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Lagrangian neoclassical transport theory applied to the region near the magnetic axis

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Neoclassical transport theory around the magnetic axis of a tokamak is studied, in which relatively wide “potato” orbits play an important role in transport. Lagrangian formulation of transport theory, which has been investigated to reflect finiteness of guiding-center orbit widths to transport equations, is developed in order to analyze neoclassical transport near the axis for a low-collisionality plasma. The treatment of self-collision term in Lagrangian formulation is revised to retain momentum conservation property of it. With directly reflecting the orbital properties of all the types of orbits in calculation, the ion thermal conductivity around the axis is found to decrease than from that predicted by conventional neoclassical theory. This result supports recent numerical simulations which show the reduction of thermal conductivity near the magnetic axis.

Keywords: neoclassical transport, potato orbit, finite-width effect

I. INTRODUCTION

Recently, neoclassical transport in the core region of tokamaks again attracts much attention. It is well-known that there appear non-standard guiding-center orbits near the magnetic axis called “potato” orbits. Typical orbit width of potato particles is as large as \((q^2 \rho^2 R_0)^{1/3}\), where \(q\) is the safety factor, \(\rho\) is the Larmor radius, and \(R_0\) is the major radius, respectively. In recent tokamak experiments in reversed-shear configuration accompanied by the internal transport barrier (ITB), the measured ion thermal conductivities in the core region sometimes become lower than those predicted by a conventional neoclassical transport theory. In fact, the standard neoclassical transport theory constructed in the small-orbit-width (SOW) approximation is not applicable to the near-axis region, and the orbital properties of potato particles should be considered in analyzing transport in this region. Then, several transport theories have been presented to include the effect of potato particles and Monte Carlo simulations (so called the \(\delta f\)-method) have also been carried out to calculate the ion thermal conductivity \(\chi_i\) in the near-axis region. However, there exist differences in the resultant \(\chi_i\)'s depending on the model used in analytical calculations, and neoclassical transport theory in the near-axis region is not completed yet.

Neoclassical transport theory has usually been discussed in Eulerian representation. Then, the extension of the theory to the near-axis region has also been discussed in an Eulerian manner. However, to include orbital properties in the transport theory, Lagrangian formulation was found to be suitable for a collisionless (banana-regime) plasma. In this approach, transport phenomena are described by a reduced drift-kinetic equation in the space of three constants-of-motion (COM) along a collisionless particle orbit in a tokamak. The previous works proved that Lagrangian formulation can reproduce the results obtained from the standard Eulerian formulation built in the SOW limit.

The present article is the first application of Lagrangian formulation to the near-axis region in which the finite-orbit-width (FOW) effect becomes really important. To utilize Lagrangian transport theory, we improve the treatment of like-particle collision term in the formulation to retain the momentum conservation property. In contrast to the other calculations using some analytical approximations, our calculation reflects quantitatively the properties of all types of particles appearing near the magnetic axis. It is found that the ion thermal conductivity \(\chi_i\) obtained by Lagrangian transport theory becomes significantly lower than that predicted by conventional Eulerian theory. Our result supports the recent results of both Monte Carlo simulations and experiments in the core region.

In Sec. II, analysis of guiding-center orbit is reviewed, and the classification of orbit types in the COM space is presented. The reduced kinetic equation and collision operator in the COM space are derived in Sec. III, and the transport equation is obtained by solving the kinetic equation in Sec. IV. We also discuss how to compare the neoclassical flux between Lagrangian and Eulerian representations. Transport coefficients, especially the ion thermal conductivity in the near-axis region is calculated in Sec. V.
II. PARTICLE ORBIT NEAR THE MAGNETIC AXIS

As pointed out in recent works, the guiding-center orbit near the magnetic axis is not so simple as in conventional analysis, in which particle orbit has been classified as "passing" or "banana". Let us explain key points in analyzing orbit here.

Consider a guiding-center motion projected on the poloidal cross-section \((r, \theta)\) of a tokamak. The magnetic field strength is given as \(B = B_0 [1 - (r/R_0) \cos \theta] \) and \(q\)-value is assumed to be constant. The guiding-center velocity of a particle is expressed as \(v = v_{\parallel} b + v_{\perp} a \), where \(b = B_i/B\) and \(a\) is the drift velocity in the direction across the magnetic field lines. To investigate orbit topology, the poloidal angular velocity \(\theta\) is considered. In the low-\(\alpha\) approximation, it is given by

\[
\dot{\theta} = (v_{\parallel} b + v_{\perp} a) \cdot \nabla \theta \\
\approx \frac{1}{q R_0} \left[ -\frac{q}{r \Omega_0} \left( v_{\parallel}^2 + \frac{e^2}{2} \right) \cos \theta \right].
\]  

(1)

where \(\Omega_0 = e B_0 / R_0\). One can see that the contribution of the second term, which arises from \(v_{\perp} \cdot \nabla \theta\), is proportional to \(r^{-1}\) and then it becomes significant when a particle is approaching the magnetic axis. This fact means that orbit topology cannot be classified as in a usual way by counting only the turning points of \(v_{\parallel}\). We have shown that the proper way of classifying orbit is to count both the turning points of \(v_{\parallel} = v_{\parallel} / |v_{\parallel}|\) and \(\sigma_\theta = \theta / |\theta|\) along a particle orbit.

The criterion of classifying orbit types is shown in Table I, and examples of orbits are shown in Figs. 1 and 2. Here, we use some new names of orbit types which are characteristic of the near-axis region. Outer- and inner-circulating particles are localized on either side of the magnetic axis, though they do not change \(\sigma_\parallel\) like passing particles. Kidney orbits, \(^{18}\) encircle the axis, though they are trapped in the magnetic mirror like bananas. We distinguish concave-kidney orbit from kidney orbit, according to the turning points of \(\sigma_\theta\).

Such a detailed classification as above is needed to identify particle orbit in the \((E, \mu, \langle r \rangle)\) space, in which we will discuss Lagrangian formulation from the next section. Here, \(E\) is the energy, \(\mu\) is the magnetic moment, and \(\langle r \rangle\) is the radial position averaged over one poloidal period. The region of each orbit type in the \((\langle r \rangle, \lambda_0)\) plane for \(E = 10\) keV hydrogen ions is shown in Figs. 3 and 4. Here, \(\lambda_0 = \mu B_0 / E\) is the normalized magnetic moment. The circle in Fig. 3 corresponds to the fattest banana orbit \(B\) in Fig. 1, of which width is \(2(q^2 \rho^2 R^3)\)^{1/3} where \(\rho = \rho_0 / \Omega_0\). Potato orbits, of which typical width becomes \(\Delta_r \sim (q^2 \rho^2 R^3)^{1/3}\), correspond to those appearing around

\[
\langle \langle r \rangle, \lambda_0 \rangle \sim \left( (q^2 \rho^2 R^3) \right)^{1/3} (1 \pm (q \rho / R_0)^{1/3})
\]  

(2)

in Figs. 3 and 4. For the convenience of notation, we introduce a typical small radius

\[
r_p = 2(q^2 \rho^2 R^3)^{1/3},
\]  

(3)

where \(\rho_0 = \rho_{th}/\Omega_0\) is the Larmor radius of thermal ions. In this article, we use "potato particles" to describe those which appear \(0 < r < r_p\) with their orbit width being \(\Delta_r \sim r_p\).

One of the important features in Figs. 3 and 4 is that there are some overlaps in regions of orbit types around the solid-line part of the boundary \(/\). In overlapped regions, particle orbit cannot be identified only by the value \((E, \mu, \langle r \rangle; \sigma_\parallel)\). Therefore the criterion in Table I should be adopted to identify orbits.

As approaching \(/\), a banana (con cave-kidney) orbit bifurcates into a kidney and a counter-passing (inner-circulating) orbit as shown in Fig. 5. Such barely-transit particles are almost stagnated at \((r, \theta) = \langle \langle r \rangle, \pi \rangle\). We call the solid-line part of \(/\) "the transition boundary" hereafter. There is the other type of stagnated particles appear on the boundary \(\sim\). They are outer-circulating particles stagnated at \((r, \theta) = \langle \langle r \rangle, 0 \rangle\) and move only in the toroidal direction. Conventionally, such particles have been regarded as banana particles in the limit \(\rho = 0\), but in fact stagnation occurs when \(\dot{\theta}\) becomes zero on \(Z = 0\) plane, and stagnated particles have finite \(\dot{\rho}\). Note that on the dashed-line part of \(/\) (the left side from the triangle mark in Fig. 3), bifurcation of orbit types does not occur. This boundary corresponds to inner-circulating orbits with zero-width like outer-circulating particles at \(l\).

One advantage in using \((E, \mu, \langle r \rangle)\) or \((\langle v \rangle)\) as a set of COM variables is that, in the collisionless limit, the position-like variable \(\langle r \rangle\) changes continuously when crossing the transition boundary. On the other hand, \(\gamma_0\) used in Ref. 15, which is the minor-radius of one of two crossing points of orbit with \(Z = 0\) plane, changes discontinuously at the transition boundary. It is also practical to choose \(\langle r \rangle\) as the position-like variable in that \(\langle r \rangle\) is the most suitable value to represent the lowest-order approximation of the particle position.

Finiteness of orbit width appears on the region of each orbit type. In the zero-width limit, banana particles exist in the range \(1 - \langle r \rangle \leq \lambda_0 \leq 1 + \langle \dot{r} \rangle\), where \(\langle r \rangle / R_0\). In reality, however, this simple
analysis is not valid for the region $0 < \langle r \rangle < r_p$ in which potato orbits appear. Moreover, because of the finiteness of orbit width, there are no particle exist at $\langle r \rangle < r_p$. The existence of outer- and inner-circling particles can be found only if the finiteness of orbit width is considered, and they have not been treated in the conventional neoclassical transport theory, nor in the recent studies treating the near-axis region. However, it will affect transport around the axis because some of them have large orbit width $\Delta \sim r_p$.

In the forthcoming sections, we will derive Lagrangian formulation of neoclassical transport theory which can include the orbit properties near the magnetic axis discussed here.

