ISSN 0915-633X

NATIONAL INSTITUTE FOR FUSION SCIENCE

Proof of Non-invariance of Magnetic Helicity in Ideal Plasmas and a General Theory of Self-organization for Open and Dissipative Dynamical Systems

Y. Kondoh, T. Takahashi and J. W. Van Dam

(Received - Apr. 9, 2002)

NIFS-729 Apr. 2002

RESEARCH REPORT
NIFS Series

TOKI, JAPAN

Proof of non-invariance of magnetic helicity in ideal plasmas and a general theory of self-organization for open and dissipative dynamical systems

KONDOH Yoshiomi, TAKAHASHI Toshiki, and James W. Van Dam[1]

Dept. of Electronic Engineering, Gunma University, Kiryu, Gunma 376-8515, Japan

e-mail: *kondohy@el.gunma-u.ac.jp*

Abstract

It is proved that the magnetic helicity is not invariant, even in an ideal plasma. A novel general theory is presented in which a variety of self-organized states in open and dissipative dynamical systems with various fluctuations can be found. This theory is based on the principle that the self-organized states must be those states for which the rate of change of global autocorrelations for multiple dynamical field quantities, which depend on multidimensional mutually independent variables, is minimized. One of the important points of this theory is that the original generalized dynamic equations are embedded in the final equivalent definition for the self-organized states, and therefore the equations deduced from the final equivalent definition include all the time evolution characteristics of the dynamical system of interest. Since states derived from the Euler-Lagrange equations with the use of variational calculus have minimal rates of change of the global autocorrelations, they are most stable and unchangeable compared with other states.

Keywords: self-organization, global autocorrelations, open or closed dynamic systems, continuous fluctuations, relaxation phenomena

1. Introduction

After J. B. Taylor published his famous theory [1] to explain the appearance of the reversed field pinch configuration [2], magnetic helicity has been believed to play an important role as a global invariant in the self-organization process and relaxation phenomena of magnetized plasmas. However, another model—the partially relaxed state model—has been proposed to explain the reversed field pinch experimental data [3, 4]; this model was theoretically derived by allowing partial losses of the magnetic flux and the helicity [5, 6]. The partially relaxed state model and the mode transition point of the self-organized state were deduced from the energy integral, without the assumption that magnetic helicity is invariant. A subsequent version of this theory of self-organization was developed, based on autocorrelations of physical quantities, which includes the Taylor state as a limiting case [8, 9].

In the present paper, we prove that the global magnetic helicity is not an invariant even in ideally conducting MHD plasmas and that therefore the Taylor relaxation process never occurs physically in real experimental plasmas and simulations. Furthermore, we present a novel general theory for how to find self-organized states in open and dissipative dynamical systems. This theory is applicable to various nonlinear dynamical systems and reproduces the Taylor state as a limiting case [5-12]. We also show some applications of the present theory to dissipative Korteweg-de Vries solitons and dissipative MHD plasmas.

2. Proof of the non-invariance of global magnetic helicity even in ideal MHD plasmas

J. B. Taylor's theory is based on Maxwell's equations for the electric and magnetic fields:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad . \tag{1}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \ . \tag{2}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} . \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0 \quad . \tag{4}$$

Using the vector and scalar potentials A and ϕ, one may rewrite Eqs. (1) and (4) equivalently as

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} , \tag{5}$$

$$\mathbf{B} = \nabla \times \mathbf{A} . \tag{6}$$

With the use of Eqs. (5) and (6), the global magnetic helicity K [2], defined in a volume V bounded by an ideally conducting surface, and its time derivative are given as follows:

$$K = \int_V \mathbf{A} \cdot \mathbf{B} \; dV , \tag{7}$$

$$\frac{\partial K}{\partial t} = \frac{2}{\mu_0} \int_V \mathbf{E} \cdot \mathbf{B} \; \mathrm{d}V + \frac{1}{\mu_0} \oint_S (\mathbf{E} \times \mathbf{A}) \cdot \mathrm{d}\mathbf{S} , \tag{8}$$

Taylor's theory involves the following two conjectures:

(1) Since magnetic fields are frozen in an ideal MHD plasma during its local flow, the global magnetic helicity K is considered to be conserved. (This is because K is considered to be a topological quantity of the magnetic field lines.) The argument is as follows: By using the simplified Ohm' s law of $\eta \boldsymbol{j} = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$ and putting resistivity $\eta = 0$, one finds that the volume integral of $\boldsymbol{E} \cdot \boldsymbol{B}$ for every flux tube is zero. Then the volume integral term in Eq. (8) vanishes. Since $\boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S}$ is zero at an ideally conducting surface, the time derivative of K is zero in an ideal MHD plasma. Thus, by this argument, K is considered to be conserved as an invariant in ideal MHD plasmas.

(2) When the resistivity η is small but finite, reconnection of magnetic field lines can take place. However, it is conjectured that the global helicity K can be treated as an invariant during the relaxation process in a non-ideal MHD plasma, because the resistive decay of the total magnetic energy inside an ideally conducting wall is faster than the decay of K. Using variational calculus with the global constraint K = const., J. B. Taylor derived the relaxed state of $\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B}$ from the Euler-Lagrange equation [1]. This result had been previously obtained by S. Chandrasekhar

and L. Woltjer for states with minimum dissipation of magnetic energies, also with the use of variational calculus [13].

Taylor's logic, described in the preceding paragraph, is, however, not based on either a variational principle or an energy principle. It is commonly known that the use of either a variational principle [14] or an energy principle [15] leads to dynamical equations that give the time evolution of the dynamical system of interest, as shown in classical mechanics theory [14] and the well-known dynamical equations for perturbed elements in an ideal MHD plasma [15].

