$, defined by

$$\widetilde{V}_j(\mathbf{T}^3) = \bigoplus_{i=0}^{j-1} W_i(\mathbf{T}^3) \qquad (57)$$

for $j = 1$, 2, 3,..., or equivalently defined by the orthogonal complement of $V_0(\mathbf{T}^3)$ in $V_j(\mathbf{T}^3)$. The obtained sequence retains the following conditions of MRA by definition:

$$\widetilde{V}_j(\mathbf{T}^3) \subset \widetilde{V}_{j+1}(\mathbf{T}^3), \qquad (58)$$
$$f(x) \in \widetilde{V}_j(\mathbf{T}^3) \iff f(2x) \in \widetilde{V}_{j+1}(\mathbf{T}^3), \qquad (59)$$

for $\forall j \geq 1$ and

$$\mathrm{clos}_{L^2(\mathbf{T}^3)}\left\{\bigoplus_{j=1}^{\infty} \widetilde{V}_j(\mathbf{T}^3)\right\} = L_0^2(\mathbf{T}^3). \qquad (60)$$

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It is easy to see that the dimension of $\widetilde{V}_j(\mathbf{T}^3)$ is $2^{3j} - 1$. The last condition is easy to prove. Because all the functions in the space are, by definition, orthogonal to all the constant functions that belong to $V_0(\mathbf{T}^3)$, that is, $\int_{\mathbf{T}^3} f(x)\mathrm{d}x = 0$ for $^\forall f(x) \in \mathrm{clos}_{L^2(\mathbf{T}^3)} \left\{ \bigoplus_{j=1}^{\infty} \widetilde{V}_j(\mathbf{T}^3) \right\}$. Hereafter we will call the nested sequence $\left\{ \widetilde{V}_j(\mathbf{T}^3) \right\}$ the MRA of $L_0^2(\mathbf{T}^3)$ or the zero-mean MRA of $L^2(\mathbf{T}^3)$.

In the previous section, we proved that the helical pull up operators are unitary in the strict sense when they act on $L_0^2(\mathbf{T}^3)$, instead of $L^2(\mathbf{T}^3)$. Thus the nested sequences that are given by the helical pull up of the zero-mean MRA of $L^2(\mathbf{T}^3)$, $\left\{ \mathcal{P}_s^\dagger \widetilde{V}_j(\mathbf{T}^3) \right\}$, satisfies the conditions

$$\mathcal{P}_s^\dagger \widetilde{V}_j(\mathbf{T}^3) \subset \mathcal{P}_s^\dagger \widetilde{V}_{j+1}(\mathbf{T}^3) \subset L_s^2(\mathbf{T}^3), \qquad (61)$$

$$u(x) \in \mathcal{P}_s^\dagger \widetilde{V}_j(\mathbf{T}^3) \Longleftrightarrow u(2x) \in \mathcal{P}_s^\dagger \widetilde{V}_{j+1}(\mathbf{T}^3), \quad (62)$$

for $^\forall j \geq 1$, and

$$\mathop{\mathrm{clos}}_{L_\chi^2(\mathbf{T}^3)} \left\{ \bigoplus_{j=1}^{\infty} \mathcal{P}_s^\dagger \widetilde{V}_j(\mathbf{T}^3) \right\} = L_s^2(\mathbf{T}^3), \qquad (63)$$

for $s = \Sigma+$, $\Sigma-$ and D. We will call each nested sequence the MRA of $L_s^2(\mathbf{T}^3)$ or the s-mode helical MRA. Because the pull up operator is unitary, the dimension of $\mathcal{P}_s^\dagger \widetilde{V}_j(\mathbf{T}^3)$ is $2^{3j} - 1$ for $s = \Sigma+$, $\Sigma-$ and D.

Intentionally we do not discuss the Riesz basis condition, which is one of the properties that constitutes the definition of multiresolution approximation, and r-regular condition till now. On the latter condition we will discuss in the next section.

In order to discuss the former condition, we define the term homogeneous basis. Consider a Hilbert space on $\mathbf{T}^m$, say H, and the finite group of residues k/N modulo 1, Γ_N, where m, $N \in \mathbf{N}$ and $k \in \mathbf{Z}^m$. We will say H has homogeneous basis if there is such a natural number N and a function $\phi(x)$ whose orbit under the action of Γ_N, $\{\phi(x - k/N) ; k \in \mathbf{Z}^m\}$ is orthonormal complete basis of H. For example, each $V_j(\mathbf{T})$ has homogeneous basis which is given by the orbit of the scaling function $2^{j/2}\phi(2^j x)$ by Γ_{2^j}.

