Probability of Statistical L-H Transition in Tokamaks

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(Received - Aug. 13, 2002.)

NIFS-739 Aug. 2002

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Probability of Statistical L-H Transition in Tokamaks

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Abstract

A statistical model of bifurcation of radial electric field $E_r$ is analyzed in relation with L-H transitions of tokamaks. A noise from micro fluctuations leads to random noise for $E_r$. The transition of $E_r$ occurs in a probabilistic manner. Probability density function and ensemble average of $E_r$ are obtained, when hysteresis of $E_r$ exists. Forward- and backward-transition probabilities are calculated. The phase boundary is shown. Due to the suppression of turbulence by $E_r$, shear, the boundary deviates from the Maxwell's construction rule.

Keywords: L-H transition, statistical theory, probabilistic transition, transition probability, radial electric field, probability density function, random noise, hysteresis
Structural formation in inhomogeneous magnetized plasma has been one of the main issues in modern plasma physics. An important example is the H-mode transition [1] in toroidal plasmas. The key is the bifurcation of radial electric field $E_r$ [2, 3] and its mutual interaction with turbulence which has a micro scale length (such as the ion gyroradius $\rho_i$ or collisionless skin depth $\delta = c/\omega_p$) [4]. Theory has made progresses in explaining the existence of bifurcation in $E_r$ of meso-scale (a hybrid between the plasma radius $a$ and micro-scales [5, 6]) including zonal flow [7]. (See reviews, e.g., [5, 8, 9].) A further breakthrough is needed. First, the statistical and stochastic properties of L-H transition must be clarified. This is because $E_r$ and fluctuations do not satisfy the laws of thermodynamical equilibrium. Efforts have been made to establish the far-non-equilibrium statistical law of micro turbulence,[9-15] and the role of nonlinear noise source was found important. The analyses must be extended to the L-H transition phenomena. The other is an experimental test of theories. Experiments have shown recently that the change of $E_r$ occurs in a short time (a few times of $qR/c_s$, $q$: safety factor, $R$: major radius and $c_s$: ion sound velocity) [16] as has been predicted [2]. This supports the model based on hard bifurcation. On the other hand, a test of observing a hysteresis by use of slow change of parameters (longer than the energy confinement time) has not shown clear hysteresis.[17]

In this article, we present a statistical model of the electric bifurcation of the L-H transitions in toroidal plasmas. Nonlinearity of micro-fluctuations statistically induces a random noise of the meso-scale $E_r$. Being kicked by this random noise, transitions between the L- and H-states occur in a probabilistic manner. A Langevin equation is formulated including the mechanism for hysteresis of $E_r$. The probability density function (PDF) of $E_r$ is obtained, and the ensemble average is given. The flux of probability density is calculated, and the transition probability and back-transition probability are obtained. The ensemble average of $E_r$ does not show a hysteresis although a deterministic model includes the hysteresis. The phase boundary of the statistical view is given by the condition that the H-mode and L-mode states have an equal probability. This is an extension of the Maxwell's construction rule. The phase boundary shifts to the ridge of cusp for H-to-L transition, due to the suppression of fluctuation by the $E_r$ shear. The competition between the life time (inverse of the transition probability) and the time for the change of global parameters determines whether the hysteresis is observed experimentally or not.

We consider a thin layer near the tokamak edge. We analyze the dynamics of the radial electric field $E_r$ (averaged over the magnetic surface) in the presence of micro fluctuations. The radial extent of $E_r$ has the scale length $\ell$. (We assume a spatial and temporal scale separation between $E_r$ and micro-fluctuations, i.e., $\ell \gg \rho_p, \delta$.) The dynamical equation for $E_r$ has been expressed in terms of the charge-conservation equation combined with Poisson's equation as
\[ \varepsilon_0 \varepsilon_\perp \frac{d}{dt} E_r = -J_r \equiv -\tilde{J}_r - J^p_r \]

where \( J_r \) is the radial current density, \( \varepsilon_0 \) is the vacuum susceptibility and 
\( \varepsilon_\perp = \left( 1 + 2q^2 \right) c^2 \nu_\perp^2 \) is a dielectric constant of the magnetized toroidal plasma. The radial current has two components: a time-averaged component, \( \bar{J}_r \), and the rapidly-varying part \( J^p_r \). The former, i.e., the time average part (deterministic part) \( \bar{J}_r \) has various origins including the bulk viscosity of ions, ion orbit loss, and eddy damping (or zonal flow excitation) for \( E_r \) by microfluctuations.[5-9] The latter is induced by the convective nonlinearity in the vorticity equation \( \vec{\nabla} \cdot \vec{\nabla} \psi \) associated with micro fluctuations. It changes with the characteristic autocorrelation time of micro-fluctuations \( \tau_{ac} \), which is much shorter than the typical evolution time of \( E_r \). In this article, the term \( J^p_r \) is considered to be a random noise. The time-average part \( \bar{J}_r \) dictates the deterministic picture of bifurcations, and the noise part \( J^p_r \) gives a random kick for \( E_r \) and causes a probabilistic nature in transitions. The scale length \( \ell \) is treated as a constant parameter in the statistical evolution of the magnitude of \( E_r \), and Eq.(1) is rewritten as a Langevin equation of the magnitude of \( E_r \) as