We now prove that the first conjecture of Taylor, (1) above, is not physically accessible, even for an ideal MHD plasma. In order to find the time change of K in an ideal MHD plasma, we must return to Eq. (8) and check the volume integral term more carefully. From the scalar product of the generalized Ohm's law for a fully ionized plasmas in the limit of zero resistivity, we can derive the volume integral term of Eq. (8) as follows,

$$\frac{2}{\mu_0}\int_V \mathbf{E}\cdot\mathbf{B}\ \mathrm{d}V = \int_V \{\ [\frac{c^2}{{\omega_{\mathrm{pe}}}^2}\frac{\partial\ \mathbf{j}}{\partial\ t} - \frac{1}{\mu_0 e {n_{\mathrm{e}}}^2}(p_{\mathrm{e}} - \frac{m_{\mathrm{e}}Z}{m_{\mathrm{i}}}p_{\mathrm{i}})\nabla n_{\mathrm{e}}]\cdot\mathbf{B}\ \}\mathrm{d}V\,, \tag{9}$$

where the usual notations for plasma physics quantities, such as the number density of electrons n_e and electron and ion pressures p_e and p_i, are used. Also, the boundary conditions $\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}=0$ and $\nabla\cdot\boldsymbol{B}=0$ at an ideally conducting wall were used in obtaining Eq. (9). Since all the volume integral terms on the right-hand side of Eq. (9), determined by local physical quantities, can usually have either positive or negative values for turbulent plasmas, we easily confirm that the time rate of change of K can be either positive or negative (the possibility of being zero is statistically negligible). Therefore one cannot definitely conclude that K is a physical invariant, even within an ideal MHD plasma. On the other hand, we can apply the following Poynting law deduced directly from the Maxwell equations, Eqs. (1)−(4), for any MHD plasmas:

$$\oint_S (\mathbf{E}\times\mathbf{B})\cdot\mathrm{d}\mathbf{S} = -\mu_0\int_V [\ \mathbf{j}\cdot\mathbf{E} + \frac{\partial}{\partial t}(\frac{B^2}{2\mu_0} + \frac{\varepsilon_0 E^2}{2})\ ]\ \mathrm{d}V\,.$$

Since the left-hand side of this law is zero at an ideal conducting wall, any changes in the electric and magnetic field energies are converted to charged particle energy by the $\boldsymbol{j} \cdot \boldsymbol{E}$ term so as to conserve the total energy, in either an ideal or a non-ideal MHD plasma.

We next clarify the meaning of the second conjecture, (2) above. Since K is never invariant during local plasma flows or relaxation even in an ideal MHD plasma, the global constraint that uses K has no power to limit the relaxation process itself within an MHD plasma. The mathematical procedure in conjecture (2) is simple calculus to find a group of solutions having minimum magnetic energy within a wider set of solutions having the same value of K. There certainly do exist relaxation processes in plasmas, but the self-organization process itself has no connection with the conjecture that K is a global invariant. Topological quantities are defined clearly in such a way that various configurations of lines have their own unique number of nards, being independent with respect to any deformation of the lines that does not change the number of connection points. In other words, after the configuration of the lines (for example, the magnetic field lines in a plasma) is determined, the value of a topological quantity can be calculated, but the value itself has no power inversely to determine the configuration of the lines (e.g., field lines).

Without using topological quantities such as K, we are able to derive the Taylor state of $\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B}$, starting from the fundamental definition of the self-organized states. The theory for how to derive the condition for realization of self-organized states, with the Taylor state of $\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B}$ included as the limiting case of uniform resistive MHD plasmas, will be presented in the next section.

3. General theory to find self-organized states in nonlinear dissipative dynamic systems

We develop here a novel basic formulation of a general theory to find self-organized states that is an extension of the theory in Ref. [9]. It should be emphasized that the present theory, which uses auto-correlations for dynamical quantities, is not based on either a variational principle or an energy principle, and also that the auto-correlations are not time invariants.

We consider a set of N dynamical variables $\boldsymbol{q} \equiv \boldsymbol{q}[\xi^k] \equiv \{ q_1[\xi^k], \ldots, q_N[\xi^k] \}$, with M-dimensional independent variables $[\xi^k]$ $(k=1,2,\ldots M)$, which may include time, space, and velocity in distribution functions, or prices, amount of materials, budgets for production systems, and other such various variables. Using generalized symbolic dynamical operators, we may write the general nonlinear $N-$set simultaneous equations for an open or a closed dynamical system as

$$\partial q_i[\xi^k] / \partial \xi^j = D_i{}^j[\boldsymbol{q}], \tag{10}$$

where $D_i{}^j[\boldsymbol{q}]$ $(j=1,2,\ldots N)$ represents dissipative or non-dissipative, linear or nonlinear operators for the change of a dynamical variable q_i along an independent variable ξ^j. After multiplying $q_i[\xi^k]$ on both sides of Eq. (10) and integrating over the independent variables ξ^k $(k \neq j)$, we obtain "conservation laws" for all the dynamical variables q_i as follows:

$$\frac{\partial}{\partial \xi^j} \int \frac{1}{2} q_i{}^2 \left| J_{k\neq j} \right| \prod\nolimits_{k\neq j} \mathrm{d}\xi^k = \int q_i D_i{}^j[\mathbf{q}] \left| J_{k\neq j} \right| \prod\nolimits_{k\neq j} \mathrm{d}\xi^k \tag{11}$$

Here, the dynamical system of interest always has fluctuations of the dynamical variables $q_i[\xi^k]$ along the axis of the variable ξ^j. The fluctuations may have several characteristic lengths in different orders along ξ^j, one of which is expressed as τ_c. The characteristic length τ_c may give the ordering of the relaxation time scale. From the standpoint of observations, the self-organized relaxed states are identified by the following definition with the use of autocorrelations between the dynamical variables $q_i[\xi^j]$ and $q_i[\xi^j + (\Delta\xi^j/\tau_c)]$, where the increase of ξ^j is normalized as

$$\min \left| \frac{\int q_i[\xi^k] \left\{ q_i[\ \xi^k + \dfrac{\Delta \xi^j}{\tau_c}] \right\} \left| J_{k\neq j} \right| \prod\nolimits_{k\neq j} \mathrm{d}\xi^k}{\int (q_i[\xi^k])^2 \left| J_{k\neq j} \right| \prod\nolimits_{k\neq j} \mathrm{d}\xi^k} - 1 \right| \ . \tag{12}$$