III. KINETIC EQUATION IN LAGRANGIAN FORMULATION

A. Reduction of the kinetic equation

Consider an axisymmetric configuration. We use the magnetic coordinate system $(\psi, \theta, \zeta)$, where $\psi$, $\theta$, $\zeta$ is the poloidal flux, the poloidal angle, and the toroidal angle, respectively. The electromagnetic field is represented as $B = I \nabla \zeta + \nabla \zeta \times \nabla \psi$ and $E = -\nabla \Phi(\psi)$, where $I = RB$, and we assume that the field is time-independent. Three constants of motion in an axisymmetric configuration are

$$E = \frac{m_a e^2}{2} + e_a \Phi = \frac{m_a v^2}{2} + \mu B - e_0 \Phi,$$

$$\mu = \frac{m_a v_a^2}{2B},$$

$$P_\zeta = \nu - \frac{m_a a^2}{e_0} \cdot \nu \cdot \nabla \zeta,$$

where $v_a = v$, $v_\perp = |v - v_e| b$, and the subscript $a$ denotes particle species. The start point of Lagrangian formulation of neoclassical transport theory is the drift kinetic equation in an Eulerian representation in the $(x, E, \mu)$ space.

$$\frac{\partial}{\partial t} f_a(x, E, \mu, t) + x \cdot \frac{\partial f_a}{\partial x} = C_{ab},$$

where $\cdot = d/dt$, $x$ is the guiding-center position and $C_{ab}$ is a collision operator. Note that eq. (5) is independent of the gyrophase $\phi$. We change the independent variables in eq. (5) into three constants of motion in the collisionless limit $(z_1, z_2, z_3)$, and the other three variables $(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)$. One can choose an arbitrary set of independent variables $(z, \tilde{z})$. In this paper, we choose $(z_1, z_2, z_3) = (\theta, \zeta, \phi)$, while $z_1 = E$, $z_2 = \mu$, and $z_3 = \langle v \rangle$ instead of $P_\zeta$ $(\nu)$ represents the averaged radial position of a particle orbit. The orbit average operator for any function $a(z, \tilde{z})$ is defined as

$$\langle a \rangle \equiv \frac{1}{4\pi^2 \tau_p} \int d\theta d\zeta d\phi a(z, \tilde{z}),$$

where

$$\tau_p \equiv \int \frac{d\theta}{\theta},$$

is the poloidal period of an particle orbit. Note that the integral is carried out along one poloidal circuit of the particle orbit. Note also that we can use $z_3 = \langle r \rangle$ instead of $\langle v \rangle$ when it is convenient. By using the set of variables $(z, \tilde{z})$, eq. (5) is transformed into

$$\frac{\partial}{\partial t} f_a(z, \theta, t) + \theta \frac{\partial f_a}{\partial \theta} = C_{ab},$$

where the property $\partial/\partial \phi = \partial/\partial z_3 = 0$ is used.

We introduce here an ordering parameter $\delta_c$ as

$$\delta_c = \nu_c^{-1} \tau_p \ll 1,$$

where $\nu_c^{-1}$ is a typical collision frequency. This assumption corresponds to the condition that the plasma is in the collisionless regime, or the banana regime. In eq. (8), $\partial f_a/\partial \theta$ and $C_{ab}$ are assumed to be $O(\delta_c)$ so that the variables $(E, \mu, \langle v \rangle)$ can really be the constants of motion through the lowest order in $\delta_c$. Expanding $f_a$ with $\delta_c$, the lowest part becomes
\[ \frac{\partial f_0}{\partial \theta} = 0, \]  
\text{(10)}

where we omit the subscript \( a \). On the other hand, from the conservation of volume in the phase space, one has

\[ \frac{1}{J_z} \frac{\partial}{\partial \theta} \left( J_z \frac{dz}{dt} \right) + \frac{1}{J_z} \frac{\partial}{\partial \theta} \left( J_z \frac{d\bar{z}}{dt} \right) = 0, \]  
\text{(11)}

where \( J_z(z, \theta) \) is the Jacobian of the transform \( (x, v) \to (z, \bar{z}) \). Since \( dz/dt = 0 \) to \( O(\delta^2) \), one obtains

\[ J_z(0) = J_0(\mathcal{E}, \mu, \langle \psi \rangle), \]  
\text{(12)}

\[ f_0 = f_0(\mathcal{E}, \mu, \langle \psi \rangle). \]  
\text{(13)}

Next, by using eq. (11), \( O(\delta^2) \) part of eq. (8) can be written as

\[ \frac{\partial}{\partial \theta} f_0(z, t) + \frac{1}{J_z} \frac{\partial}{\partial \theta} \left( J_z \frac{df_0}{dt} f_0 \right) = \frac{1}{J_z} \frac{\partial}{\partial \theta} \left( J_z \frac{\partial z}{\partial \mathcal{V}} \mathbf{\Gamma}(f_0) \right), \]  
\text{(14)}

where the right hand side is derived from the fact that the collision term can be written in the divergence form in the velocity space \( \mathcal{C}(f) = \nabla_v \cdot \mathbf{\Gamma}(f) \). The last procedure is to take the orbit average of eq. (14). It yields

\[ \frac{\partial \tilde{f}}{\partial t} = \frac{1}{J_z} \frac{\partial}{\partial \theta} \left( J_z \langle \frac{\partial z}{\partial \mathcal{V}} \mathbf{\Gamma}(\tilde{f}) \rangle \right) = \tilde{\mathcal{C}}, \]  
\text{(15)}

where \( \tilde{f} = f_0 \) is used to emphasize that \( \tilde{f} \) is a function of \( (\mathcal{E}, \mu, \langle \psi \rangle, t) \), and

\[ J_z(\mathcal{E}, \mu, \langle \psi \rangle) \equiv 4\pi^2 J_0 \tau_p \]  
\text{(16)}

is the Jacobian in the \( (\mathcal{E}, \mu, \langle \psi \rangle) \) space. The collision term is also averaged over a particle orbit. Thus we obtain a reduced drift-kinetic equation in the \( (\mathcal{E}, \mu, \langle \psi \rangle) \) space.

**B. Jacobian**

Here, we derive the explicit form of the Jacobian \( J_z \) defined in eq. (16). First, consider the transform from Cartesian coordinate system \((x, v)\) to the guiding-center variables \((\psi, \theta, \zeta, \mathcal{E}, \mu, \phi)\). The Jacobian of the transform is\(^{30}\)

\[ J = \frac{1}{(\mathbf{B} \cdot \nabla \theta)} \frac{B_*}{m^2 |v||}. \]  
\text{(17)}

where \( B_* = B[1 + (\gamma/m\Omega) \mathbf{b} \cdot \nabla \times \mathbf{b}] \). By changing \( \psi \) to its orbit averaged value \( \langle \psi \rangle \), we obtain the set of variables \((z, \bar{z}) = (\mathcal{E}, \mu, \langle \psi \rangle, \theta, \zeta, \phi)\). Therefore, the Jacobian \( J_z \) can be written as \( J_z = J[\partial \psi/\partial \psi(\langle \psi \rangle)] \). To determine \( J_z \), we use the conservation of \( P_z \) (of its gyro-averaged form)

\[ P_z = \psi - \frac{I}{\Omega} v_\parallel = \text{const}. \]  
\text{(18)}

Taking orbit average of both sides, it becomes

\[ \psi - \frac{I}{\Omega} v_\parallel = \langle \psi \rangle - \left\langle \frac{I}{\Omega} v_\parallel \right\rangle. \]  
\text{(19)}

Differentiating both sides by \( \langle \psi \rangle \), we have

\[ \left[ 1 - \frac{\partial}{\partial \psi} \left\langle \frac{I}{\Omega} v_\parallel \right\rangle \right] \frac{\partial \psi}{\partial \langle \psi \rangle} = 1 - \frac{\partial}{\partial \langle \psi \rangle} \left\langle \frac{I}{\Omega} v_\parallel \right\rangle. \]  
\text{(20)}

Note that all partial derivatives in eq. (20) are taken with \( \mathcal{E}, \mu, \) and \( \theta \) being kept constant. Next, by using the equation of guiding-center motion,\(^{30}\) one obtains

\[ \dot{\theta} = v \cdot \nabla \theta = \frac{v_\parallel \mathbf{B} \cdot \nabla \theta}{B_*} \left[ 1 - \frac{\partial}{\partial \dot{\psi}} \left\langle \frac{I}{\Omega} v_\parallel \right\rangle \right], \]  
\text{(21)}
Then, combining eqs. (17), (20) and (21) yields

\[ J_z = \frac{1}{m^2|\theta|} |1 - \delta_*| . \]  

(22)

where

\[ \delta_* = \frac{\partial}{\partial (\psi)} \left\langle \frac{I}{\Omega} \right\rangle \right|_{\langle \psi \rangle, \theta^*} \]  

(23)

Thus one can confirm that \( J_z|\theta| = J_0 \) in eq. (12) is independent of \( \theta \). Finally, combining eqs. (16) and (22), we obtain the Jacobian in the \((E, \mu, \langle \psi \rangle)\) space

\[ J_z = \frac{4\pi^2}{m^2} \tau_p |1 - \delta_*| . \]  

(24)

In a numerical calculation, the poloidal period \( \tau_p \) can easily be determined. As concerns \( \delta_* \), it should be noted that from eq. (19),

\[ \left\langle \frac{I}{\Omega} \right\rangle = \left. \frac{I_{\|}}{\Omega} \right|_{\langle \psi \rangle, \theta^*} . \]  

(25)

where \((\psi, \theta) = (\langle \psi \rangle, \theta^*)\) is the position at which a particle crosses its averaged flux surface \( \psi = \langle \psi \rangle \). We call it “the averaging point” of an orbit. Then, eq. (24) is interpreted as

\[ \delta_* = \left( \frac{\partial}{\partial \psi} + \frac{\partial \theta^*}{\partial (\psi)} \frac{\partial}{\partial \theta} \right) \left. \frac{I_{\|}}{\Omega} \right|_{\langle \psi \rangle, \theta^*} . \]  

(26)

where \( \partial \theta^*/\partial (\psi) \) represents the displacement of the averaging point. Fortunately, however, we can estimate that \( 1 - \delta_* \approx 1 \) for almost all particles as shown in Appendix A. Though we retain the term \( 1 - \delta_* \) in the derivation of transport equation hereafter, it is approximated to be unity in the numerical calculations.

An important property of \( J_z \) arises from the factor \( \tau_p \) for particles which are stagnated on \( Z = 0 \) plane. Remember that there are two types of the stagnated orbit. One type is the outer-circulating orbit stagnated at \( \langle \psi, \theta = (\langle \psi \rangle, 0) \) (on the \( l1 \) boundary in Fig. 3), and the other is stagnated orbit at \( \langle \psi, \theta = (\langle \psi \rangle, \pi) \) (on the transition boundary \( l2 \)). Approaching the \( l1 \) boundary, orbits resemble a pendulum motion in the Z direction with a infinitesimal oscillation. Therefore, \( \tau_p \) remains finite on \( l1 \). On the other hand, \( \tau_p \to \infty \) when approaching the \( l2 \) boundary. Then, we have

\[ \lim_{\mu \to l1} J_z = \text{finite}, \]  

(27a)

\[ \lim_{\mu \to l2} J_z = \infty. \]  

(27b)

We have shown that, in the \((E, \mu, \langle \psi \rangle)\) space, there are some overlaps in regions of orbit types. Then, \( J_z(z) \) and \( f(z) \) are generally multi-valued functions of \( z \) depending on the orbit types. We introduce the sign \( \sigma_i \) to indicate the orbit type of each particle. The notation \( J_z(z) \) and \( f(z) \) implicitly mean that they also depend on \( \sigma_i \): \( J_z = J_z(z, \sigma_i) \), etc.