The dimension of each subspace of zero-mean MRA of $L^2(\mathbf{T}^3)$ $\widetilde{V}_j(\mathbf{T}^3)$ is $2^{3j} - 1$. Because there is no such an integer p that satisfies $2^{3j} - 1 = p^3$, $\widetilde{V}_j(\mathbf{T}^3)$ never has any homogeneous basis. So is its helical pull up $\mathcal{P}_s^\dagger \widetilde{V}_j(\mathbf{T}^3)$

Proposition 7 *Each subspace of the MRA of $L_s^2(\mathbf{T}^3)$, $\mathcal{P}_s^\dagger \widetilde{V}_j(\mathbf{T}^3)$ ($s = \Sigma+$, $\Sigma-$, D and $j \in \mathbf{N}$), cannot have any homogeneous basis.*

Intuitively this is a consequence of a practical requirement for the signal processing or the numerical analysis that the vector field $u(x)$ should be regarded as an element of $L_H^2(\mathbf{T}^3)$ if all of its values on the grid points located at $x = k/2^j$ are identical. In mathematical words, if there exists a vector-valued homogeneous basis $\{\phi_{jl}(x)\}$, where j and l are indices for the resolution and the location, respectively,

it should be able to approximate not only the elements of $L_s^2(\mathbf{T}^3)$ but also those of $L_H^2(\mathbf{T}^3)$. Thus the requirement is restated as the following postulation.

Postulation 8 *If $u(x)$ is approximated as $u_j(x) = \sum c_j \phi_{jl}(x)$, where c_j is a constant for each finite resolution class j, i.e. the inner products $c_{j,l} = \langle \phi_{jl}, u \rangle_\chi$ are dependent only on j and independent of l, u belongs to $L_H^2(\mathbf{T}^3)$.*

Defining the helical pull up of the scaling function of j-th resolution $\phi_j(x)$ by

$$\phi_{j,s}(x) := \widehat{\phi}(0) h_s(0) + \mathcal{P}_s^\dagger \phi(x), \qquad (64)$$

where $h_s(0)$ is an arbitrary uniform vector field with amplitude one, one can obtain the set of function that is given by $\{\Gamma_{2^j} \phi_{j,s}(x)\}$. It approximates any constant vector field in such a way that is stated above.

[6] singularity of $\Sigma\pm$-mode helical vectors and spatial coherence of solenoidal helical wavelet

In this section the spatial coherence of helical wavelet is discussed.

MRA is called r-regular when the generating scaling function ψ_0 and the associated mother wavelets ψ_ϵ satisfy the conditions

$$\forall m \in \mathbf{Z}, \exists C_m < \infty \text{ s.t. } |\partial^\alpha \psi_\epsilon(x)| \leq C_m (1 + |x|)^{-m} \tag{65}$$

for every multi-index α satisfying $|\alpha| \leq r$. Our interest here is whether the regularity of the helical wavelet suffers from the singularities of the $\Sigma\pm$-mode helical vectors. For this purpose, in this section we will discuss spatial coherence in terms of the vector valued function on $\mathbf{R}^3$ defined by the Fourier integral of $L^2(\mathbf{R}^3)$ function:

$$\psi_{\lambda s}(x) = \int_{\mathbf{R}^3} (\mathcal{F}\psi_\lambda)(k) h_s(k) \exp(2\pi i k \cdot x) \mathrm{d}k. \tag{66}$$

We will call the integral transform integral helical pull up (to s-mode) hereafter. Vector fields on $\mathbf{T}^3$, which we have discussed, are given by periodification of the function. We will assume $(\mathcal{F}\psi_\lambda)(0) = 0$ in order to avoid the arbitrariness of the definition at $k = 0$.