Using Taylor expansion for Eq. (12), we obtain the following equivalent definition of the self-organized states from the first order of $\Delta\xi^j/\tau_c$,

$$\min\left|\frac{\int \frac{1}{2}\frac{\partial}{\partial\xi^j}\left\{ q_i[\xi^k]^2\right\}|J_{k\neq j}|\prod_{k\neq j}\mathrm{d}\xi^k}{\frac{\tau_c}{\Delta\xi}\int\{ q_i[\xi^k])^2|J_{k\neq j}|\prod_{k\neq j}\mathrm{d}\xi^k}\right|. \tag{13}$$

Substituting the original dynamical equations (10) into Eq. (13), we obtain the final definition of the self-organized states as follows:

$$\min\left|\frac{\int q_i[\xi^k]D_i[\mathbf{q}]|J_{k\neq j}|\prod_{k\neq j}\mathrm{d}\xi^k}{\frac{\tau_c}{\Delta\xi}\int(q_i[\xi^k])^2|J_{k\neq j}|\prod_{k\neq j}\mathrm{d}\xi^k}\right|. \tag{14}$$

It should be emphasized that all of the dynamical laws, characterized by the nonlinear simultaneous system of equations, Eq. (10), are embedded in the equivalent definition, Eq. (14).

The mathematical expressions of Eqs. (13) and (14) are obtained by variational calculus with the use of functionals F_i with Lagrange multipliers λ_i as follows:

$$F_i = \int\{\frac{\partial}{\partial\xi^j}(\frac{1}{2}q_i[\xi^k]^2) + \frac{\tau_c}{\Delta\xi^j}\lambda_i q_i[\xi^k]^2\}|J_{k\neq j}|\Pi_{k\neq j}\,\mathrm{d}\xi^k \tag{15}$$

$$= \int\{q_i[\xi^k](D_i^{\,k}[\bar{q}] + \frac{\tau_c}{\Delta\xi^j}\lambda_i q_i[\xi^k])\}|J_{k\neq j}|\Pi_{k\neq j}\,\mathrm{d}\xi^k, \tag{16}$$

$$\delta F = \int \delta q_i[\xi^k{}_{k\neq j}](D_i^{\,k}[\bar{q}] + \frac{\tau_c}{\Delta\xi^j}\lambda_i q_i[\xi^k])$$

$$+ q_i[\xi^k]\delta\{D_i^{\,k}[\bar{q}] + \frac{\tau_c}{\Delta\xi^j}\lambda_i q_i[\xi^k{}_{k\neq j}]\}|J_{k\neq j}|\Pi_{k\neq j}\,\mathrm{d}\xi^k = 0 \tag{17}$$

$$\delta F^2 = \int\{\delta q_i[\xi^k{}_{k\neq j}](\delta D_i^{\,j}[\bar{q}] + \frac{\tau_c}{\Delta\xi^j}\lambda_i\ \delta q_i[\xi^k{}_{k\neq j}])\}|J_{k\neq j}|\Pi_{k\neq j}\,\mathrm{d}\xi^k \geq 0, \tag{18}$$

where the functional given in Eq. (16) is used; δF_i and $\delta^2 F_i$ are, respectively, the first and the second variations of F_i; and the variational calculus is performed with respect to the dynamical variables q_i that depend on the variables ξ^k except $k=j$. By means of repeated partial integration and application of the boundary conditions, we eventually obtain the simplest expression for the terms of the operator $D_i^{\,j}[q]$, which we denote by $D_i^{\,j\#}[q]$; this reduction had been previously reported in [7, 8]. In terms of this notation, the condition for the marginal minimum for arbitrary variation $\delta q_i[\xi^k{}_{k\neq j}]$ is given, from Eq. (18), by

$$\delta\ D_i^{\,j\#}\ \ [\bar{q}]+\frac{\tau_c}{\Delta\xi^J}\lambda_i\ \ \delta q_i\ \ [\xi^k{}_{k\neq j}]\ \ =0 \tag{19}$$

Substituting Eq. (19) into Eq. (17), we obtain the Euler-Lagrange equation for arbitrary variation δq_i as follows:

$$D_i^{\,j\#}[\bar{q}]+\frac{\tau_c}{\Delta\xi^J}\lambda_i q_i[\xi^k]=0 \tag{20}$$

Equation (19) can be written as an eigenvalue equation with boundary conditions for $\delta q_i[\xi^k{}_{k\neq j}]$, viz., $D_i^{\,j\#}\{[u_{im}[\xi^k]]$ $+(\tau_c/\Delta\xi^j)\lambda_{im}\,u_{im}[\xi^k]=0$, where $u_{im}[\xi^j]$ and λ_{im} are the normalized eigenvalue solutions and their eigenvalues, respectively, with the appropriate normalization written as $\int u_{im}\cdot u_{in}|J_{k\neq j}|\Pi_{k\neq j}\mathrm{d}\xi^k=\delta_{mn}$, as was also reported in [7, 8]. Substituting one of these eigensolutions into Eq. (18) and using the eigenvalue equation, we obtain from Eq. (18) the following:

$$\delta^2 F=\frac{\tau_c}{\Delta\xi^J}(\lambda_{im}-\lambda_i)\int(u_{im}[\xi^k])^2\left|J_{k\neq j}\right|\Pi_{k\neq j}\ \mathrm{d}\xi^k\geq 0\,. \tag{21}$$

Since Eq. (21) is required for all eigenvalues, we obtain the following condition for the self-organized state with the minimum rate of change:

$$0\ <\ \lambda_i\ <\ \lambda_{i1}, \tag{22}$$

where λ_{i1} is the smallest positive eigenvalue and λ_i is taken to be positive.