C. Collision operator

To obtain transport equations in the \((E, \mu, \langle \psi \rangle)\) space, we need to evaluate the change rate of COM by collisions. First, consider the collision term in Eulerian representation \(^{21}\)

\[ C_{ab} = K_{ab} \frac{\partial}{\partial \psi} \cdot \int d^3v' U(v - v') \cdot \left[ \frac{\partial f_b(v)}{\partial \psi} f_b(v') - \frac{m_a}{m_b} f_a(v) \frac{\partial f_b(v)}{\partial \psi} \right] . \]  

(28)

where

\[ U(v) = \frac{1}{v^1} - \frac{vv}{|v|^3} . \]

\[ K_{ab} = \frac{\tau^2 \tau^2}{8\pi^2 \tilde{m}_e} \ln \Lambda \frac{\tilde{m}_e}{\tilde{m}_e^2} . \]
Substituting eq. (28) into the averaging operator eq. (6), we obtain the exact description of the orbit-averaged collision term\textsuperscript{13}

\[
\tilde{C}_{ab} = \frac{1}{T_i} \frac{\partial}{\partial \mathbf{r}} \left[ J_i \left( A f_i(z) + D \cdot \frac{\partial}{\partial \mathbf{z}} \bar{f}_i(z) \right) \right].
\] (29)

where

\[
A = -K_{ab} \left\langle \frac{\partial \mathbf{z}}{\partial \mathbf{v}} \right\rangle \left\langle d^3 \mathbf{v}' \bar{f}_i(z') \frac{\partial}{\partial \mathbf{v}} \right\rangle.
\] (30a)

\[
D = K_{ab} \left\langle \frac{\partial \mathbf{z}}{\partial \mathbf{v}} \right\rangle \left\langle d^3 \mathbf{v}' \bar{f}_i(z') \frac{\partial}{\partial \mathbf{v}} \right\rangle.
\] (30b)

Note that $\mathbf{z}'$ in eq. (30) is a functional depending on $\theta$

\[
\mathbf{z}' = (\mathbf{E}', \mu', \psi') = (\mathbf{E}', \mu', \psi) + \bar{v}'(z, \mathbf{E}, \mu, \theta).
\]

where $\mathbf{z}'$ arises from the finiteness of particle orbit width. In this sense, the averaged collision term eq. (29) has a non-locality. Moreover, since eq. (29) is a integro-differential function, it is not suitable to solve analytically. Therefore, we need some approximation to handle collision operator in Lagrangian formulation.

From here on, we consider only the ion transport because the FOW effect near the magnetic axis is important for ions. We neglect the ion-electron term $C_{ei}$ since it is smaller than the ion-ion term $C_{ii}$ by a factor $\sqrt{m_i/m_e}$. An easy approximation for collision term is the Lorentz operator, which had been used in the fundamental study of Lagrangian formulation by Bernstein and Moliag,\textsuperscript{11} but it does not conserve momentum. It is well-known that the momentum conservation property of like-species collisions plays an important role in transport theory. Therefore, we use here a model collision operator which conserves the parallel momentum locally so that the transport equation may reproduce the result obtained from Eulerian formulation in the SOW limit.

The model collision operator is given in the following form\textsuperscript{22}

\[
C_i(f_i) = \nu_i \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{v}^2 - \mathbf{v} \mathbf{v}) \cdot \frac{\partial}{\partial \mathbf{v}} f_i + \nu_i \frac{m_i}{T_i} \frac{\partial n_i}{\partial \mathbf{v}} f_i f_i. \] (31)

where $n_i$ is a functional of $f_i$, and $f_i f_i$ is a local Maxwellian. Collision frequency $\nu_i$ is defined as

\[
\nu_i = \frac{3 \sqrt{\pi}}{4 \tau e^2 \mathbf{r}^2} \Psi(c_i).
\] (32)

where $c_i = v/v_i$, and

\[
\tau_i^{-1} = \frac{\nu_i Z_i^2 e^2}{3\pi \sqrt{\pi} \mathbf{m}_i^2 v_i \mathbf{e}_{hi}}.
\] (33a)

\[
g(c) = \left( 1 - \frac{1}{2c^2} \right) \Psi(c) + \frac{1}{2c} \Psi'(c).
\] (33b)

\[
\Psi(c) = \frac{2}{\sqrt{\pi}} \int_0^c dxe^{-x^2}.
\] (33c)

Here, $a_{i,j}$ is determined to conserve the parallel momentum

\[
\int d^3 c_i \mathbf{c} C_i(f_i) = 0.
\] (34)

Then, substituting eq. (31) into this equation yields

\[
a_{i,j} = \frac{\tau_i}{2n_i \mathbf{c}_{hi}} \int d^3 c_i \mathbf{c} \mathbf{c}_i f_i.
\] (35)

where

\[
\mathbf{c}_{hi} = \int_0^\infty \mathbf{c} e^{\mathbf{c}_h^2} \mathbf{c} \mathbf{y}(c).
\] (36)

It is convenient to rewrite eq. (31) in the divergence form, by noting $\partial f_i/\partial \mathbf{v} = -(n \mathbf{v}/T) f_i$.\n
6
\[ C_i(f_\nu) = \frac{\nu_i}{2} \frac{\partial}{\partial \nu} \left( \left( V(\nu) \frac{\partial}{\partial \nu} f_\nu - \frac{m_i u_{i\perp}}{T_i} \mathbf{w} f_{i,M} \right) \right). \]  

(37)

where \( V(\nu) = \nu^2 \mathbf{v} \cdot \mathbf{v} \) and \( \mathbf{w} = \nu^2 \mathbf{b} - \nu \mathbf{v} \). Since \( \mathbf{V} \cdot \mathbf{v} = 0 \) and \( \mathbf{w} \cdot \mathbf{v} = 0 \), the model collision operator eq. (37) also conserves particle number and energy.

Finally, by taking the orbit-average of eq. (37), we obtain the orbit-averaged model collision operator

\[ \bar{C}_i(f_\nu) = \frac{1}{T_i} \frac{\partial}{\partial \nu} \int_0^1 \frac{\nu_i}{2} \left( \frac{\partial f_\nu}{\partial \nu} \left( V(\nu) \frac{\partial f_\nu}{\partial \nu} - \frac{m_i u_{i\perp}}{T_i} \mathbf{w} f_{i,M} \right) \right). \]

(38)

We neglect here the variation of \( \nu_i \) along an ion orbit because, though typical ion orbit width becomes as large as \( \Delta \nu \sim \nu_i \), there, experiments show that density and temperature profiles near the axis are flat. Note also that we neglect the variation of \( \nu_i \) along a particle orbit in averaged collision terms. To ensure this approximation, if it assumed that

\[ E \gg |\Delta, \nu| \frac{d\Phi}{d|\nu|}. \]

(39)

and \( \nu_i \) is evaluated as

\[ \nu_i = \sqrt{\frac{2}{m_i} \left[ E - \nu_i \Phi(\nu_i) \right]} \]

(40)

In eq. (38), we need to evaluate \( \partial \mathbf{z}/\partial \nu \). It is immediately shown that

\[ \frac{\partial E}{\partial \nu} = m \mathbf{v}, \quad \frac{\partial \mu}{\partial \nu} = \frac{m \mathbf{v}}{B} \]

We also need the expression of \( \partial(\nu)/\partial \nu \). This factor is important in Lagrangian formulation, because it measures the rate of excursion in average radial position of a particle by scattering in the velocity space. \( \partial(\nu)/\partial \nu \) can be obtained by taking partial derivative on both sides of eq. (39). It gives

\[ \frac{\partial(\nu)}{\partial \nu} = -\frac{I}{\Omega} \mathbf{b} + \frac{\partial}{\partial \nu} \left( \frac{I_{\nu\perp}}{\Omega} \right) \]

\[ = -\frac{I}{\Omega} \mathbf{b} + \left( \frac{\partial E}{\partial \nu} \frac{\partial}{\partial E} + \frac{\partial \mu}{\partial \nu} \frac{\partial}{\partial \mu} + \frac{\partial(\nu)}{\partial \nu} \frac{\partial}{\partial(\nu)} \right) \left( \frac{I_{\nu\perp}}{\Omega} \right). \]

(41)

In a previous work, this factor is treated approximately as a step-function

\[ \frac{\partial(\nu)}{\partial \nu} = \begin{cases} -\frac{I}{\Omega} \mathbf{b} & \text{(Banana),} \\ 0 & \text{(Passing)}. \end{cases} \]

(42)

in the SOW limit. However, the estimation above needs some explanations. Note that \( \partial(\nu)/\partial \nu \) can be decomposed as \( a_1 \mathbf{v} + a_2 \mathbf{b} \). Since \( \mathbf{V} \cdot \mathbf{v} = \mathbf{w} \cdot \mathbf{v} = 0 \), we need to retain only the \( \mathbf{b} \) component of the derivative. Noting eqs. (23) and (25), we obtain

\[ (1 - \delta_\nu) \frac{\partial(\nu)}{\partial \nu} = -\frac{I}{\Omega} (1 - \Delta) \mathbf{b}. \]

(43)

where

\[ \Delta_\nu = -\frac{ev_i}{I} \frac{\partial}{\partial \mu} \left( \frac{I_{\nu\perp}}{\Omega} \right) \]

(44)

For the convenience of notation, we introduce a factor \( \gamma \) as follows

\[ \frac{\partial(\nu)}{\partial \nu} = -\gamma (z, \theta, \nu, \frac{I}{\Omega}) \mathbf{b} \]

(45)

Later, it is shown that we do not need the explicit form of \( \Delta_\nu \) to calculate transport coefficients as pointed out in Ref. 14, and the estimation in eq. (42) works well in the SOW limit.
IV. NEOCLASSICAL FLUXES

A. Derivation of transport equation

We now expand the reduced kinetic equation (15) by a small ordering parameter

$$\delta_0 \equiv \frac{\Delta_0}{L} \ll 1,$$  \hspace{1cm} (46)

where $\Delta_0$ is a typical orbit width and $L$ is a typical gradient scale length of plasma pressure. Though $\Delta_0$ for ions becomes large in the near-axis region, the condition (46) can be satisfied since $L$ also becomes large there.

The orderings we put in eq. (15) is as follows. $\delta \dot{f}/\delta t$ is assumed to be $O(\delta_0^3)$ as is often called “transport-ordering”. For the parallel flow, we put a plausible ordering that $u_{1||}/v_{thi} \sim O(\delta_0)$. Concerning partial derivatives $\partial/\partial z$, $\partial/\partial E$ and $\partial/\partial \mu$ are treated as $O(\delta_0^2)$, while $\partial / \partial \langle \psi \rangle \sim O(\delta_0^1)$. Collision operator is then expanded in $\delta_0$ as $\tilde{C} = \tilde{C}^{(0)} + \delta_0 \tilde{C}^{(1)} + \delta_0^2 \tilde{C}^{(2)} + \cdots$. With the expansion $\tilde{f} = \tilde{f}_0 + \delta_0 \tilde{f}_1 + \delta_0^2 \tilde{f}_2 + \cdots$, the $O(\delta_0)$ part of eq. (15) for ion becomes

$$\tilde{C}_t^{(0)}(\tilde{f}_0) = \frac{1}{J_t} \frac{\partial}{\partial \mu} \frac{J_t \nu_t}{2} \left( \frac{\partial \mu}{\partial \nu} \cdot \mathbf{V} + \frac{\partial \mu}{\partial \mathbf{E}} \cdot \frac{\partial \tilde{f}_0}{\partial \mu} \right) = 0. \hspace{1cm} (47)$$

Because $(\partial \mathbf{E}/\partial \nu) \cdot \mathbf{V} = 0$, only $\mu$-derivative appears in eq. (47). Then, any distribution function $\tilde{f}_0$ independent of $\mu$ is the solution of this equation. However, we adopt here the averaged collision operator of its exact form shown in eq. (29) for $\tilde{C}_t^{(0)}$. Then the solution of $\tilde{f}_0$ becomes the local Maxwellian

$$\tilde{f}_0 = \bar{n}_i \left( \frac{m_i}{2\pi T_i} \right)^{3/2} \exp \left[ -\frac{\mathbf{E} - e_i \Phi}{T_i} \right]. \hspace{1cm} (48)$$

where $\bar{n}_i$, $T_i$, and $\Phi$ are defined as functions of $\langle \psi \rangle$.