\[ \frac{d}{d\tau} X + \Lambda X = w(\tau) g, \]

where normalization is introduced for the electric field and time as \( X = \varepsilon \rho \varepsilon E / T \) and 
\( \tau = t / 2qR \) and \( w(\tau) \) is a white-noise. \( (\rho \varepsilon : \text{ion gyroradius at poloidal magnetic field, } T : \text{plasma temperature.}) \) The damping term, \( \Lambda X = \left( 1 + 2q^2 \right)^{1/2} \left( qR \varepsilon \rho \varepsilon \right) \bar{J}_r \) is the normalized current. The term \( g \) denotes the noise current \( J^p_r \).

Let us consider the L-H electric bifurcation where the bulk viscosity of ions, ion orbit loss and zonal flow excitation with shear viscosity damping have the key roles.[2, 3, 7, 18] One has

\[ \Lambda X = \text{Im} \left( X + i\nu_\ast \right) \cdot \left( X + X_{nc} \right) + \frac{v_b}{\left( v_b + \alpha X^4 \right)^{1/2}} \exp \left( -\left( v_b + \alpha X^4 \right)^{1/2} \right) \gamma_{zonal} X \]

where \( Z(X) \) is the plasma dispersion function, \( X_{nc} \) is the neoclassical drive and is of the order of \( -\rho \varepsilon n_e^{-1} dn_e / dr \), \( \nu_\ast = v_\ast qRc_s^{-1} \) is the normalized ion collision frequency, 
\( v_b = \varepsilon^{-3/2}\nu_\ast, \varepsilon = a/R, \alpha \) denotes the orbit squeezing [3] and \( \gamma_{zonal} \) is the zonal flow excitation rate combined with shear viscosity damping.[7] Zeros of \( \Lambda \) and relation with L-H transition have been discussed in literature.[2, 3] When \( \Lambda X = 0 \) has one solution, the solution describes either the L-mode state or H-mode state. When multiple solutions exist, the bifurcation has a hysteresis and the hard transition is possible to occur. A thin
curve in Fig. 1(a) illustrates the solution of deterministic model, $\Delta X = 0$, as a function of the gradient on the $X_{nc}$ for a fixed value of $v_p$. Bifurcation and hysteresis of the radial electric field are shown.

The magnitude of $J_r$ is evaluated as follows. The nonlinearly-driven current, $J_r = m_B^2 / (4 \pi^2 \nabla^2)$ (averaged over the magnetic surface), is given as sum of radial-Fourier components, $J_r = \sum_{d_z} J_r(d_z)$, where $d_z$ is a radial wavelength of a randomly-excited current. One component is given as $|J_r(d_z)| = m_B^2 / (4 \pi^2 \nabla^2)$ for electrostatic fluctuations, where $\Phi$ and $k_0$ are the amplitude of electrostatic potential perturbation and a characteristic wave number of micro-fluctuations, respectively. (When the finite-ion-gyro radius effect is included, $\phi$ is screened by a numerical factor.) Time-varying current $J_r(d_z)$ with various values of $d_z$ can be simultaneously excited. Each $d_z$-component $J_r(d_z)$ is considered to be statistically independent, so that an average of the sum of $J_r(d_z)$ over the length $l$ is estimated as $\langle |J_r| \rangle = \sqrt{\langle J_r \rangle} / \langle |J_r| \rangle$ after the law of large numbers, i.e., $|J_r| = m_B^2 / (4 \pi^2 \nabla^2)^{1/2} k_0^2 \Phi^2 / \sqrt{\tau_{ac}} w(t)$.