On the other hand, when we use Eq. (17), we obtain another functional F and its first variation δF, which are equivalent to Eqs. (22) and (23), and also Euler-Lagrange equations for arbitrary variations $\delta q_i[\xi^k{}_{k\neq j}]$, as follows,

$$\delta F=\int\{\delta\ q_i[\xi^k{}_{k\neq j}](\frac{\partial q_i[\xi^J{}_{J\neq k}]}{\partial\xi^J}+\frac{\tau_c}{\Delta\xi^J}2\lambda_i\ \ q_i[\xi^k{}_{k\neq J}])\}\left|J_{k\neq j}\right|\Pi_{k\neq j}\ \mathrm{d}\xi^k=0, \tag{23}$$

$$\frac{\partial q_i[\xi^k{}_{k\neq j}]}{\partial\xi^J}+\frac{\tau_c}{\Delta\xi^J}2\lambda_i\ \ q_i[\xi^k{}_{k\neq j}]\ =0. \tag{24}$$

We can easily obtain the following solutions of Eq. (24):

$$q_i[\xi^k] = \exp(-2\lambda_i \xi^j)\ q_i[\xi^j{}_0, \xi^k{}_{k \neq j}], \tag{25}$$

where $\xi^j{}_0$ is the initial value of ξ^j.

The new theory presented above is a natural logical extension of the theories reported in [7-9] to the general nonlinear set of N simultaneous equations expressed by Eq. (10), for any open or closed dynamical system.

4. Application of general theory for self-organization

In this section we present two typical applications of the present theory: first, to solitons described by the Korteweg-deVries equation with a dissipative term and, second, to dissipative MHD plasmas.

4-1. K-dV solitons with dissipation

Here, we present analytical self-organized soliton solutions of the following Korteweg-deVries equation with a dissipation term,

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = \nu \frac{\partial^2 u}{\partial x^2} \ , \tag{26}$$

where ν is the viscosity. Fluctuations may be neglected in the soliton solutions, and therefore we obtain the following functional F from Eq. (16) to first order of Δt, integrated over the periodic length $[a, b]$:

$$F = \int_a^b \left\{ u[\nu \frac{\partial^2 u}{\partial x^2} - 6u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}] + \lambda u^2 \right\} dx \tag{27}$$

Using partial integration and the applying the periodic conditions at $x = a$ and b, we obtain the following first variation δF:

$$\delta\ F = 2\int_a^b \{\ \delta\ u(\nu \frac{\partial^2 u}{\partial x^2} + \lambda u)\ \} dx = 0 \ . \tag{28}$$

The Euler-Lagrange equation for arbitrary variations of δu is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\lambda}{\nu} u = 0 \ . \tag{29}$$

If we denote the solution of Eq. (29) as $u^*(t, x)$, we obtain the following:

$$u^*(t,x) = \exp[\pm i(\frac{\lambda}{\nu})^{1/2} x]\ u^*(t,x_0) \quad . \tag{30}$$

On the other hand, if we use Eq. (15), we obtain another variation δF, an Euler-Lagrange equation, and its solution, corresponding to Eqs. (23), (24) and (25), as follows:

$$\delta\ F = \int_a^b \{\ \delta\ u\ (\frac{\partial u}{\partial t} + 2\lambda\ u)\ \} dx\,. \tag{31}$$

$$\frac{\partial u}{\partial t} + 2\lambda\ u = 0\,. \tag{32}$$

$$u^*(t,x) = \exp(-2\lambda t)\ u^*(t_0,x) \quad . \tag{33}$$

Combining Eqs. (30) and (33), we derive the final analytical self-organized solution as follows:

$$u^*(t,x) = \exp[-2\lambda t \pm i(\frac{\lambda}{\nu})^{1/2} x]\ u^*(t_0,x_0)\,. \tag{34}$$

The meaning of this analytical solution is that after nonlinear dissipative relaxation that is associated with the dominant operator changing from the nonlinear term to the dissipative term, the final self-organized state comes to have a sinusoidal profile without propagation, with only the amplitude of the profile decreasing in time. The analytical solution Eq. (34) coincides very well with the simulations reported in [12], and therefore we have demonstrated that the present new theory of self-organization is very useful for various dynamical systems.

4-2. Compressible resistive and viscid MHD fusion plasmas

Here, we present self-organized states in compressible, resistive, and viscid MHD plasmas described by the following generalized Navier-Stokes equation, Faraday's law, and the energy conservation law:

$$\rho_m \frac{\partial \mathbf{u}}{\partial t} - \rho_m (\mathbf{u}\cdot\nabla)\ \mathbf{u} + \rho\ \mathbf{E} + \mathbf{j}\times\mathbf{B} - \nabla p + \nu\ \nabla^2\mathbf{u} + \frac{\nu}{3}\nabla(\nabla\cdot\mathbf{u}) \tag{35}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times[\ \mathbf{u}\times\mathbf{B} - \eta\mathbf{j}\ \ -\frac{1}{en}(\mathbf{j}\times\mathbf{B} - \nabla p) - \frac{m_e}{e^2 n}\frac{\partial \mathbf{j}}{\partial t}\ ] \tag{36}$$

$$\frac{\partial}{\partial t}(\frac{p}{\gamma-1}) = -\frac{1}{(\gamma-1)}[(\mathbf{u}\cdot\nabla)p + \gamma p(\nabla\cdot\mathbf{u})] + [\mathbf{j}\cdot(\mathbf{E} + \mathbf{u}\times\mathbf{B}) + \nabla\cdot(\kappa\nabla T)]\ , \tag{37}$$