The definition of $\bar{n}_i$ is not equal to the flux-surface averaged density $n_i(\langle \psi \rangle)$. First, consider the particle number per unit $\langle \psi \rangle$ as follows

$$\mathcal{N}_i(\langle \psi \rangle) = \sum_{\sigma_i} \int d\mathbf{E} d\mu J_{\sigma_i} \tilde{f}_0(\mathbf{E}, \mu, \langle \psi \rangle; \sigma_i). \hspace{1cm} (49)$$

Then, $\bar{n}_i(\langle \psi \rangle)$ is defined as

$$\bar{n}_i = \frac{d\mathcal{N}_i}{dV} \bigg|_{\langle \psi \rangle} \mathcal{N}_i(\langle \psi \rangle) \lambda_n(\langle \psi \rangle), \hspace{1cm} (50)$$

where $V$ is the volume enclosed by a $\psi = \text{const}$ surface. The numerical factor $\lambda_n$ is given by

$$\lambda_n(\langle \psi \rangle) = \frac{dV}{dV} \left( \frac{2\pi T_i}{m_i} \right)^{3/2} \left( \sum_{\sigma_i} \int d\mathbf{E} d\mu J_{\sigma} \exp \left[ -\frac{\mathbf{E} - e_i \Phi}{T_i} \right] \right)^{-1}. \hspace{1cm} (51)$$

If $\langle \psi \rangle$ is away from the magnetic axis and the orbit width is narrow, $\lambda_n \rightarrow 1$ and $\bar{n}_i(\langle \psi \rangle) \approx n_i(\langle \psi \rangle)$. When approaching $\langle \psi \rangle \rightarrow 0$, however, $\lambda_n$ becomes large because the integral region in the $(\mathbf{E}, \mu)$ plane is small there. Note that $\mathcal{N}_i$ is nearly proportional to $\lambda_n^{-1}$ near the axis, assuming that the density profile in the real space is flat in the range $r < r_p$. This assumption is valid when considering the core region with ITB. Then, it is a simple and plausible assumption that $\bar{n}_i(\langle \psi \rangle) \approx n_i(\langle \psi \rangle)$, even in the near-axis region.

Before proceeding to the $O(\delta_0^2)$ equation, let us consider the order expansion of the momentum-restoring term in eq. (38). Since $u_{1||}$ and $f_{1||}$ vary along a particle orbit, we expand it as

$$\frac{u_{1||} f_{1||}}{T_i} \bigg|_{\langle \psi \rangle} \equiv \frac{f_{1||}}{B_i T_i(\langle \psi \rangle)} + \Delta_3(\mathbf{z}, \langle \psi \rangle). \hspace{1cm} (52)$$

Here, $f_{1||}(\langle \psi \rangle)$ is the lowest-order approximation of $u_{1||}$ chosen properly so that $\Delta_3$ becomes $O(\delta_0)$. The reason why we expand it in this way is an analogy with the fact that the neoclassical flux in the radial direction, in Eulerian representation, is proportional not to $\langle u_{1||} \rangle$, but to $\langle f_{1||} / B \rangle$, where $\langle \cdots \rangle_\psi$ means the flux-surface average. Then, substituting eq. (52) into eq. (38) yields
\[
C_i(\tilde{f}_i) = \frac{1}{J_c} \frac{\partial}{\partial \tilde{z}} \left[ \frac{1}{ \Omega_\tilde{z} } \int \frac{\partial \tilde{f}_i}{\partial \tilde{z}} \delta \tilde{z} \Omega_\tilde{z} \right] - \frac{m_i \Delta_{\parallel} \Delta_{\parallel}}{\Omega_\tilde{z} T_i} \left[ \langle I_{\parallel} B \Omega_\parallel \cdot \tilde{w} \rangle \tilde{f}_{\parallel,0} \right. \\
- m_i \left. \left[ \frac{\partial \tilde{w}_0}{\partial \tilde{z}} \cdot \Delta_{\parallel} \right] \tilde{f}_{\parallel,0} \right].
\]  

(53)

Now consider the \(O(\delta_i^1)\) equation of kinetic equation \(C_i^{(1)}(\tilde{f}_i, \tilde{f}_0) = -C_i^{(1)}(\tilde{f}_0)\), or written it down explicitly.

\[
\frac{\nu_i}{J_c} \frac{\partial}{\partial \mu} \left[ J_c \left( \frac{\mu_i \nu_i^2}{B} \right) \frac{\partial \tilde{f}_{i,0}}{\partial \mu} \right] = -\frac{\nu_i}{J_c} \frac{\partial}{\partial \mu} \left( \langle I_{\parallel} \Omega_{\parallel} v_{\parallel} \rangle \frac{\partial \tilde{f}_{i,0}}{\partial \langle \varphi \rangle} \right) \\
+ \frac{m_i \Omega_\parallel \Delta_{\parallel} \langle I_{\parallel} \rangle \tilde{f}_{\parallel,0}}{\Omega_\parallel T_i}.
\]  

(54)

To solve eq. (54) for \(\tilde{f}_{i,0}\), it is rewritten in terms of driving forces \(F_k\) as

\[
C_i^{(0)}(\tilde{f}_{i,0}) = \tilde{f}_{i,0} \sum_k F_k \frac{\partial \phi_k}{\partial \mu}.
\]  

(55)

where

\[
F_1 = \frac{\partial}{\partial \langle \varphi \rangle} \left[ \frac{d \ln \bar{n}_\parallel}{d \langle \varphi \rangle} + \frac{\epsilon_\parallel}{T_i} \frac{d \Phi}{d \langle \varphi \rangle} \right],
\]  

(56a)

\[
F_2 = \frac{\partial}{\partial \langle \omega \rangle} \left[ \frac{\partial}{\partial \langle \omega \rangle} \right] T_i d \langle \varphi \rangle,
\]  

(56b)

\[
F_3 = \frac{m_i \Omega_\parallel \Delta_{\parallel} \parallel_{\parallel}}{\Omega_\parallel T_i},
\]  

(56c)

and

\[
\alpha_1 = -J_c \nu_i \frac{\Omega_\parallel \langle I_{\parallel} \rangle}{\Omega_\parallel} v_{\parallel},
\]  

(57a)

\[
\alpha_2 = -\left( \frac{\epsilon_\parallel}{3} - \frac{3}{2} \right) J_c \nu_i \frac{\Omega_\parallel \langle I_{\parallel} \rangle}{\Omega_\parallel} r_{\parallel},
\]  

(57b)

\[
\alpha_3 = -J_c \nu_i \frac{\Omega_\parallel \langle I_{\parallel} \rangle}{\Omega_\parallel}.
\]  

(57c)

Introducing the perturbed distribution function \(g_k(\mathcal{C}_i, \mu, \langle \varphi \rangle)\) which satisfies

\[
\tilde{C}_i^{(n)}(g_k) = \tilde{f}_{i,0} \frac{\partial \alpha_k}{\partial \mu}.
\]  

(58)

\(\tilde{f}_{i,1}\) can be expressed as

\[
\tilde{f}_{i,1} = \sum_{k=1}^3 \eta_k F_k.
\]  

(59)

Thus the first order equation is found to have a similar form to that by Bernstein et al.\(^{11}\) though we successfully include the momentum-restoring term by introducing an additional driving force \(F_3\).

Next, consider \(O(\delta_i^2)\) part of the reduced kinetic equation

\[
\frac{\partial \tilde{f}_{i,0}}{\partial \tau} = C_i^{(2)}(\tilde{f}_{i,0}) + C_i^{(1)}(\tilde{f}_{i,1}) + C_i^{(1)}(\tilde{f}_{i,2}) - C_i^{(1)}(\tilde{f}_{i,0}) - \frac{\partial}{\partial \tau} \left[ \frac{1}{J_c} \frac{\partial}{\partial \langle \varphi \rangle} \right] J_c \nu_i \frac{\mu_i}{e_i} \left[ \langle I_{\parallel} \rangle \Omega_{\parallel}^2 \right] \frac{\partial \tilde{f}_{i,0}}{\partial \langle \varphi \rangle} + \frac{m_i \Omega_\parallel \Delta_{\parallel} \parallel_{\parallel}}{\Omega_\parallel T_i} \langle I_{\parallel} \parallel \rangle \tilde{f}_{\parallel,0} \\
+ \left( \frac{\epsilon_\parallel}{3} \right) \frac{\partial \tilde{f}_{i,0}}{\partial \mu} - \frac{\partial}{\partial \mu} J_c \nu_i \mu_i \left[ \langle I_{\parallel} \rangle \right] \frac{\partial \tilde{f}_{i,0}}{\partial \langle \varphi \rangle} \\
+ \left( \frac{m_i \Omega_\parallel \Delta_{\parallel} \parallel_{\parallel}}{\Omega_\parallel T_i} \right) \frac{\partial \tilde{f}_{i,0}}{\partial \mu} - m_i \langle v_{\parallel} \Delta_{\parallel} \rangle \tilde{f}_{\parallel,0} = 0.
\]  

(60)
By taking a moment with $\mathcal{E}$ and $\mu$, we obtain the particle transport equation in the $\langle \psi \rangle$ direction

$$\frac{\partial}{\partial \bar{n}} N_i + \frac{\partial}{\partial \langle \psi \rangle} J_i^0 = 0, \quad (61)$$

where the particle flux $J_i^0$ is given as

$$J_i^0 = \sum_{k=1}^{3} (S_{i1}^{k} + S_{i2}^{km}) F_k. \quad (62)$$

An important point is that the transport coefficients are separated into the explicit and implicit parts, $S_{jk}^{exp}$ and $S_{jk}^{imp}$, respectively. They are given as follows

$$S_{i1}^{exp} = -\left\{ \nu_i \frac{\mu}{e_i} \left( \frac{P_i^2}{\Omega_i \gamma^2} \right) \cdot \hat{f}_{i0} \right\}, \quad (63a)$$

$$S_{i2}^{exp} = -\left\{ \nu_i \frac{\mu}{e_i} \left( \frac{P_i^2}{\Omega_i \gamma^2} \right) \cdot \left( c_i^2 - \frac{3}{2} \right) \hat{f}_{i0} \right\}, \quad (63b)$$

$$S_{i3}^{exp} = -\left\{ \nu_i \frac{\mu}{e_i} \left( \frac{R_i^2}{\Omega_i \gamma^2} \right) \cdot \hat{f}_{i0} \right\}, \quad (63c)$$

$$S_{i4}^{imp} = \left\{ \frac{\alpha_i}{J_i} \frac{\partial g_k}{\partial \mu} \right\}. \quad (63d)$$

where the inner-product is defined as $\{a, b\} = \sum_{\sigma} \int d\mathcal{E} d\mu J_{\sigma} ab$.