The fact that $J_r$ changes much faster than $E_r$ enables us to approximate it as a white noise

$$J_r = m_B^2 / (4 \pi^2 \nabla^2)^{1/2} k_0^2 \Phi^2 / \sqrt{\tau_{ac}} w(t),$$

where $\sqrt{\tau_{ac}}$ is explicitly written for the dimension. (A detailed argument of modelling of noise term is given in [11, 14].) When $\tau_{ac}$ is much shorter than the response time of $E_r$, the statistical average of micro-fluctuations is calculated by treating $E_r$ as a constant parameter. In this dc-limit, fluctuation level has been given as $|\Phi|^2 = \left[1 + \omega_B^2 \tau_{ac}^{-1} \right] |\Phi|^2_{\text{L}}$, where $|\Phi|^2_{\text{L}}$ is the fluctuation level in the L-mode state, $\omega_B = B^{-1} dE_r / dr$ is the $E \times B$ shearing rate,[4, 5] Using an evaluation $dE_r / dr = E_r / \ell$, one has $\omega_B^2 \tau_{ac}^{-1} = \tau_{ac}^2 B^{-2} \ell^{-2} E_r^2$. In the following, $|\Phi|^2_{\text{L}}$ and global plasma parameters (like temperature) are treated as control parameters. The amplitude of the noise is a nonlinear function of $X$, and is explicitly given as

$$\xi = \sqrt{\tau_{ac}} \frac{R^2 k_0^2 \Phi^2 / \ell_{\text{L}}^2}{a_R \ell / \ell_{\text{L}}} \frac{1}{1 + U X^2},$$

where $\Phi = \sqrt{\Phi}/T$, $\tau_{ac} = \tau_{ac} \ell / \omega_R$ and $U = (\tau_{ac} \ell / \ell_{\text{L}})^2$.

Statistical property of radial electric field $X$ (the PDF of $X$, $P(X)$, ensemble average, transition probability between the L- and H-modes) is studied. The Fokker-Planck equation of $P(X)$ is deduced from the Langevin equation (2) as

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Fig.1 (a) Electric field $X$ as a function of global gradient $X_{NC}$ (for fixed collision frequency $\nu_s = 0.1$). The solution of $\Delta X = 0$ is characterized by the cusp catastrophe (thin line). Ensemble average of the electric field $\langle X \rangle$ is shown by the thick solid line. Transition in ensemble average takes place at $X_{NC} = X_{NC}^c$. $X_{NC} = X_{NC}^{\infty}$ denotes the condition of $\int J \, dE = 0$. (b) PDF in a stationary state. Solid line is for $X_{NC} = 0.4 \approx X_{NC}^c$, dotted line for $X_{NC} = 0.38$ (L-mode is dominant), and broken line for $X_{NC} = 0.42$ (H-mode is dominant). (Parameters are: $\alpha = 0.5$, $q = 3$, $\varepsilon = a/R = 1/3$, $U = 3$ and $\Gamma = 5$.)

\[
\frac{\partial}{\partial \tau} P + \frac{\partial}{\partial X} \left( \Lambda + g \frac{\partial}{\partial X} g \right) P = 0.
\]  

(6)

The stationary solution $P_{eq}(X)$ is expressed as $P_{eq}(X) \propto g^{-1} \exp \left( -S(X) \right)$ by use of the nonlinear potential as

\[
S(X) = \int^X 4AXg^{-2} \, dX.
\]  

(7)

The minimum of $S(X)$ (apart from a correction $\ln g$), i.e., zero of $\Lambda$, predicts the probable state of $X$. The probable state agrees with the one in deterministic model, and the dominant (i.e., the most probable) state is determined by the statistical theory in which noise is kept. $S(X)$ can have two minima at $X = X_L$ and $X = X_H$, which are separated by the local maximum at $X = X_m$. Figure 1(b) illustrates PDF $P_{eq}(X)$ for various values of parameter $X_{NC}$. The PDF has two peaks, representing the hysteresis. However, the state $X = X_L$ is dominant if $X_{NC} < X_{NC}^{\infty}$ holds ($X_{NC}^{\infty} = 0.4$ for the parameters or Fig.1(a)), and $X = X_H$ is dominant if $X_{NC} > X_{NC}^{\infty}$. The ensemble average $\langle X \rangle = \int X P_{eq}(X) \, dX$ changes smoothly as the global control parameter varies. The ensemble average $\langle X \rangle$ is illustrated by a thick curve in Fig.1(a). When one solution of bistable branches is chosen
as an initial condition, many transitions in between $X_H$ and $X_L$ branches occur in a long time, and $P(X)$ reaches to $P_{eq}(X)$.