where ρ_m is mass density of the plasma, and the generalized Ohm's law is used in Eq. (1) to obtain Eq. (36). Since $\tau_c/\Delta t = 1$ can be used under the assumption that self-organized states have negligibly small fluctuations and displacement current compared to the other terms for fusion plasmas, the three first variations δF_u for Eq. (35), δF_B for Eq. (36) and δF_p for Eq. (37) are written as follows, using Eq. (17):

$$\begin{aligned}\delta F_{\mathrm{u}} = \int_V & \{\delta\mathbf{u}\cdot[-\frac{\rho_{\mathrm{m}}}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) + [\nabla\cdot(\rho_{\mathrm{m}}\mathbf{u})]\mathbf{u} + \rho\eta\mathbf{j} + (1+\frac{\rho}{en})[\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} - \nabla p] \\ & - 2\{\nabla\times(\nu\nabla\times\mathbf{u}) + (\nabla\cdot\mathbf{u})\nabla\nu\} + \frac{m_{\mathrm{e}}\rho}{\mathrm{e}^2 n}\frac{\partial\mathbf{j}}{\partial t} + 2\lambda_{\mathrm{u}}\mathbf{u}] + \delta\mathbf{B}\cdot\{\frac{1}{\mu_0}[\mathbf{u}\times(\nabla\times\mathbf{B})] - \frac{1}{\mu_0}\nabla\times(\mathbf{u}\times\mathbf{B}) \\ & + \nabla\times(\rho\eta\mathbf{u}) - \nabla\times(\frac{\rho}{en}\mathbf{u}\times\mathbf{B}) + (\frac{\rho}{en}\mathbf{u}\times\mathbf{j})\} + \delta p\ [\nabla\cdot(\frac{\rho}{en}\mathbf{u}) + (\nabla\cdot\mathbf{u})]\}\mathrm{d}V. \end{aligned} \tag{38}$$

$$\begin{aligned}\delta F_{\mathrm{B}} = \int_V & \frac{1}{\mu_0}\{\ \delta\ \mathbf{B}\cdot[\nabla\times(\mathbf{u}\times\mathbf{B}) - 2\nabla\times(\frac{\eta}{\mu_0}\nabla\times\mathbf{B}) - \nabla\times(\frac{1}{\mu_0 en}[(\nabla\times\mathbf{B})\times\mathbf{B}]) - \nabla\times\frac{m_{\mathrm{e}}}{\mathrm{e}^2 n}\frac{\partial\mathbf{j}}{\partial t} \\ & + (\nabla\times\mathbf{B})\times\ \mathbf{u} + \nabla\times[(\nabla\times\mathbf{B})\times(\frac{1}{\mu_0 en}\mathbf{B})] + 2\lambda_{\mathrm{B}}\mathbf{B}] - \delta\ \mathbf{u}\cdot[(\nabla\times\mathbf{B})\times\mathbf{B}]\}\mathrm{d}V. \end{aligned} \tag{39}$$

$$\begin{aligned}\delta F_{\mathrm{p}} = \int_V & [\frac{\delta p}{\gamma-1}\{-\frac{1}{(\gamma-1)}[(\mathbf{u}\cdot\nabla)p + 2\gamma\ p\ (\nabla\cdot\mathbf{u})] + \eta(\frac{\nabla\times\mathbf{B}}{\mu_0})^2 + \nabla\cdot(\kappa\ \nabla T) + \nabla\cdot(p\ \mathbf{u}) \\ & - (\frac{\nabla\times\mathbf{B}}{\mu_0})\cdot\nabla(\frac{p}{en}) + \frac{m_{\mathrm{e}}}{2\mathrm{e}^2 n}\frac{\partial j^2}{\partial t} + 2\lambda_{\mathrm{p}}\frac{p}{\gamma-1}\} + \delta\mathbf{u}\cdot[(2\gamma-1)\frac{p}{\gamma-1}\nabla(\frac{p}{\gamma-1})] \\ & + \frac{\delta\mathbf{B}}{\mu_0}\cdot[2\eta\nabla(\frac{p}{\gamma-1})\times(\frac{\nabla\times\mathbf{B}}{\mu_0}) + \frac{2p}{\gamma-1}\nabla\eta\times(\frac{\nabla\times\mathbf{B}}{\mu_0}) - \frac{p}{(\gamma-1)en}\frac{\nabla T}{T}\times\nabla\mathrm{p}] \\ & + \frac{\partial}{\partial \mathrm{t}}\frac{\delta\mathbf{B}}{\mu_0}\cdot\nabla\times(\frac{p}{(\gamma-1)}\frac{m_{\mathrm{e}}}{\mathrm{e}^2 n}]\mathrm{d}V = 0. \end{aligned} \tag{40}$$

Note that Eq. (40) was divided by μ_0 to make Eqs. (38) - (40) have the same dimension of energy in M.K.S.A unit. All of Eqs. (38) - (40) are derived by using vector formulae, Gauss' theorem, the conservation equation for mass, and also the boundary conditions of $\delta\boldsymbol{u} = 0$, $\delta\boldsymbol{j} = 0$ and $\delta\boldsymbol{B} = 0$ at the confinement wall. We note here that $\partial(\delta\boldsymbol{B})/\partial t = 0$ because of $\delta\boldsymbol{B} = \delta\boldsymbol{B}(\boldsymbol{r})$. Adding from Eq. (39) to Eq. (40), we obtain the following :