To obtain the energy transport equation, the moment to be taken is $\int d\mathcal{E} d\mu J_{\sigma} W$, where $W = \mathcal{E} - e_i \Phi(\langle \psi \rangle)$. Define here

$$Q_i = \sum_{\sigma} \int d\mathcal{E} d\mu J_{\sigma} W \hat{f}_{i0}. \quad (64)$$

which is the sum of the kinetic energy of particles with the same $\langle \psi \rangle$. The use of partial integrals yields the energy transport equation

$$\frac{\partial}{\partial \bar{n}} Q_i + \frac{\partial}{\partial \langle \psi \rangle} \left[ J_2^i + \frac{3}{2} J_i^0 \right] = -e_i J_i^0 \frac{d\Phi}{d\langle \psi \rangle}. \quad (65)$$

where $J_2^i$ represents the conductive ion heat flux

$$J_2^i = \sum_{k=1}^{3} (S_{21}^{k} + S_{22}^{km}) F_k. \quad (66)$$

where

$$S_{21}^{exp} = -\left\{ \nu_i \frac{\mu}{e_i} \left( \frac{P_i^2}{\Omega_i \gamma^2} \right) \cdot \left( c_i^2 - \frac{3}{2} \right) \hat{f}_{i0} \right\}, \quad (67a)$$

$$S_{22}^{exp} = -\left\{ \nu_i \frac{\mu}{e_i} \left( \frac{P_i^2}{\Omega_i \gamma^2} \right) \cdot \left( c_i^2 - \frac{3}{2} \right)^2 \hat{f}_{i0} \right\}, \quad (67b)$$

$$S_{23}^{exp} = -\left\{ \nu_i \frac{\mu}{e_i} \left( \frac{R_i^2}{\Omega_i \gamma^2} \right) \cdot \hat{f}_{i0} \right\}, \quad (67c)$$

$$S_{24}^{imp} = \left\{ \frac{\alpha_i}{J_i} \frac{\partial g_k}{\partial \mu} \right\}. \quad (67d)$$

The right hand side of eq. (65) describes work done by the radial ion current.

**B. Properties of transport coefficients**

An important property of the transport coefficients $S_{jk}$ is the symmetry of the implicit part $S_{jk}^{imp} = S_{kj}^{imp}$. This can be shown by as follows. First, integrating both sides of eq. (58) by $\mu$ yields...
\[ \alpha_s = \frac{J \nu_1}{J_0} \left( \frac{m \nu_1^2}{B} \right) \mu \frac{\partial g}{\partial \mu} \]  

where the integral constant is zero from the boundary condition. Then, one obtains

\[ S_{1}^{m} = \sum_{\sigma} \int dE d\mu \frac{\partial g}{\partial \mu} \nu_1 \left( \frac{m \nu_1^2}{B} \right) \mu \frac{\partial g}{\partial \mu} \frac{\partial g}{\partial \mu} \]  

This equation is symmetric in \( j \) and \( k \), therefore implicit coefficients are symmetric.

One can also find that the explicit part has a symmetry \( S_{1}^{*} = S_{2}^{*} \). Then, it is natural to seek the third flux which satisfies the symmetry \( S_{3}^{*} = S_{4}^{*} \).

\[ J_{3} = \sum_{k=1}^{3} (S_{3}^{*} - S_{4}^{*}) F_{k} \]  

Now we show that the proper definition of the third flux is

\[ J_{3} = \frac{2\nu_{1} K_{1}}{r_{e}} \left( \frac{I_{r_{e}}}{\Omega_{r_{e}}} \right) \frac{dV}{d\nu} \]  

Noting that \( \langle \cdots \rangle_{\psi} \) means the flux-surface average, we have

\[ J_{1} = 4\pi \int_{0}^{2\pi} \frac{d\theta}{B} \frac{B}{\nabla \theta} \frac{I_{r_{e}}}{\Omega_{r_{e}}} \int dE d\mu' \frac{B_{s}}{m_{l}^{2} v_{l}^{2}} \nu_{l} \langle \psi \rangle_{\psi} f_{\psi} \left| \frac{d\psi}{d\nu} \right| \]  

Next, distribution function is expanded as

\[ \frac{\partial f}{\partial \mu} \approx \frac{\partial}{\partial \mu} \left[ f_{\psi} - \Delta \frac{\partial f_{\psi}}{\partial \mu} + f_{s} \right] = \frac{I_{r_{e}}}{\Omega_{r_{e}}} \frac{\partial f_{\psi}}{\partial \mu} + \frac{\partial f_{s}}{\partial \mu} \]  

where

\[ \Delta = \psi - \langle \psi \rangle = \frac{I_{r_{e}}}{\Omega_{r_{e}}} - \left( \frac{I_{r_{e}}}{\Omega_{r_{e}}} \right) \]  

represents the deviation from an instantaneous particle position to its averaged flux-surface, and we use the relation \( \partial \psi / \partial \nu = -\partial \Delta \psi / \partial \nu \). Using eqs. (20) and (21), and changing the order of integrals, we have

\[ J_{1} = -\sum_{\sigma} \int dE d\mu \frac{4\pi^{2} T_{p}}{m_{l}^{2}} \left[ 1 - \delta_{s} \right] \frac{\nu_{l} \mu}{r_{e}} \int \frac{d\theta}{B} \frac{I_{r_{e}}}{\Omega_{r_{e}}} \frac{\partial f_{s}}{\partial \mu} \]  

Note here that the implicit coefficients \( S_{3}^{m} \) are written as follows. By its definition,

\[ S_{3}^{m} = \left\{ \frac{\alpha_{3}}{J_{3}} \frac{\partial g_{l}}{\partial \mu} \right\} = -\sum_{\sigma} \int dE d\mu J_{\sigma} \left( \frac{I_{r_{e}}}{\Omega_{r_{e}}} \right) \mu \frac{\partial g_{l}}{\partial \mu} \]  

Then, comparing eqs. (74) and (75), one finds that the explicit coefficients for \( J_{1} \) are given as follows
Thus we obtain the $3 \times 3$ symmetric coefficients for both explicit and implicit parts.

To close transport equations, we must eliminate the additional driving force $F_3$ introduced to include the parallel momentum balance. For this purpose, we choose $\bar{u}_{||}$ as follows

$$\bar{u}_{||} = \frac{\Omega_{a0}}{I_0(h^2)_0}(\Omega)_{\psi}^{-\frac{1}{2}}\left(\frac{I_{u_{||}}}{\Omega}\right)_{\psi}. \quad (77)$$

The factor $(h^2)_0$, where $h = B_0/B$, is needed to retain the ambipolarity in the SOW limit, as shown in Appendix C. By using this definition, eq. (70) can be solved for $F_3$. It yields

$$F_3 = -i(S_{13}F_1 + S_{23}F_2). \quad (78)$$

$$\beta = -\left(\frac{\bar{n}_0K_1}{\tau_\parallel} \frac{dV}{dp} (h^2)_\psi - S_{30} \right)^{-1}. \quad (79)$$

where $S_{jk} = S_{jk}^e + S_{jk}^m$ is the total transport coefficients. Finally, neoclassical fluxes are rewritten in the following form

$$\begin{bmatrix} J^i_1 \\ J^i_2/T_i \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}. \quad (80)$$

where

$$A_{11} = S_{11} - \beta S_{13}^2, \quad (81a)$$

$$A_{12} = S_{12} = S_{13}S_{23}, \quad (81b)$$

$$A_{22} = S_{23} - \beta S_{23}^2. \quad (81c)$$

Thus the resulting transport matrix $A_{jk}$ in Lagrangian formulation is shown to be Onsager-symmetry as same as in Eulerian formulation.

Next, let us calculate the total coefficients $S_{jk}$. The perturbed distribution functions $g_k$ are needed to calculate the implicit part. From eq. (68), we obtain

$$\frac{\partial g_k}{\partial \mu} = \frac{m_i}{(m_i^2/B)} \frac{\partial}{\partial \mu} \left( \frac{I_{v_{||}}}{\Omega} \right) f_{10}. \quad (82a)$$

$$\frac{\partial g_2}{\partial \mu} = \frac{m_i}{(m_i^2/B)} \frac{\partial}{\partial \mu} \left( \frac{I_{v_{||}}}{\Omega} \right) f_{10}. \quad (82b)$$

$$\frac{\partial g_3}{\partial \mu} = \frac{m_i}{(m_i^2/B)} \frac{\partial}{\partial \mu} \left( \frac{I_{v_{||}}}{\Omega} \right) f_{10}. \quad (82c)$$

From here on, we use the approximation $I(\psi) = I_0 = R_0B_0$, which corresponds to the low-$\beta$ plasma. For $j, k = 1, 2$ using eq. (41) yields

$$-S_{jk} = \frac{p_0^3}{e_i\Omega_{a0}} \left\{ \frac{\nu_\mu}{(1 - \Delta_c)^2} \left( \frac{\Delta_c}{(1 - \Delta_c)^2} - \frac{\Delta_c}{(h^2)_{10}} \right) \left( \frac{c_i^2 - 3}{2} \right)^{j-k-2} f_{10} \right\} \quad (83)$$

$$= \frac{3\pi R_0^2 \tilde{n}_0 \nu_{a0} q R_0}{8\tau_\parallel B_0} \sum_{\sigma_i} \int dz \lambda \lambda_0 \tilde{g}_{\sigma_i} \left( \frac{c_i^2 - 3}{2} \right)^{j-k-2} g(\lambda) \lambda_0 \left( \frac{I_{v_{||}}}{\Omega} \right) \quad (83)$$

where $x = \exp(-\Sigma/T_i)$, $\lambda_0 = \mu B_0/\Sigma$, and $\tilde{g}_{\sigma_i} = \tilde{g}_{\sigma_i} / q R_0$, respectively. We call $S_{jk}$ the normalized transport coefficients. In the equation above, the terms which are proportional to $\Delta_c$ and $\Delta_c^2$ are exactly canceled. In a similar way, one has

$$12$$
\[ S_{24} = -\sum_{c} \int d\varepsilon d\lambda_{0} \tau\rho c_{0} \left( \frac{\alpha^{2}}{2} - \frac{3}{2} \right) y(c_{0}) \lambda_{0} \left( \langle h \rangle - \frac{\langle h \rangle^{2}}{\langle h, h \rangle} \right). \]  

\[ S_{13}^{in} = \sum_{c} \int d\varepsilon d\lambda_{0} \tau\rho c_{0} y(c_{0}) \lambda_{0} \frac{\langle h \rangle^{2}}{\langle h \rangle}. \]  

Note that eq. (83) differs from eq. (84) only by the factor \((1 - \alpha)^{-1} \approx 1\). Therefore, this difference is neglected in the final calculations, and the approximation \(S_{13} = S_{11}\) and \(S_{24} = S_{12}\) are used. As a complement, we define

\[ \tilde{\beta} = -\left( \frac{8B_{0}K_{i}}{3\pi q R_{0}} \frac{dV}{d\psi} \langle h^{2} \rangle - \tilde{S}_{34} \right)^{-1}. \]

so that eq. (78) can be rewritten as \(F_{3} = -\tilde{\beta}(S_{11}F_{1} + S_{34}F_{2})\).