Our afraid is as follows. In order to construct solenoidal basis by helical pull up from an orthonormal complete scalar function basis $\{\psi_\lambda ; \lambda \in \Lambda\}$, the conditions

$$k \cdot h_{\Sigma\pm}(k) = 0, \tag{67}$$

$$|h_{\Sigma\pm}(k)| = 1, \tag{68}$$

$$\sum_{\lambda \in \Lambda} |\mathcal{F}\psi_\lambda(k)|^2 = 1, \tag{69}$$

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are required for every k The first condition says that the vector field is tangential to spheres $|k| = const$. It is a well known result of the differential topology that two-sphere S^2 is not parallelizable manifold so that the vector fields on S^2 have at least two singular points[12]. Thus $h_{\Sigma\pm}(k)$ must have singular points for every $|k|$'s. The second condition says that the vector fields should behave like the function $\mathrm{sgn}(x)$ around the singular points. The third condition together with the two previous ones leads to a conclusion that there must exist such functions that have singular points in their supports Therefore, even if ψ_λ belongs to the Schwartz class $\mathcal{S}(\mathbf{R}^3)$, the vector function $\mathcal{F}\psi_\lambda h_{\Sigma\pm}$ breaks the continuity and does not belong to $\mathcal{S}$ when it has singular points in its support. Furthermore, these properties leads to an expectation that the inverse Fourier transform of $\mathcal{F}\psi_\lambda h_{\Sigma\pm}$ has algebraic tail like the Fourier transform of sgn function.

Asymptotic analysis of the Fourier transform of $\Sigma\pm$-mode helical vector at large r shows the far field behaviour as r^{-2} around $z \sim 0$ plane (see Appendix A). The same exponent of algebraic tail is obtained by a simple scaling argument and the two-dimensional distribution of the tail is required by the convergence of wavelet function under the periodification operation (see Appendix B).

Thus it is plausible that, in the orthonormal solenoidal basis on $\mathbf{R}^3$ which is constructed by the integral helical pull up of a scalar basis, there exist such functions that have algebraic tail. Thus we conjecture that there exist no such orthonormal solenoidal wavelet basis that all the species of wavelets decreases rapidly at large r.

In the preceding part of this section we will examine the behaviour at large r numerically using Littlewood-Paley type MRA and its helical pull up.

According to the recipe by Yamada and Ohkitani [13], the Fourier image $\hat\phi(k)$ of the scaling function $\phi(x)$ is given by

$$\hat\phi(k) = \sqrt{g(k)g(-k)}, \qquad (70)$$

where

$$g(k) = \frac{h(\frac{2}{3} - k)}{h(k - \frac{1}{3}) + h(\frac{2}{3} - k)}, \qquad (71)$$

$$h(k) = \begin{cases} \exp(-(\pi k)^{-2}) & k > 0, \\ 0 & k \leq 0. \end{cases}$$

By definition, this scaling function is infinitely differentiable and following properties are satisfied:

$$\begin{cases} \hat\phi(k) = 1 & \text{for } -\frac{1}{3} \leq k \leq \frac{1}{3}, \\ 0 < \hat\phi(k) < 1 & \text{for } -\frac{2}{3} < k < -\frac{1}{3}, \ \frac{1}{3} < k < \frac{2}{3}, \\ \hat\phi(k) = 0 & \text{for } k \leq -\frac{2}{3}, \ \frac{2}{3} \leq k, \end{cases} \qquad (72)$$

and

$$\sum_{j \in Z} \left| \hat\phi(k + j) \right|^2 = 1. \qquad (73)$$

Then, the mother wavelet $\psi(x)$ is obtained by the Fourier transform of the image,

$$\hat\psi(k) = \sqrt{\hat\phi\left(\frac{k}{2}\right)^2 - \hat\phi(k)^2} \exp(-\pi i k). \qquad (74)$$

Three dimensional wavelet functions are constructed by tensor products of one dimensional ones. There are seven species of mother wavelets ψ_ϵ ($\epsilon = 1, 2, \ldots, 7$) given by

$$\psi_\epsilon(x, y, z) = \psi_\xi(x)\psi_\eta(y)\psi_\zeta(z), \quad (\xi, \eta, \zeta = 0 \text{ or } 1), \qquad (75)$$

where $\psi_0(x) := \phi(x)$ is the one dimensional scaling function, $\psi_1(x) := \psi(x)$ the one dimensional mother wavelet, and the label ϵ is determined as $\epsilon = \xi + 2\eta + 4\zeta$.

It is obvious from the definition given above that Meyer's wavelet belongs to Schwarz class. Because its Fourier transform has compact support, are bounded and infinitely differentiable. Thus the Littlewood-Paley type MRA (Meyer's wavelet) is r-regular for any positive integer r.