$$\delta F_{\mathrm{u}} + \delta F_{\mathrm{B}} + \delta F_{\mathrm{p}}$$
$$= \int_V \{\delta\mathbf{u}\cdot[-\frac{\rho_{\mathrm{m}}}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) + [\nabla\cdot(\rho_{\mathrm{m}}\mathbf{u})]\mathbf{u} + \rho\eta\mathbf{j} + \nabla\times(\rho\eta\mathbf{u}) - \nabla\times(\frac{\rho}{en}\mathbf{u}\times\mathbf{B}) + (\frac{\rho}{en}\mathbf{u}\times\mathbf{j})$$
$$+ \frac{\rho}{\mu_0 en}(\nabla\times\mathbf{B})\times\mathbf{B} - (1+\frac{\rho}{en})\nabla p - 2\{\nabla\times(\nu\nabla\times\mathbf{u}) + (\nabla\cdot\mathbf{u})\nabla\nu\} + (2\gamma-1)\frac{p}{\gamma-1}\nabla(\frac{p}{\gamma-1})$$
$$+ \frac{m_{\mathrm{e}}\rho}{\mathrm{e}^2 n}\frac{\partial\mathbf{j}}{\partial t} + 2\lambda_{\mathrm{u}}\mathbf{u}] + \delta\mathbf{B}\cdot\frac{1}{\mu_0}[\nabla\times(\mathbf{u}\times\mathbf{B}) - 2\nabla\times(\frac{\eta}{\mu_0}\nabla\times\mathbf{B}) - \nabla\times\frac{m_{\mathrm{e}}}{\mathrm{e}^2 n}\frac{\partial\mathbf{j}}{\partial t}$$
$$+ 2\eta\nabla(\frac{p}{\gamma-1})\times(\frac{\nabla\times\mathbf{B}}{\mu_0}) + \frac{2p}{\gamma-1}\nabla\eta\times(\frac{\nabla\times\mathbf{B}}{\mu_0}) + (\nabla\times\mathbf{B})\times\mathbf{u} - \frac{p}{(\gamma-1)en}\frac{\nabla T}{T}\times\nabla\mathrm{p}] + 2\lambda_{\mathrm{B}}\mathbf{B}]\}$$
$$\frac{\delta p}{\gamma-1}[-\frac{1}{(\gamma-1)}[(\mathbf{u}\cdot\nabla)p + 2\gamma p(\nabla\cdot\mathbf{u})] + \eta(\frac{\nabla\times\mathbf{B}}{\mu_0})^2 + \nabla\cdot(\kappa\nabla T) + \nabla\cdot(p\mathbf{u}) + (\gamma-1)(\nabla\cdot\mathbf{u})$$
$$+ (\gamma-1)[\nabla\cdot(\frac{\rho}{en}\mathbf{u}) + (\nabla\cdot\mathbf{u})] - (\frac{\nabla\times\mathbf{B}}{\mu_0})\cdot\nabla(\frac{p}{en}) + \frac{m_{\mathrm{e}}}{2\mathrm{e}^2 n}\frac{\partial j^2}{\partial t} + 2\lambda_{\mathrm{p}}\frac{p}{\gamma-1}]\}\mathrm{d}V = 0, \qquad (41)$$

From Eq. (41), we obtain three necessary conditions for self-organized states given by Euler-Lagrange equations for arbitrary variations of $\delta\boldsymbol{u}$, $\delta\boldsymbol{B}$, and δp as follows:

$$2\{\nabla\times(\nu\nabla\times\mathbf{u}) - (\nabla\cdot\mathbf{u})\nabla\nu\}\frac{\rho_{\mathrm{m}}}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) - [\nabla\cdot(\rho_{\mathrm{m}}\mathbf{u})]\mathbf{u} + \rho\eta\mathbf{j} - \nabla\times(\rho\eta\mathbf{u})$$
$$+ \nabla\times(\frac{\rho}{en}\mathbf{u}\times\mathbf{B}) - (\frac{\rho}{\mu_0 en}\mathbf{u}\times(\nabla\times\mathbf{B})) - \frac{\rho}{\mu_0 en}(\nabla\times\mathbf{B})\times\mathbf{B}$$
$$+ (1+\frac{\rho}{en})\nabla p - (2\gamma-1)\frac{p}{\gamma-1}\nabla(\frac{p}{\gamma-1}) - \frac{m_{\mathrm{e}}\rho}{\mathrm{e}^2 n}\frac{\partial\mathbf{j}}{\partial t} = 2\lambda_{\mathrm{u}}\mathbf{u} \qquad (42)$$

$$2\nabla\times(\frac{\eta}{\mu_0}\nabla\times\mathbf{B}) - \nabla\times(\mathbf{u}\times\mathbf{B}) + \nabla\times\frac{m_{\mathrm{e}}}{\mathrm{e}^2 n}\frac{\partial\mathbf{j}}{\partial t} - 2\eta\nabla(\frac{p}{\gamma-1})\times(\frac{\nabla\times\mathbf{B}}{\mu_0}) - \frac{2p}{\gamma-1}\nabla\eta\times(\frac{\nabla\times\mathbf{B}}{\mu_0})$$
$$-(\nabla\times\mathbf{B})\times\mathbf{u} + \frac{p}{(\gamma-1)en}\frac{\nabla T}{T}\times\nabla\mathrm{p}] = 2\lambda_{\mathrm{B}}\mathbf{B} \qquad (43)$$

$$\frac{1}{(\gamma-1)}[(\mathbf{u}\cdot\nabla)p + 2\gamma p(\nabla\cdot\mathbf{u})] - \eta(\frac{\nabla\times\mathbf{B}}{\mu_0})^2 - \nabla\cdot(\kappa\nabla T) - \nabla\cdot(p\mathbf{u}) - (\gamma-1)(\nabla\cdot\mathbf{u})$$
$$-(\gamma-1)[\nabla\cdot(\frac{\rho}{en}\mathbf{u}) + (\nabla\cdot\mathbf{u})] + (\frac{\nabla\times\mathbf{B}}{\mu_0})\cdot\nabla(\frac{p}{en}) - \frac{m_{\mathrm{e}}}{2\mathrm{e}^2 n}\frac{\partial j^2}{\partial t} = 2\lambda_{\mathrm{p}}\frac{p}{\gamma-1} \qquad (44)$$