In the SOW limit, \(\langle hv \rangle^{2} = 0\) for banana particles, while \(\langle hv \rangle^{2} \approx \langle h, h \rangle\) for passing ones. Therefore, transport coefficients other than \(S_{13}\) are determined mainly by the banana part. Because of this separation of the contribution to transport between banana and passing, the approximation of \(\partial \langle \psi \rangle / \partial v\) as in eq. (42) used in Ref. 16 ends up in the same result as the SOW limit of eqs. (83) to (85). However, by using the exact solution for \(S_{13}\), we can include the contribution for neoclassical transport not qualitatively, but quantitatively, from all the orbit-types of particles appearing in the near-axis region. The magnitude of contribution of each particle is evaluated by the factor \(\langle h \rangle - \langle h \rangle^{2} / \langle h, h \rangle\).

In the present analysis, we improved the treatment of collision term to retain the momentum conservation law. Since we consider only the ion-ion self collisions here, \(J_{1}\) must vanish in the SOW limit. It is shown in Appendix C that \(A_{11}\) and \(A_{22}\) become zero in this limit, and therefore \(J_{1}\) vanishes intrinsically.

Thus Lagrangian transport theory applied to the region away from the axis, where potato particles do not appear and the SOW limit is valid, reproduces conventional Eulerian transport theory.

C. Comparison with Eulerian transport theory

The representation of fluxes in eqs. (61) and (65) from Lagrangian formulation are different from those in the standard neoclassical transport theory based on Eulerian representation. The former describes the change in \(\dot{N}_{i}\) and \(Q_{i}\), which are functions of averaged particle position \(\langle \cdot \rangle\), while the latter describes the change in \(n_{i}\) and \(p_{i} = m_{i}T_{i}\) through radial fluxes averaged on a magnetic surface \(\psi\). Then, the comparison of neoclassical flux between these two representations is not straightforward. Let us consider this problem here.

As a preparation, we introduce a normalizing factor for \(Q_{i}\), as follows

\[ \lambda_{q}(\langle \psi \rangle) = \frac{3}{2} \frac{n_{i}T_{i}}{m_{i}} V' Q_{c}^{-1} = \frac{3T_{i}}{2} \left( \frac{2\pi T_{i}}{m_{i}} \right)^{1/2} V' \left( \sum_{c} \int d\varepsilon d\mu J_{i} \exp \left[ \frac{-W}{T_{i}} \right] \right)^{-1}. \]

where \(V' = dV/d\psi = \langle \dot{\psi} \rangle\), \(\lambda_{q}\) has a similar property to \(\lambda_{u}\) in eq. (51). The particle flux and the heat flux are redefined as

\[ \Gamma_{c} = J_{i} / V'. \]

\[ q_{c} = J_{c} / V'. \]

so that they represent fluxes per unit cross-section. Note that \(q_{c}\) differs from its general definition by \(\Gamma_{i}\) since we adopt \(F_{3} = d\ln T_{i} / d\psi\) as a driving force rather than the pressure gradient. Then transport equations are rewritten as follows

\[ \frac{\partial n_{i}}{\partial t} + \frac{\lambda_{u}}{V' \partial \langle \psi \rangle} \left( V' \Gamma_{i} \right) = 0. \]

\[ \frac{\partial}{\partial t} \left( \frac{3}{2} n_{i} T_{i} \right) + \frac{\lambda_{q}}{V' \partial \langle \psi \rangle} \left[ V' \left( q_{c} - \frac{3}{2} \Gamma_{i} T_{i} \right) \right] = -\lambda_{f} r_{c, 1} \frac{d\Phi}{d\langle \psi \rangle}. \]

where we assume that the time variation of \(\lambda_{u}\) and \(\lambda_{q}\) is slow compared with that of \(N_{i}\) and \(Q_{i}\). These transport equations have the same dimensions as those of a standard Eulerian representation. In the SOW limit, \(\lambda_{u}\) and \(\lambda_{q}\) becomes unity, and eqs. (89) and (90) reduce to the Eulerian representation.
As will be shown in Sec. VC, $\lambda_n$ and $\lambda_q$ is nearly unity around the region $\langle r \rangle \sim r_p$ though the finiteness of the potato width significantly affects the transport coefficients there. Only in the region $\langle r \rangle \lesssim q_{\psi,0}$ they become much larger than unity. Then, the qualitative difference in $\Gamma$ and $q_{\psi,0}$ between two representations are not so much significant as far as we are interested in the neoclassical transport around $\langle r \rangle \sim r_p$.

V. CALCULATION OF THERMAL CONDUCTIVITY

A. Definition of collisionless regime

In this section, we calculate the ion thermal conductivity in the near-axis region. As a preparation, let us reconsider the definition of the collisionless regime in which Lagrangian approach is valid.

Usually, the collisionless (or banana) regime is defined as follows

$$
\tau_c \simeq \frac{qBR_0}{v_{th}, \sqrt{\epsilon}} \cdot \frac{\nu_{\psi,0}}{\epsilon} \cdot \frac{\nu_{\psi}}{\epsilon} \\
\Rightarrow \delta_c \sim \frac{\nu_{\psi} qBR_0}{v_{th} \epsilon^{3/2}} \ll 1.
$$

(91)

since banana particles exist in the range $|\psi| < \epsilon \sqrt{\epsilon}$. In Eulerian representation, collisions cause diffusion only in the velocity space, and then the collisionless regime can be defined as above. In Lagrangian representation, however, collisions bring about diffusion directly in the $\langle \psi \rangle$ direction through the factor $\partial\langle \psi \rangle / \partial \psi$. This fact means that the effect of scattering on each particle differs according to this factor.

Remember here that, as mentioned in Sec. IVB, transport coefficients can be obtained by using the estimation that

$$
\frac{\partial \langle \psi \rangle}{\partial \tau} \sim \left\{ \begin{array}{cl}
\frac{I}{\Omega} & \text{(banana),} \\
\frac{I}{\delta} & \text{(well - passing).}
\end{array} \right.
$$

(92)

In the near-axis region, potato particles can be assumed to have $\partial \langle \psi \rangle / \partial \psi \approx -I/\Omega$ like bananas. The transition of orbit topology of a potato particle occurs when their averaged radial position changes as large as its orbit width $\Delta_r \sim r_p$, or

$$
\Delta_r \sim \frac{I}{\Omega_0} \left( \frac{q\rho_{\psi,0}}{R_0} \right)^{1/3}.
$$

(93)

To change $\langle \psi \rangle$ as large as $\Delta_r$, pitch-angle scattering of the magnitude

$$
\Delta r_\parallel \sim \Delta_r \left[ \frac{\partial \langle \psi \rangle}{\partial \psi} \right]_\parallel \left( \frac{q\rho_{\psi}}{R_0} \right)^{1/4} v_{th,1}
$$

is needed. Therefore, on the analogy of eq. (91), the effective collision frequency for potato ions can be defined as

$$
\nu_{\psi,eff} = \frac{1}{\tau_c} \left( \frac{R_0}{q\rho_{\psi,0}} \right)^{2/3}.
$$

(94)

On the other hand, $\tau_p$ for potato particles is estimated as

$$
\tau_p^{\text{pot}} \sim \frac{qBR_0}{v_{th}, \sqrt{\epsilon}} \left( \frac{R_0}{q\rho_{\psi,0}} \right)^{1/3}.
$$

(95)

From eq. (94) and (95), we obtain

$$
\delta_c^{\text{pot}} = \nu_{\psi,eff} \tau_p^{\text{pot}} \ll 1 \\
\Rightarrow \frac{T_i}{\epsilon} \gg \frac{4Z^2 (\bar{n}_e B_0 R_0^2)^{2/5}}{(m_e/m_p)^{1/5}}
$$

(96)

where $T_i$ (keV), $\bar{n}_e$ ($10^{20}$m$^{-3}$) and $m_p$ is the mass of proton. For example, if $B_0 = 4T$, $R_0 = 4m$ and $\bar{n}_e = 1 \times 10^{20}$m$^{-3}$ for hydrogen ion, then $T_i \gg 5\text{keV}$ is needed for the collisionless assumption. Note that
the criterion eq. (96) corresponds to that from the usual definition of $\delta$, in eq. (91) evaluated at $r \approx r_p$ as mentioned in Ref. 13.

In reality, barely-transit potato particles have much longer $r_p$ than the estimation in eq. (95). Then, particles around the transition boundary $\Omega_2$ in Fig. 3 break the collisionless-assumption $\delta_r \ll 1$. Treating these collisional particles in Lagrangian transport theory like the banana-plateau transition in the standard Eulerian theory is not considered here. Therefore, our calculation corresponds to the collisionless limit of neoclassical transport.

### B. Ion heat flux

In Eulerian transport theory, the ion heat flux is expressed as follows

\[
\frac{q_i}{T_i} = -n_i \chi^{(r)}_i \frac{d}{dr} \ln T_i.
\]

where $\chi^{(r)}_i$ is the ion thermal conductivity in the $r$ direction. Here, $q_i$ in the form as above is the result of (i) neglecting ion-electron collisions, and (ii) $\Gamma_i = 0$ because of the momentum conservation in ion-ion self collisions. In Lagrangian approach, however, the condition $\Gamma_i = 0$ is not intrinsic. In reality, the momentum-restoring term $S_{11}F_3$ in $J_i$ cannot exactly cause $S_{11}F_1 + S_{12}F_2$, especially in the region near the magnetic axis. This is because of the non-local nature of Lagrangian formulation, and partially because only the lowest order expansion of $\varphi_j$ is included. In the present calculation, we rewrite the transport equation in eq. (80) to eliminate $F_1$

\[
\begin{align*}
\frac{q^{(r)}}{T_i} &= \frac{S_{12}}{S_{11}} \Gamma^{(r)}_i - \bar{n}_i \chi^{(r)}_i \frac{d}{d\langle r \rangle} \ln T_i, \quad (88) \\
\chi^{(r)}_i &= -\frac{3q_i^2 \rho_0^2}{32\pi \langle e \rangle^2 \tau_i} \left[ S_{22} - \frac{S_{12}^2}{S_{11}} \right]. \quad (99)
\end{align*}
\]

where $q$, $\rho_0$, and $\tau_i$ are evaluated at $r = \langle r \rangle$, and we change here the radial coordinate from $\langle \varphi \rangle$ to $\langle r \rangle$. The ion heat conductivity $\chi^{(r)}_i$ defined in this way is compared to $\chi^{(r)}_i$, though $\Gamma^{(r)}_i$ does not vanish here.

It is well-known that $\chi^{(r)}_i \propto q_i^2 \rho_0^2 / (n^2 e^2 \tau_i)$ in Eulerian theory, while the apparent dependence of $\chi^{(r)}_i$ is $q_i^2 \rho_0^2 / (n^2 e^2 \tau_i)$. However, $S_{12}$ away from the magnetic axis is proportional to $\sqrt{\langle r \rangle}$, and its density has the same dependency as $\chi^{(r)}_i$ there.