According to our definition of helical vectors, x and y components of type $\epsilon = 4$ helical Meyer wavelets $\mathcal{P}_{\Sigma\pm}^\dagger \psi_0(x)\psi_0(y)\psi_1(z)$ have singular points in their support and do not belong to Schwarz class. Spatial coherence of these two components are not obvious. It should be remarked that the Fourier transform of these components remain bounded and compact supported so that Sobolev's imbedding theorem guarantees the analyticity of them. All the components of helical mother wavelets except these two retain the r-regular conditions.

In order to evaluate the localization of wavelet functions, we introduce *coherence spectrum* defined by

$$I(\psi_{j\epsilon ls}(x), c, r) := \int_0^{2\pi} \int_0^\pi \left| \psi_{j\epsilon ls}(x - c) \right|^2 \times r^2 \sin\vartheta \, d\vartheta \, d\varphi, \qquad (76)$$

where the origin of spherical coordinate system (r, ϑ, φ) is taken at c.

In Fig.1, the coherence spectra of the type-ϵ wavelet functions are depicted. The other parameters that define the wavelet are the resolution class j, location l, and helicity s. Those of the tested wavelet are set to $j = 6, l = 0$ and $s = +$, respectively. The center of the spectrum c is taken at $(2^{-j-1}\xi, 2^{-j-1}\eta, 2^{-j-1}\zeta)$ for the type-ϵ scalar or helical mother wavelet, where scalar Meyer wavelet $\psi_{6\epsilon 0}(x)$ takes its maximum. The function is evaluated numerically on the 256^3 number of grids given by $x = (i_x/N, i_y/N, i_z/N)$ where $N = 256$, i_x, i_y and i_z are integers that satisfy $-N/2 < i_x, i_y, i_z \leq N/2$. The spectrum is approximated by the sum

$$\int_{r_j - \Delta}^{r_j + \Delta} I(\psi_\lambda(x), c, r_j) dr$$

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$$\sim \sum_{r_j - \Delta \leq |x - c| \leq r_j + \Delta} \frac{|\psi_\lambda(x - c)|^2}{N^3}, \quad (77)$$

where the radius r_j and the shell thickness Δ are $r_j = j/N$ and $\Delta = 1/2N$, respectively.

In Fig.1(a) the spectra of scalar wavelet functions are depicted. All the wavelets show rather exponential behaviour at large r with oscillations. Helical wavelets, except for the type-4 one, show almost the same behaviour as the scalar ones (see Fig.1(b)). The oscillations at large r are slightly modified. There are two remarkable features in the spectrum of the type-4 helical wavelet. One is that the spectrum decreases more slowly than those of the other wavelets. The functional form of the spectrum seems rather algebraic at medium scales about $0.05 < r < 0.3$. The other is that the relative amplitude of the oscillations is smaller than that of other helical wavelets. In Fig.1(c), the contribution of each component of type-4 wavelet to the spectrum is depicted. It is obvious from the figure that x- and y-components contribute to the features given above. The z-component, on the other hand, shows the same rapidly decreasing features as others.

On the whole, the type-4 helical wavelet is less localized and less oscillating at large r than the others. These numerically investigated features are consistent with the results of asymptotic analysis and scaling argument given in the appendices.

[7] concluding remarks

In the present paper, we proved the Hodge-Beltrami decomposition of vector fields on $\mathbf{T}^3$. Then, we have shown the construction procedure of the orthonormal complete basis of the solenoidal fields on $\mathbf{T}^3$. This procedure has a merit that it requires only conventional numerical algorithms (i.e. fast Fourier transform and fast wavelet transform) and no novel one in calculating the wavelet coefficients.

Based on this procedure, multiresolution approximation (MRA) of vector fields on $\mathbf{T}^3$ is constructed. It was shown that there exists no scaling function which has close relation to the assumption that if a vector field is constant in numerical simulation, it can be regarded as an approximation of a uniform field. Nonexistence of a single scaling function which generates MRA might have relation to the idea of multiwavelet [14].

We have conjectured that, in any orthonormal divergence-free three-dimensional vector wavelet basis, there must be such kind of wavelet function that behaves algebraically at large $|x|$ so that it breaks the r-regular condition. This conjecture has its root in the facts that the helical vector must be tangential to S^2 for every point, and that S^2 is not a parallelizable manifold. We feel that the requirement of exponential decay imposed in the work by Battle and Federbush is inappropriate if it is required for all the

functions that constitutes an orthonormal solenoidal basis [5]. In two or four dimensional spaces, existence of the r-regular solenoidal vector wavelet basis is plausible because S^1 and S^3 are parallelizable.