We can easily deduce the Taylor state $\nabla\times\boldsymbol{B} = \lambda\boldsymbol{B}$ as the limiting case in which the self-organized state is assumed to have no fluid flow, no pressure and no charge, i.e., $\boldsymbol{u}=0$, $p=0$, $\rho=0$ and to be spatially uniform and constant ν and η. In this special case, Eqs. (42) and (43) become simply $\nabla\times\nabla\times\boldsymbol{B}=\lambda^2\boldsymbol{B}$, which includes the Taylor state $\nabla\times\boldsymbol{B}=\lambda\boldsymbol{B}$, i.e., the Beltrami flow field, as a partial solution in a wider set of solutions. In other words, we can derive the Taylor state $\nabla\times\boldsymbol{B}=\lambda\boldsymbol{B}$ without using the concept of the magnetic helicity or the helicity of the generalized vorticity [16, 17], but rather as a limiting case from the present general theory, which includes the original dynamical equations characterizing the time evolution of the dynamical system. In general cases, we see from Eqs.(42)–(44) that self-organized states are balanced profiles among pressure, fluid flow, charge separation, temperature gradient depending on the profiles of ν and η. It is important to note here that the present theory leads to reasonable self-organized states with time change terms as seen in from Eqs.(42)–(44), i.e., the present dissipative dynamical system should definitely repeat relaxation process leading to self-organized configurations which would slowly change, because of dissipative terms, and therefore reach unstable profiles. We can find the self-organized configurations of all physical quantities of interest by solving the three Euler-Lagrange equations of Eqs.(42)–(44) under given shape of confinement wall and boundary conditions. When we assume spatially uniform and constant ν and η, we can obtain simpler self-organized states from Eqs.(42) and (43) which are characterized by two Beltrami equations of $\nabla\times\boldsymbol{u}=\alpha_0\boldsymbol{u}$ and $\nabla\times\boldsymbol{B}=\lambda\boldsymbol{B}$, as reported in [17]. We emphasize again that theories that use the concept of helicity invariance, such as reported in [1, 16, 17], are neither a variational principle [14] nor an energy principle [15], which would usually lead to dynamical equations for the time evolution of the dynamical system of interest. Those theories are mathematical formulations based on variational calculus. They use the conjecture that topological quantities related to the helicity are constant and invariant in ideal plasmas, even though this expectation is not guaranteed, as was proved in Sec. 2. Also, they conjecture that these topological quantities decrease more slowly than energies in slightly non-ideal plasmas and, postulating the global constraints to be the topological quantities, find a group of solutions with minimum energy compared with other solutions that have the same topological quantities. We have shown, however, that the topological quantities themselves can be determined after the solutions are selected in reality.

5. Concluding remarks

We have proved in Sec. 2 that the value of the magnetic helicity may increase or decrease, even in an ideal MHD plasma, and that therefore the global constraint using helicity has no power to limit the relaxation process in plasmas. In Sec. 3, we have established a novel general theory to find self-organized states, which includes all dynamical laws given by the nonlinear simultaneous general dynamic equations, Eq. (1), because those equations are embedded in the present formulation. Self-organized states are stable, because other states will have higher rates of change in their configurations.

We have presented a typical application of the present theory to dissipative KdV soliton propagation and shown that our theory agrees very well with corresponding computer simulations.

We have applied our theory to compressible, resistive, and viscid MHD plasmas described by the generalized Navier-Stokes' equation, Faraday's law, and the energy conservation law. It has been clarified generally that the dissipative dynamical system should definitely repeat the relaxation process leading to self-organized configurations which would slowly change, because of dissipative terms, and therefore reach unstable profiles.

We have shown that the Taylor state can be derived as a limiting case from the present theory without any concept of topological quantities. A similar conclusion had been reported earlier, albeit in a more simple form [9]. It would be worthwhile to mention that this general theory may be applicable not only to a plasma modeled as a multi-fluid, but also to the Boltzmann equation for distribution functions.

Acknowledgement

The authors would like to appreciate Associate Prof. Y. Tomita, Prof. T. Hayashi, Prof. T. Sato, at NIFS, and Prof. S. Shiina at Nihon University, Japan for their valuable comments. This work has been done under the collaborating study programs at NIFS through discussion with Associate Prof. Y. Tomita and Prof. T. Hayashi at Theory and Computer Simulation Center in NIFS.

[1] Permanent address: Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas 78712, USA.

References

[1] J. B. Taylor: Phys. Rev. Lett. **33** (1974) 1139.

[2] H. A. B. Bodin and A. A. Newton: Nucl. Fusion **20** (1980) 1255

[3] Y. Kondoh: Nucl. Fusion **21** (1981) 1607.

[4] Y. Kondoh: Nucl. Fusion **22** (1982) 1372.

[5] Y. Kondoh: J. Phys. Soc. Jpn. **54** (1985) 1813.

[6] Y. Kondoh, T. Yamagishi and M. S. Chu, Nucl. Fusion **27** (1987) 1473.

[7] Y. Kondoh: J. Phys. Soc. Jpn. **58** (1989) 489.

[8] Y. Kondoh, Phys. Rev. E **48** (1993) 2975.

[9] Y. Kondoh, Phys. Rev. E **49** (1994) 5546.

[10] Y. Kondoh, Y. Hosaka, J. Liang, R. Horiuchi and T. Sato, J. Phys. Soc. Jpn. **63** (1994) 546.

[11] Y. Kondoh and J. W. Van Dam, Phys. Rev. E **52** (1995) 1721.

[12] Y. Kondoh, M. Yoshizawa, A. Nakano and T. Yabe, Phys. Rev. E **54** (1996) 3017.

[13] S. Chandrasekhar and L. Woltijer, Proc. Natl. Acad. Sci. (U.S.A.) **44** (1958) 285

[14] H. Goldstein, *Classical Mechanics, Sec.II*, Addison-Wesley Publishing Com.., Mass.,U.S.A.(1957).