### C. Calculation result and discussion

Numerical calculation of transport coefficients in eq. (83) is implemented by using Monte Carlo integration method. In the calculation, test particles which have a given $\langle r \rangle$ are generated randomly and uniformly in the phase space $(r, \lambda_0, \sigma_0)$, where $x$ and $\lambda_0$ are defined in Sec. IV.B. And, all the functions in the integrand of $S_{12}$, which we write $F_{12}(r, \lambda_0, \langle r \rangle; \sigma_0)$ here, are calculated by tracing each particle orbit. Then, transport coefficients at $\langle r \rangle$ are given as

\[
S_{12}(\langle r \rangle) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} F_{12}(x_n, \lambda_{0n}, \langle r \rangle; \sigma_{1n}).
\]

where $N$ is total number of test particles and $(x_n, \lambda_{0n}, \sigma_{1n})$ is the position of $n$-th test particle in the phase space. We can include all the types of orbit to transport coefficients. Note that we approximate $1 - \delta_n = 1$ so that $S_{11} = S_{14}$ and $S_{12} = S_{23}$.

As a example, we calculate the ion thermal conductivity $\chi^{(r)}_i$ under the conditions $B_0 = 4\mathrm{T}$, $q = 3$, $T_i = 20\mathrm{keV}$ and $\bar{n}_i = 1 \times 10^{22} \mathrm{m}^{-3}$. The radial electric field $dE/dr$ is neglected. In this case, typical potato particles appear in the region $\langle r \rangle \lesssim r_p = 0.24\mathrm{m}$. According to eq. (96), the plasma is well in the collisionless regime. The calculation result of $\chi^{(r)}_i$ is plotted in Fig. 6. For a comparison, $\chi^{(r)}_i$ by a standard Eulerian theory in the banana regime is also plotted. Note that $\chi^{(r)}_i$ is obtained by regarding $r$ as $\langle r \rangle$.

A significant reduction in $\chi^{(r)}_i$ can be seen in the region $\langle r \rangle \lesssim r_p$. The main reason of this reduction explained in Lagrangian approach is that potato particles, which mainly contribute to the radial transport, cannot exist in the region $\langle r \rangle \lesssim r_p/4$ when observed in the COM space $(E, \mu, \langle r \rangle)$, as is shown in Fig. 3.
The FOW effect is thus included in the calculation by reflecting the real population of potato particles near the magnetic axis.

In Fig. 7, \( \chi \) defined in eq. (87) is plotted. One can see that \( \chi \) becomes much larger than unity only on the inner-most point \( \langle r \rangle = 2\rho_0 = 0.03m \) in this case. As mentioned in Sec. IV C, simple comparison of \( \chi \), between Lagrangian and Eulerian formulations is possible as long as \( \lambda_\parallel \approx 1 \). Then it can be said that the reduction of the ion thermal conductivity occur not only when it is observed in the \( \langle r \rangle \)-coordinate, but also in the real space around \( r \sim r_\rho/2 \). On the other hand, it will be an underestimation that \( \chi_1 \sim 0.1 \) at \( r = 0.03m \) because it is almost the same level as the classical transport \( \chi_0 \sim \rho_0^2/c_s r_\rho \), and also because of \( \lambda_\parallel \gg 1 \) there.

In Fig. 7, we also plot \( A_1/2 \) which is approximately the ratio of \( \Gamma_s \), to \( q_s \). Away from the axis it is almost zero and then \( \Gamma_s \) can be neglected, while it becomes finite around \( \langle r \rangle = r_\rho/2 \). It is the FOW effect that cause the finite particle flux by ion-ion collisions. Since the electron particle flux is negligible compared to the ion flux, radial electric field \( E_r \) will develop to satisfy ambipolarity \( \Gamma_s = 0 + O(\sqrt{m_e/m_i}) \). In our present formulation, however, the ambipolar electric field cannot be calculated correctly, because it requires to solve \( dE_r/dr \) from the particle flux equation, which in turn affects transport coefficients \( A_{jk}(r,t) \) through the orbit-squeezing effect of potato particles. It is a future work to determine neoclassical \( E_r \) in the core region.

Next, to investigate the degree of contribution from each orbit type to transport coefficients, we plot in Fig. 8 the factor \( H_{\parallel} = (\langle r \rangle - \langle h|r|\rangle)^2/\langle v_\parallel^2 \rangle \) for particles at \( \langle r \rangle = 0.12m \) and \( E = 20keV \). Since \( S_{jk} \) contains \( H_{\parallel} \), one can see that not only banana particles but also all the potato particles, that is, kidney, outer-circling, and inner-circling particles appearing around the transition boundary, contribute transport to the same degree. In the numerical calculation, the factor \( H_{\parallel} \) is evaluated without any approximation by using Monte Carlo integration method.

The reducive tendency of \( \chi_1 \) in the near-axis region is the common feature of recent simulation results. Our \( \chi^{(p)} \) also shows a similar dependence on \( \langle r \rangle \) to these simulations, for example, to the fitting formula by Lin et al based on a simple random-walk model. This suggests that the neoclassical transport in the near-axis region can be explained by the random-walk diffusion process of the potato-center \( \langle r \rangle \). However, there is a difference between ours and the others in that the reduction of \( \chi_1 \) compared to the standard neoclassical value begins at \( r \sim r_\rho \) in our calculation, while it begins from somewhat more outer position \( r \sim 2r_\rho \) in the other simulations. This may be caused either by the differences of the profiles, of the treatment of collision terms, or of the algorithms used in each calculation. More detailed comparison between Lagrangian formulation and \( df \) simulations will be done in a future work.

VI. SUMMARY

Lagrangian formulation of neoclassical transport theory is applied to the near-axis region \( \langle r \rangle \sim r_\rho \) to include the effect of potato particles in the calculation of ion thermal conductivity. In a collisionless regime plasma, a significant reduction of \( \chi_1 \) compared to the standard neoclassical level is found as shown in Fig. 6. By introducing the factor \( \lambda_\parallel \), we show that the direct comparison of \( \chi_1 \) between Eulerian and Lagrangian representations is possible at \( r \sim r_\rho \).

Since the reduction of \( \chi_1 \) occurs from \( \langle r \rangle \sim r_\rho \propto q^{2/3} \), the finiteness of potato orbit on neoclassical transport will be important in a reversed-shear configuration in which \( q \)-value becomes very high at the near-axis region. The development of ambipolar radial electric field \( E_r \) and its effect on \( \chi_1 \), will be studied in a future work.

In the present article, we show that the Lagrangian transport theory is of practical use in treating transport phenomena in which the orbital property of particles is really important. This approach will also be useful to analyze other problems treating FOW effects, such as the bootstrap current near the magnetic axis, or the neoclassical transport around the internal transport barrier in tokamaks where the gradient scale length of plasma pressure becomes comparable to typical banana width.

ACKNOWLEDGMENTS

One of the authors (S. S.) would like to thank Dr. N. Nakajima for useful discussion on numerical calculation method. This work is supported by JSPS Research Fellowships for Young Scientists, and in part by the Grant-in-Aid from Japanese Ministry of Education, Science, Sports, and Culture, No. 12680597.
APPENDIX A: ESTIMATION OF $\delta$.

From the definition of $\delta_*$, we have to evaluate
\[
\delta_* = \left( \frac{\partial}{\partial \psi} \theta^* - \frac{\partial \theta^*}{\partial \psi} \frac{\partial}{\partial \theta} \right) \alpha_* .
\]  
(A1)

where $\alpha_* = I v_\parallel / \Omega$ evaluated at the averaging point $(\psi, \theta) = (\psi^*, \theta^*)$. The difficulty lies in the evaluation of $\partial \theta^*/\partial \psi$. First, let us consider the case for orbits which have the turning points of $\dot{\theta}$, that is, for banana, outer-circulating, and inner-circulating orbits. The averaging point of these orbits can be approximated by the turning points. Therefore, from the equation of $\theta$ in eq. (21), $\partial \alpha_*/\partial \psi = 1$ must be satisfied on the averaging point. Then, we have
\[
\frac{\partial \theta^*}{\partial \psi} \approx \frac{\partial}{\partial \theta} \left( \frac{\partial \alpha_*}{\partial \psi} \right) .
\]  
(A2)

Substituting eq. (A2) into (A1), we have
\[
\delta_* = \frac{m_i B}{\mu B} \frac{\partial B}{\partial \psi} \frac{\partial^2 B}{\partial \psi \partial \theta} - \frac{\partial B}{\partial \theta} \frac{\partial^2 B}{\partial \psi^2} \left( \frac{\partial B}{\partial \psi} \right)^2 - \frac{2m_i^2}{\mu B} \right)
\]  
(A3)

Note that eq. (A3) is evaluated at the averaging point. Since $v_\parallel \approx 0$ at the turning point of $\dot{\theta}$, one can see that $\delta_* \approx 0$ for banana and outer, inner-circulating orbits. As concerns kidney orbits, we can evaluate $\partial \theta^*/\partial \psi$ by approximating the averaging point by the turning point of $v_\parallel$. Then one also finds that $\delta_*$ can be neglected for kidney orbits.

For passing particles, we cannot determine the averaging point $\theta^*$ in a simple way as above. However, for well-passing particles, $\partial \theta^*/\partial \psi$ in eq. (A1) can be negligible, and one can estimate
\[
\delta_* \approx \frac{\partial \alpha_*}{\partial \psi} \approx \alpha_* \frac{1}{B} \frac{\partial B}{\partial \psi} \sim \frac{q \rho}{(r)} .
\]  
(A4)

which is negligible when considering $(r) \sim r_p \gg q \rho$.

Thus the approximation $1 - \delta_* \approx 1$ is ensured for all types of orbit.

APPENDIX B: INTEGRAL IN THE $(\mathcal{E}, \mu)$ PLANE

In the derivation of transport equations, integrals in the $(\mathcal{E}, \mu)$ plane appear. We prove some properties used in the integral here. Consider a integral of a function $F$
\[
\sum_{\sigma_i} \int d\mathcal{E} \int d\mu J_{\mathcal{E}, \mu}(\mathcal{E}, \mu, (\psi); \sigma_i) F(\mathcal{E}, \mu, (\psi); \sigma_i) .
\]  
(B1)

where $J_{\mathcal{E}}$ and $F$ also depend on the orbit type $\sigma_i$. Integral regions in the $\mu$ direction is shown in Fig. 9. $\mu_1$, $\mu_2$, and $\mu_3$ correspond to the boundaries $1, 2$, and $3$ in Fig. 3, respectively. The kidney region is laid between $\mu_2$ and $\mu_3$, and it is overlapped with a part of the banana and co-passing regions. The integral path in $\mu$ is taken in the direction of arrows in Fig. 9.