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Appendix A: asymptotic analysis of Fourier transform of helical vectors

In this appendix Fourier transform of the helical vectors is considered. Since the helical vectors have singularities, there exists such type of helical wavelet that does not belong to Schwarz class. In order to estimate the amplitude of helical wavelet at large r, first we calculate formally the Fourier transform of each component, and then, asymptotic analysis at large r is carried out.

In Cartesian coordinates, the basis vectors of the spherical polar coordinate system are decomposed as

$$
\begin{aligned}
e_r(\boldsymbol{k}) &= \frac{\rho\cos\beta}{\sqrt{\rho^2+\zeta^2}}\, \boldsymbol{e}_x + \frac{\rho\sin\beta}{\sqrt{\rho^2+\zeta^2}}\, \boldsymbol{e}_y \\
&\quad + \frac{\zeta}{\sqrt{\rho^2+\zeta^2}}\, \boldsymbol{e}_z, \\
e_\vartheta(\boldsymbol{k}) &= \frac{\zeta\cos\beta}{\sqrt{\rho^2+\zeta^2}}\, \boldsymbol{e}_x + \frac{\zeta\sin\beta}{\sqrt{\rho^2+\zeta^2}}\, \boldsymbol{e}_y \\
&\quad - \frac{\rho}{\sqrt{\rho^2+\zeta^2}}\, \boldsymbol{e}_z, \\
e_\varphi(\boldsymbol{k}) &= -\sin\beta\, \boldsymbol{e}_x + \cos\beta\, \boldsymbol{e}_y.
\end{aligned}
\tag{78}
$$

where (ρ, β, ζ) denotes cylindrical coordinates. We consider here the x-component of the Fourier transform of $h_{\Sigma\pm}(\boldsymbol{k})$. By operating the ninety degree rotation around z-axis, one can obtain the y-component of the Fourier transform. These two components are relevant to the singularity problem of wavelet function.

The Fourier transform of the x-component of $e_\vartheta(\boldsymbol{k})$ is given by

$$
\begin{aligned}
&\int_{\mathbf{R}^3} (e_\vartheta(\boldsymbol{k}))_x \exp\left(2\pi i \boldsymbol{k}\cdot\boldsymbol{x}\right) d\boldsymbol{k} \\
&= \int_{-\infty}^{\infty}\!\!\int_0^{2\pi}\!\!\int_0^{\infty} \exp\left(2\pi i\left(\rho r\cos(\beta-\alpha)+\zeta z\right)\right) \\
&\quad \times \frac{\rho\zeta\cos\beta}{\sqrt{\rho^2+\zeta^2}} d\rho\, d\beta\, d\zeta \\
&= \int_0^{\infty}\left(2i\int_0^{\infty}\frac{\zeta\sin(2\pi z\zeta)}{\sqrt{\rho^2+\zeta^2}}d\zeta\right) \\
&\quad \times \left(2\cos\alpha\int_0^{\pi}\cos\beta\exp(2\pi i\rho r\cos\beta)d\beta\right)\rho\, d\rho \\
&= -4\pi\cos\alpha\int_0^{\infty}\rho^2 K_1(2\pi z\rho)\, J_1(2\pi\rho r)\, d\rho, \quad (79)
\end{aligned}
$$

where we define the value at $-z$ by exchanging the sign of Eq.(79) because the integrand is an odd function. Similarly, the Fourier image of the x-component

of $e_\varphi(\boldsymbol{k})$ is

$$
\begin{aligned}
&\int_{\mathbf{R}^3} (e_\varphi(\boldsymbol{k}))_x \exp\left(2\pi i\boldsymbol{k}\cdot\boldsymbol{x}\right) d\boldsymbol{k} \\
&= -\int_{-\infty}^{\infty}\!\!\int_0^{2\pi}\!\!\int_0^{\infty} \exp\left(2\pi i\left(\rho r\cos(\beta-\alpha)+\zeta z\right)\right) \\
&\quad \times \sin\beta\rho\, d\rho\, d\beta\, d\zeta \\
&= -\int_0^{\infty}\left(\int_{-\infty}^{\infty}\exp(2\pi i\zeta z)d\zeta\right)\left(\int_0^{2\pi}\sin(\beta+\alpha)\right. \\
&\quad \left.\times \exp(2\pi i\rho r\cos\beta)d\beta\right)\rho\, d\rho \\
&= -2\pi i\,\delta(z)\,\sin\alpha\int_0^{\infty}\rho\, J_1(2\pi\rho r)\, d\rho. \quad (80)
\end{aligned}
$$