[15] I. B. Bernstein, E. A. Frieman, M. D. Kruskal and R. M. Kulsrud, Proc. Roy. Soc. **A244** (1958) 17.

[16] L. C. Steinhauer and A. Ishida, Phys. Rev. Lett. **79** (1997) 4323

[17] S. M. Mahajan and Z. Yoshida, Phys. Rev. Lett. **81** (1998) 4863.

Recent Issues of NIFS Series

NIFS-708 N Ishihara and S Kida
Dynamo Mechanism in a Rotating Spherical Shell Competition between Magnetic Field and Convection Vortices
July 2001

NIFS-709 LHD Experimental Group
Contributions to 28th European Physical Society Conference on Controlled Fusion and Plasma Physics (Madeira Tecnopolo, Funchal, Portugal, 18-22 June 2001) from LHD Experiment
July 2001

NIFS-710 V Yu Sergeev, R K Janev, M J Rakovic, S Zou, N Tamura, K V Khlopenkov and S. Sudo
Optimization of the Visible CXRS Measurements of TESPEL Diagnostics in LHD
Aug 2001

NIFS-711 M Bacal, M Nishiura, M Sasao, M Wada, M. Hamabe and H Yamaoka
Effect of Argon Additive in Negative Hydrogen Ion Sources
Aug 2001

NIFS-712 K Saito, R Kumazawa, T Mutoh, T Seki, T Watari, T Yamamoto, Y Torii, N Takeuchi C Zhang, Y Zhao, A Fukuyama, F Shimpo, G Nomura, M Yokota, A. Kato, M. Sasao, M Isobe, A V. Krasilnikov, T. Ozaki, M Osakabe, K Narihara, Y Nagayama, S Inagaki, K Itoh, T Ido, S Morita, K Ohkubo, M Sato, S Kubo, T Shimozuma, H. Idei, Y. Yoshimura, T. Notake, O Kaneko, Y Takeiri, Y. Oka, K Tsumori, K Ikeda, A Komori, H. Yamada, H. Funaba, K Y. Watanabe, S. Sakakibara, R. Sakamoto, J Miyazawa, K. Tanaka, B.J. Peterson, N. Ashikawa, S. Murakami, T. Minami, M. Shoji, S Ohdachi, S Yamamoto, H. Suzuki, K Kawahata, M. Emoto, H Nakanishi, N Inoue, N Ohyabu, Y Nakamura, S. Masuzaki, S Muto, K Sato, T Morisaki, M. Yokoyama, T Watanabe, M Goto, I Yamada, K Ida, T Tokuzawa, N Noda, K Toi, S Yamaguchi, K. Akaishi, A Sagara, K Nishimura, K. Yamazaki, S. Sudo, Y. Hamada, O. Motojima, M. Fujiwara
A Study of High-Energy Ions Produced by ICRF Heating in LHD
Sep. 2001

NIFS-713 Y. Matsumoto, S.-I. Oikawa and T Watanabe
Field Line and Particle Orbit Analysis in the Periphery of the Large Helical Device
Sep. 2001

NIFS-714 S. Toda, M. Kawasaki, N. Kasuya, K Itoh, Y Takase, A Furuya, M. Yagi and S. -I. Itoh
Contributions to the 8th IAEA Technical Committee Meeting on H-Mode Physics and Transport Barriers (5-7 September 2001, Toki, Japan)
Oct 2001

NIFS-715 A. Maluckov, N. Nakajima, M. Okamoto, S. Murakami and R Kanno
Statistical Properties of the Particle Radial Diffusion in a Radially Bounded Irregular Magnetic Field
Oct. 2001

NIFS-716 Boris V. Kuteev
Kinetic Depletion Model for Pellet Ablation
Nov. 2001

NIFS-717 Boris V. Kuteev and Lev D. Tsendin
Analytical Model of Neutral Gas Shielding for Hydrogen Pellet Ablation
Nov. 2001

NIFS-718 Boris V Kuteev
Interaction of Cover and Target with Xenon Gas in the IFE-Reaction Chamber
Nov. 2001

NIFS-719 A. Yoshizawa, N Yokoi, S -I Itoh and K. Itoh
Mean-Field Theory and Self-Consistent Dynamo Modeling
Dec. 2001

NIFS-720 V.N Tsytovich and K. Watanabe
Universal Instability of Dust Ion-Sound Waves and Dust-Acoustic Waves
Jan. 2002

NIFS-721 V N. Tsytovich
Collective Plasma Corrections to Thermonuclear Reactions Rates in Dense Plasmas
Jan. 2002

NIFS-722 S. Toda and K. Itoh
Phase Diagram of Structure of Radial Electric Field in Helical Plasmas
Jan. 2002

NIFS-723 V D Pustovitov
Ideal and Conventional Feedback Systems for RWM Suppression
Jan. 2002

NIFS-724 T. Watanabe and H. Hojo
The Marginally Stable Pressure Profile and a Possibility toward High Beta Plasma Confinement in LHD
Feb 2002

NIFS-725 S.-I. Itoh, K Itoh, M. Yagi, M Kawasaki and A. Kitazawa
Transition in Multiple-scale lengths Turbulence in Plasmas
Feb. 2002

NIFS-726 S.-I. Itoh, A. Kitazawa, M Yagi and K. Itoh
Bifurcation and Phase Diagram of Turbulence Constituted from Three Different Scale-length Modes
Apr 2002

NIFS-727 M Nagata
Preliminary Experiment on the Negative Magneto-Resistance Effect in a Weakly Ionized Discharge Plasma
Apr 2002

NIFS-728 K Akaishi and M. Nakasuga
Calculation of Hydrogen Outgassing Rate of LHD by Recombination Limited Model
Apr 2002

NIFS-729 Y. Kondoh, T Takahashi and J W. Van Dam
Proof of Non-invariance of Magnetic Helicity in Ideal Plasmas and a General Theory of Self-organization for Open and Dissipative Dynamical Systems
Apr 2002