To use Gauss’s theorem to take moments of the reduced kinetic equation (60), some boundary conditions are needed. First, consider the boundary $\mu = 0$. Here, noting that all the integrand having the form $\partial / \partial \mu$ in eq. (60) are proportional to $\mu$. Then the surface integral vanishes there. Second, consider the boundary $\mu_3$, where a co-passing particle moves into a kidney region. This transition occurs continuously, since the difference between kidney and co-passing orbits is only that the former has turning point of $v_\parallel$ and the latter does not. Then, we have
\[
\lim_{\mu \to 0} J_{\mathcal{E}}(\mu, (\psi); \sigma_i = P+) = \lim_{\mu \to \mu_2} J_{\mathcal{E}}(\mu, (\psi); \sigma_i = K) .
\]  
(B2)

Therefore, the surface integral is canceled at $\mu_3$ between co-passing and kidney, since any physical value in the integrand $F$ is also continuous on the boundary.
On the boundary $\mu_2$, a banana particle bifurcates into a kidney or a counter-passing particle as shown in Fig. 5. At the limit $\mu = \mu_2$, the stagnation point is at $(r_0, \theta) = (r, \pi)$. All the values contained in the integrand $F$ are then evaluated at the stagnation point. On the other hand, Jacobian $J_i$ becomes infinity at $\mu_2$ as is pointed out in Sec. III. Noting that the kidney and counter-passing orbit at the boundary correspond to outer- and inner-sections of a banana orbit, we have

$$\lim_{\mu \to \mu_2} J_i(\mu; \sigma_t = B) = \lim_{\mu \to \mu_2} [J_i(\mu; \sigma_t = K) + J_i(\mu; \sigma_t = P - I)].$$

Therefore, the integrand $F$ for each orbit type are required to be continuous on the boundary so that the surface integral on the boundary $\mu_2$ can be canceled between banana, counter-passing and kidney.

The last condition is on the $\mu = \mu_1$ corresponding to stagnated outer-circulating orbits. In eq. (60), one can see that all the terms within the $\partial / \partial \mu$ operator have $n = 1$ in the averaged operator $\langle \cdots \rangle$. As mentioned in Sec. II, $v_{ij}$ is not exactly zero for stagnated orbits when particle orbits are solved strictly. Then, there remains small contribution from the surface integral in the $\mu$ direction where one takes the moment of eq. (60) to obtain eqs. (61) and (65). However, this contribution vanishes in the SOW limit, since stagnated condition is $v_{ij} = 0$ in this limit. Therefore, the contribution from the surface integral to $\partial N_i / \partial t$ and $\partial Q_i / \partial t$ is considered to be negligible in this paper.

As a consequence, the perturbed distribution function $g_k$ must have continuous derivative $\partial g_k / \partial \mu$ on the boundary $\mu_2$ and $\mu_3$. This fact is used in eq. (82).

**APPENDIX C: TRANSPORT COEFFICIENTS IN THE SOW LIMIT**

To show that the particle flux $J_1$ vanishes in the SOW limit, consider the transport coefficients $\tilde{S}_{11}$ and $\tilde{S}_{12}$ away from the magnetic axis. We assume a model magnetic field $B = B_0 [1 - \delta \cos \theta]$ as in Sec. II. Orbit types considered here are passing and banana in a usual sense, and the factor $[1 - \delta_i]$ becomes unity in this limit. From the definition of the normalized transport coefficients eqs. (83) and (85), one has

$$\tilde{S}_{33} = \tilde{S}_{11} + \sum \int d \lambda_0 \lambda_0 \lambda_0(\theta).$$

Since $\theta = v_\eta \cdot B$ in the SOW limit, the integral above can be calculated as

$$\sum \int d \lambda_0 \lambda_0 \lambda_0(\theta) = \sum \int d \lambda_0 \lambda_0 \lambda_0(\theta) \exp^{-c_i(\theta)} \lambda_i \lambda_0 \int \frac{d \theta}{\gamma_0} B_0$$

$$\frac{d \theta}{B_0} \int_{\theta_1}^{\theta_2} \frac{d \theta}{B_0} \int_0^\infty \frac{d \theta}{B_0}$$

Therefore, we obtain

$$\tilde{S}_{33} = \tilde{S}_{11} + \frac{8B_0 K_1}{3qR_0 \langle h^2 \rangle}.$$  

(C2)

where we have used the flux-surface average

$$\langle h^2 \rangle \equiv \frac{1}{2\pi} \frac{dV}{d\psi} \int_0^{2\pi} \frac{d\theta}{B \cdot \nabla \theta} h^2$$

And, combining eqs. (86) and (C2) yields

$$\tilde{J} = S_{11}^{-1}.$$  

(C3)

Since $A_{11} \propto S_{11}(1 - z S_{11})$ and $A_{12} \propto S_{12}(1 - z S_{11})$, the transport matrix $A_{ik}$ other than $A_{22}$ component vanish intrinsically. Therefore, ion particle flux does not occur in Lagrangian formulation in the SOW limit when only ion-ion collision is considered.

Note here that, in the SOW limit, eq. (83) for $S_{ik}$ becomes the same form as eq. (80) in Ref. 25 since $\langle h \eta \rangle = 0$ for bananas in this limit. Then, the ion thermal conductivity obtained from our formulation reproduce the result from Eulerian formulation in the SOW limit. Though this property has already been proved by Bernstein et al, we succeed in introducing the momentum-conservation nature in Lagrangian formulation.
TABLE I. Classification of particle orbits.

<table>
<thead>
<tr>
<th>Orbit Type</th>
<th>$\sigma_i$</th>
<th>$\sigma_o$</th>
<th>Sign of $\sigma_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>passing</td>
<td>0</td>
<td>0</td>
<td>$\pm$</td>
</tr>
<tr>
<td>banana</td>
<td>2</td>
<td>2</td>
<td>$\pm$</td>
</tr>
<tr>
<td>outer-circulating</td>
<td>0</td>
<td>2</td>
<td>$\pm$</td>
</tr>
<tr>
<td>inner-circulating</td>
<td>0</td>
<td>2</td>
<td>$\pm$</td>
</tr>
<tr>
<td>kidney</td>
<td>2</td>
<td>0</td>
<td>$\pm$</td>
</tr>
<tr>
<td>concave-kidney</td>
<td>2</td>
<td>4</td>
<td>$\pm$</td>
</tr>
</tbody>
</table>

*Numbers of turning points.

*bFor ions. The sign is opposite for electrons.
FIG. 1. Examples of particle orbits for $\varepsilon = 10$keV ions in the model field $B_0 = 4T$ and $q = 3$. The circle and bar marks represent turning points of the signs $q_0$ and $q_{\parallel}$, respectively. Orbit types are A : standard banana, B : the fattest banana, C : passing, D : inner-circulating, and E : outer-circulating, respectively.

FIG. 2. Examples of kidney (F) and concave-kidney (G) orbits.
FIG. 3. The regions and the boundaries for each type of orbit in the \((r', \lambda_0)\) plane for \(E = 10\) keV ions. Orbit types are abbreviated as B: banana, P+: co-passing, P: counter-passing, OC: outer-circulating, IC: inner-circulating, respectively. The solid-line part of \(l3\) up to the triangle mark is the transition boundary. The boundaries \(l1\) and the dotted-line part of \(l2\) correspond to zero-width outer-circulating and inner-circulating orbits, respectively. \(l3\) is the boundary between the kidney and co-passing regions. The circle mark corresponds to the fattest banana orbit.

FIG. 4. Regions of K: kidney (shaded) and CK: concave-kidney (enclosed by solid lines) orbits. The kidney region overlaps with the banana, co-passing, and concave-kidney regions. A part of concave-kidney region overlaps with co-passing region.
FIG. 5. Transition in orbit types. (a) A banana orbit changes into a kidney or a counter-passing orbit according to the position on which the transition occurs. (b) Similarly, a concave-kidney changes into a kidney or an inner-circulating orbit.
FIG. 6. The ion thermal conductivity normalized by \( q_i \rho_i / \tau \). Solid line is the result from our Lagrangian formulation. Dashed line is from standard Eulerian theory by Hinton and Havelock.

FIG. 7. Dependence of \( \lambda_q \) defined in eq. (87) (solid line) and the ratio of transport coefficients \( A_{11}/A_{22} \) (dashed line) on \( \langle r \rangle \).
FIG. 8. $H_{\parallel} = (\langle h \rangle - \langle h v_2 \rangle)^2 / (\hbar c^2)$ vs. $\lambda_0$ for ions with $E = 20$ keV at $<r> = 0.12$ m. The transition boundary is at $\lambda_0 = 0.965$. Abbreviations of orbit type are the same as in Figs. 3 and 4.

FIG. 9. Integral way in the $\mu$ direction. Abbreviations of orbit types are the same as used in Figs. 3 and 4.
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NIFS-715  A. Malevich, N. Nakajima, M. Okamoto, S. Murakami and K. Kanno
Statistical Properties of the Particle Radial Diffusion in a Radially Bounded Irregular Magnetic Field
Oct. 2001

NIFS-716  Boris V. Kuteev
Kinetic Depletion Model for Pellet Ablation
Nov. 2001

NIFS-717  Boris V. Kuteev and Lev D. Tsindin
Analytical Model of Neutral Gas Sheathing for Hydrogen Pellet Ablation
Nov. 2001

NIFS-718  Boris V. Kuteev
Interaction of Cover and Target with Xenon Gas in the IFE Reaction Chamber
Nov. 2001

NIFS-719  A. Yoshizawa, N. Yokou, S. Itoh and K. Itoh
Mean-Field Theory and Self-Consistent Dynamo Modeling
Dec. 2001

NIFS-720  V. N. Tsytypovitch and K. Watanabe
Universal Instability of Dust Ion-Sound Waves and Dust-Acoustic Waves
Jan. 2002

NIFS-721  V. N. Tsytypovitch
Collective Plasma Corrections to Thermonuclear Reaction Rates in Dense Plasmas
Jan. 2002

NIFS-722  S. Toda and K. Itoh
Phase Diagram of Structure of Radial Electric Field in Helical Plasmas
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NIFS-723  V. D. Postnov
Ideal and Conventional Feedback Systems for RWM Suppression
Jan. 2002

NIFS-724  T. Watanabe and H. Hojo
The Marginally Stable Pressure PROFILE and a Possibility toward High Beta Plasma Confinement in LHD
Feb. 2002

NIFS-725  S. Itoh, K. Itoh, M. Yagi, M. Kawasaki and A. Kitazawa
Transition in Multiple-scale lengths Turbulence in Plasmas
Feb. 2002

NIFS-726  S. Itoh, A. Kitazawa, M. Yagi and K. Itoh
Bifurcation and Phase Diagram of Turbulence Consituted from Three Different Scale-length Modes
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NIFS-727  M. Higata
Preliminary Experiment on the Negative Magnetor-Resistance Effect in a Weakly Ionized Discharge Plasma
Apr. 2002

NIFS-728  K. Akaishi and M. Nakasuga
Calculation of Hydrogen Outgassing Rate of LHD by Recombination Limited Model
Apr. 2002

NIFS-729  Y. Kondeh, T. Takashima and J. W. Van Dam
Proof of Non-invariance of Magnetic Helicity in Ideal Plasmas and a General Theory of Self-organization for Open and Dissipative Dynamical Systems
Apr. 2002

NIFS-730  S. Itoh and K. Itoh
From Dressed Particle to Dressed Mode in Plasmas
May 2002

NIFS-731  R. Kanazawa, N. Nakajima and H. Takahara
Path Integral Approach for Electron Transport in Disturbed Magnetic Field Lines
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NIFS-732  H. Suganuma and S. Nishimura
How to Calculate the Neo-classical Viscosity, Diffusion, and Current Coefficients in General Toroidal Plasmas
May 2002

NIFS-733  S. Satake, M. Okamoto and H. Suganuma
Lagrangian Neoclassical Transport Theory Applied to the Region near the Magnetic Axis
June 2002