From Eq.(79) and Eq.(80), one formally obtains the Fourier transform of $(h_\pm(\boldsymbol{k}))_x$ as

$$
\begin{aligned}
&\int_{\mathbf{R}^3} (h_\pm(\boldsymbol{k}))_x \exp\left(2\pi i\boldsymbol{k}\cdot\boldsymbol{x}\right) d\boldsymbol{k} \\
&= \sqrt{2}\pi\int_0^{\infty}\left(-2\rho K_1(2\pi z\rho)\,\cos\alpha \pm \delta(z)\sin\alpha\right) \\
&\quad \times \rho\, J_1(2\pi\rho r)\, d\rho. \quad (81)
\end{aligned}
$$

Because Dirac's delta function $\delta(z)$ is zero for $|z|>0$ and asymptotic behaviour of modified Bessel's function $K_1(2\pi\rho z) \sim \exp(-2\pi\rho z)$ for $|z| \gg 1$, function is well localized in the z-direction. So we focus on the behaviour around $|z| \ll 1$. In the region $2\pi\rho z \ll 1$ and $2\pi\rho r \gg 1$, the Bessel functions are approximated as

$$
\begin{aligned}
K_1(2\pi\rho z) &\sim \frac{1}{2\pi\rho z}, \\
J_1(2\pi\rho r) &\sim \frac{1}{\pi\sqrt{\rho r}}\cos\left(2\pi\rho r - \frac{3\pi}{4}\right).
\end{aligned}
\tag{82}
$$

Substituting this expression, we obtain the asymptotic expression of Eq.(81) given by

$$
\begin{aligned}
&\int_{\mathbf{R}^3} (h_\pm(\boldsymbol{k}))_x \exp\left(2\pi i\boldsymbol{k}\cdot\boldsymbol{x}\right) d\boldsymbol{k} \\
&\sim \sqrt{\frac{2}{r}}\left(-\frac{1}{\pi}\text{p.v.}\left(\frac{1}{z}\right)\cos\alpha \pm \delta(z)\sin\alpha\right) \\
&\quad \times \int_0^{\infty}\sqrt{\rho}\,\cos\left(2\pi\rho r - \frac{3\pi}{4}\right)d\rho. \quad (83)
\end{aligned}
$$

Let us investigate the behaviour of the definite integral in Eq.(83) by replacing the semi-infinite interval $[0,\infty)$ by the finite one $[0, \rho_0]$, i.e.

$$
I(\rho_0) := \int_0^{\rho_0}\sqrt{\rho}\,\cos\left(2\pi\rho r - \frac{3\pi}{4}\right)d\rho. \tag{84}
$$

It is easily confirmed that

$$
\begin{aligned}
I(\rho_0) &= \int_0^{\rho_0}\sqrt{\rho}\,\cos\left(2\pi\rho r - \frac{3\pi}{4}\right)d\rho \\
&= \frac{1}{2\pi r}\left\{\sqrt{\rho_0}\sin\left(2\pi\rho_0 r - \frac{3\pi}{4}\right)\right. \\
&\quad \left. + \frac{1}{2\sqrt{2r}}\left[S(2\sqrt{\rho_0 r}) + C(2\sqrt{\rho_0 r})\right]\right\} \quad (85)
\end{aligned}
$$

where S and C are Fresnel integrals,

$$S(x) := \int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt,$$
$$C(x) := \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt. \tag{86}$$

For sufficiently large x, they are asymptotically evaluated as follows:

$$S(x),\ C(x) \sim \frac{1}{2} + O\left(\frac{1}{\sqrt{x}}\right). \tag{87}$$

Substituting Eq.(87) into Eq.(85), one obtains

$$I(\rho_0) = \frac{1}{2\pi r}\left\{ \sqrt{\rho_0}\sin\left(2\pi\rho_0 r - \frac{3\pi}{4}\right) \right.$$
$$+ \frac{1}{2\sqrt{2r}}\left[1 + O\left(\frac{1}{\sqrt{\rho_0 r}}\right)\right]\right\}$$
$$= \frac{1}{4\sqrt{2}\pi r^{\frac{3}{2}}} + \frac{1}{2\pi r}\left[\sqrt{\rho_0}\right.$$
$$\left.\times \sin\left(2\pi\rho_0 r - \frac{3\pi}{4}\right) + O\left(\frac{1}{\sqrt{\rho_0 r}}\right)\right], \tag{88}$$

which leads to the following approximation:

$$\int_{\mathbf{R}^3} (h_\pm(k))_x \exp(2\pi i k \cdot x)\, dk$$
$$\sim \sqrt{\frac{2}{r}}\left(-\frac{1}{\pi}\mathrm{p.v.}\left(\frac{1}{z}\right)\cos\alpha \pm \delta(z)\sin\alpha\right)$$
$$\times \lim_{\rho_0 \to \infty}\left\{\frac{1}{4\sqrt{2}\pi r^{\frac{3}{2}}} + \frac{1}{2\pi r}\left[\sqrt{\rho_0}\right.\right.$$
$$\left.\left.\times \sin\left(2\pi\rho_0 r - \frac{3\pi}{4}\right) + O\left(\frac{1}{\sqrt{\rho_0 r}}\right)\right]\right\}. \tag{89}$$

The term

$$\frac{1}{4\pi r^2}\left(-\frac{1}{\pi}\mathrm{p.v.}\left(\frac{1}{z}\right)\cos\alpha \pm \delta(z)\sin\alpha\right) \tag{90}$$

in Eq.(89) does not depend on ρ_0. Thus we can draw a conclusion that the Fourier transform of $(h_\pm(k))_x$ is a function asymptotically behaves like $(x^2 + y^2)^{-1}$ when $z \sim 0$ and $|x| \gg 1$.

Appendix B: some estimations a priori

In Appendix A we showed the corroboration of the existence of algebraic tail whose exponent is -2. The tail is shown to distribute rather two-dimensionally around $z \sim 0$. In this appendix we will show that the same conclusion is obtained by a simple scaling argument and by the requirement of convergence of wavelet function under the periodification operation.

Assuming that singularities of the helical vectors cause *algebraic tail* of helical wavelet at sufficiently

large $r = |x|$, exponent of the tail is determined by a simple scaling argument. When wavelet function in W_j is assumed to be divided into two parts, rapidly decreasing part and algebraic tail, as

$$\psi_\lambda(x) = f(x) + Ar^{-\alpha}, \tag{91}$$

for sufficiently large r, the wavelet in W_{j+1} must be given by

$$\sqrt{2^3}\psi_\lambda(2x) = \sqrt{2^3}f(2x) + 2^{\frac{3}{2}-\alpha}Ar^{-\alpha}. \tag{92}$$

The domain of Fourier integral is enlarged by two in each direction so that the contribution of the singularities of helical vectors to class W_{j+1} wavelet become twice of those in class W_j. (Remember that functional form of $h_\pm$ around the singular points does not depend on $|k|$ and the points are aligned with a one-dimensional manifold). The amplitude of the Fourier transform of $\psi(2x)$ is one-eighth of that of $\psi(x)$. Thus the factor of algebraic term should be $2^{-\frac{1}{2}}$, one-fourth of the factor of f, and the exponent $\alpha = 2$.

Scaling argument leads to a conclusion that the exponent of algebraic tail is -2. Is such behaviour of r^{-2} isotropically distributed, i.e. found in all the directions of three-dimensional space? The answer is no. They distributes two-dimensionally because of the following two reasons. One is that the singularity distributes one-dimensionally so that the Fourier transform in the directions perpendicular to the singularity suffers from it. The other is that helical wavelet on $\mathbf{R}^3$, say $\psi_R(x)$, should not diverge under the periodification operation to obtain the wavelet on $\mathbf{T}^3$. This requirement leads to an estimation of the contribution of algebraic tails from far field to the periodification sum:

$$\sum_{|l|>r_0} \psi_R(x+l) \propto \int_{r_0}^\infty \frac{1}{r^2}r^{D-1}\, dr < \infty \implies D < 2,$$
$$\tag{93}$$

where r_0 is a sufficiently large number and D is the (fractal) dimension of region where the algebraic tails spread over. Though the estimation is quite rough, this result claims that the tail must not spread over to all the directions of three-dimensional space, but should be confined in the region which is at most two-dimensional.

Figure captions

Fig 1 Coherence spectra of wavelet functions, (a) scalar Meyer wavelet. (b) helical Meyer wavelet Type of wavelet is distinguished by the line, solid line $c = 1, 2$. broken line, $c = 3, 5$ and 6, dashed line $c = 4$, dash-dotted line $c = 7$. (c) Spectra of each component of type-4 helical wavelet Solid line x-component, dash-dotted line. y-component, broken line z-component. Solid and dash-dotted lines are indistinguishable
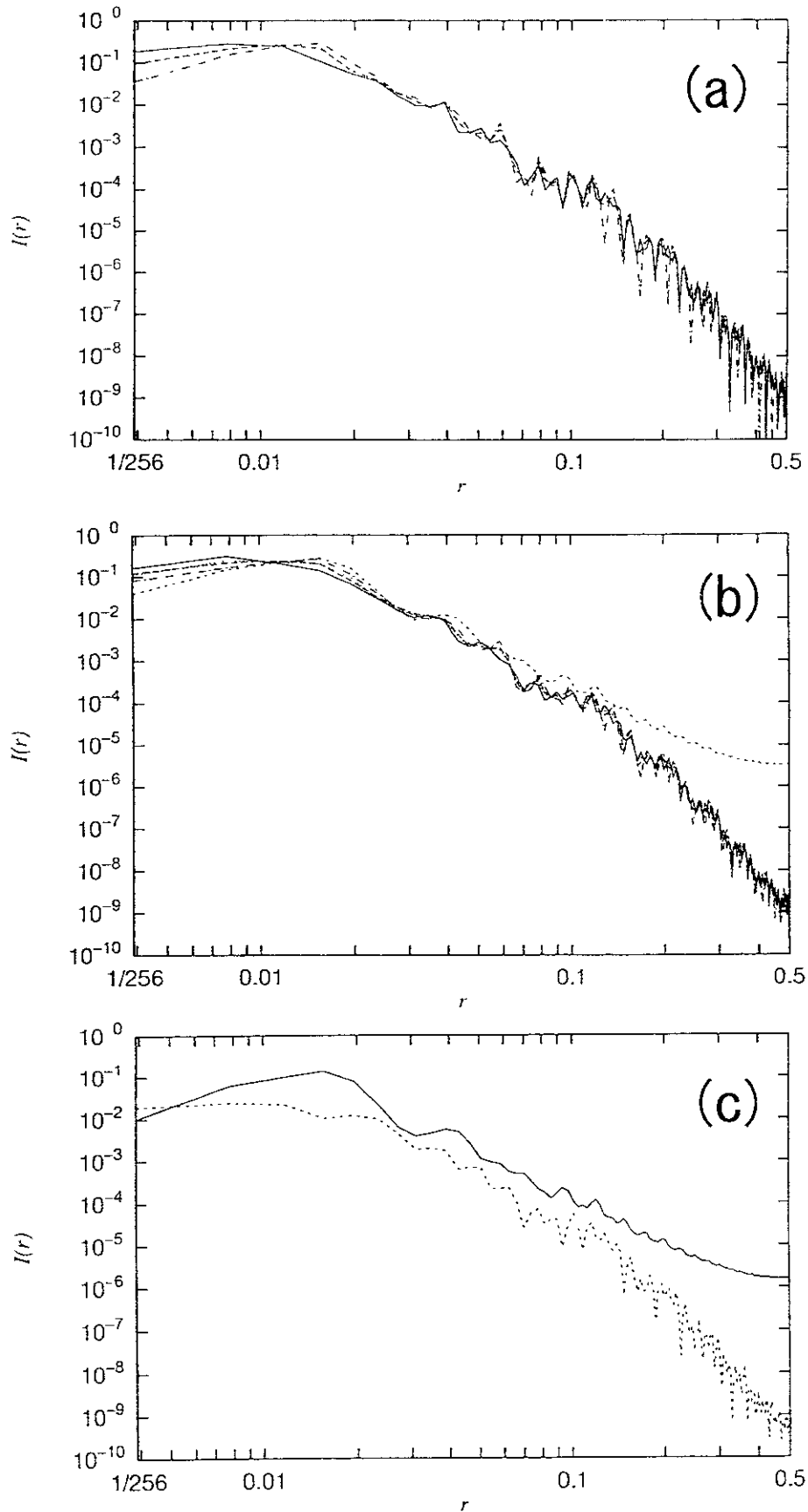
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