The transition probability is obtained by calculating a flux of probability density from the Fokker-Planck equation (6), and is expressed by use of the potential $S(X)$. [13, 15] The probabilities of the L-to-H transition and back-transition are given as

$$r_{L \rightarrow H} = \frac{\sqrt{\Lambda_L \Lambda_m}}{2\pi} \exp \left( S(X_L) - S(X_m) \right), \quad (8a)$$

$$r_{H \rightarrow L} = \frac{\sqrt{\Lambda_H \Lambda_m}}{2\pi} \exp \left( S(X_H) - S(X_m) \right), \quad (8b)$$

respectively, where the time rates $\Lambda_{L,m,H}$ are given as $\Lambda_{L,m,H} = 2X \left| \frac{\partial \Lambda}{\partial X} \right|$ at $X = X_{L,m,H}$. Note that the time rates are normalized and $\Lambda_{L,m,H}$ are of the order unity.

Therefore, the dominant dependence of the transition probability comes from the exponential parts in Eq.(8). By use of Eq.(5), the transition probability is explicitly evaluated as,

$$S(X_L) - S(X_m) = -\Gamma I_L = -\Gamma \int_{X_m}^{X_L} \Lambda X \left( 1 + UX^2 \right)^2 dX, \quad (9a)$$

$$S(X_H) - S(X_m) = -\Gamma I_H = -\Gamma \int_{X_m}^{X_H} \Lambda X \left( 1 + UX^2 \right)^2 dX \quad (9b)$$

with the coefficient $\Gamma = 2 \frac{\tau_{ac}^{-1} a^2}{\ell_c R^{-4} k_0^{-1} \rho_T^{-4} \Phi^{-4}}$. Substitution of Eq.(9) into Eq.(8) provides the transition probability and back-transition probability as

$$r_{L \rightarrow H} = \sqrt{\Lambda_L \Lambda_m} \left( 2\pi \right)^{-1} \exp \left( -\Gamma I_L \right), \quad \text{and} \quad r_{H \rightarrow L} = \sqrt{\Lambda_H \Lambda_m} \left( 2\pi \right)^{-1} \exp \left( -\Gamma I_H \right).$$

Integrals $I_H$ and $I_L$ are calculated and are of the order unity. (Details are explained in a full paper [19].)

The phase boundary between the L-mode and H-mode (e.g., $X_{NC}^c$) is defined by the condition that both are equally observed. The probability that the state is found to stay in the L-state is given as $P_L = r_{H \rightarrow L} / \left( r_{L \rightarrow H} + r_{H \rightarrow L} \right)$. That for the H-state is given by $P_H = r_{L \rightarrow H} / \left( r_{L \rightarrow H} + r_{H \rightarrow L} \right)$. The equal-probability condition, $r_{L \rightarrow H} = r_{H \rightarrow L}$, is given from Eq.(8) as

$$S(X_H) = S(X_L) + \frac{1}{2} \ln \left( \Lambda_L / \Lambda_H \right). \quad (10)$$
Apart from a weak logarithmic term, it is approximated as \( S(X_H) = S(X_L) \), i.e.,

\[
\int_{X_H}^{X_L} \Delta X \left( 1 + UX^2 \right)^2 dX = 0.
\] (11)

This result is an extension of the Maxwell's construction rule. When the noise is independent of \( X \), Eq.(11) gives that the condition \( \int_{X_H}^{X_L} \Delta X dX = 0 \) describes the boundary of phases. \( \int \Delta X dX \) corresponds to a work function \( \int J dE \), and Maxwell's construction is deduced. The correction of \( UX^2 \) in the integrand (turbulence suppression term) is important in the H-mode \( X = X_H \). By this effect, the phase boundary of ensemble average (\( X_{NC}^e \) in Fig.1(a)) deviates from the conventional criterion (denoted by \( X_{NC}^{es} \) in Fig.1(a)), and the region of the H-mode becomes wider. It is noted that the boundary does not depend on the magnitude of fluctuations. A phase diagram in a control parameter space \( (v_h, X_{NC}) \) is obtained and is explicitly given in a separate paper.[19]

We further address the problem whether a hysteresis in \( X \) is observed in experiments or not. The ensemble-averaged value \( \langle X \rangle \) is reached if the averaging time is longer than the life time, \( \tau_{life} \rightarrow H = 1 / r_L \rightarrow H \). The observation of hysteresis critically depends on the ratio between the life time of one state and the time of global parameter change, \( \tau_{global} \). If \( \tau_{global} \gg \tau_{life} \) holds, two states are well equilibrated via abundant transitions. The averaged observations do not depend on from where the global parameters have evolved. In the case of \( \tau_{global} \sim \tau_{life} \), the observed results strongly depend on from which branch the parameters have evolved, i.e., from the L-mode or from the H-mode; The hysteresis in the response of \( X \) to the global parameters is observed.

In summary, the statistical theory of the \( E \) bifurcation in the edge of tokamaks was analyzed. Micro fluctuations induce a random noise for the transition to occur in a probabilistic manner, when a hysteresis is predicted in the deterministic model analysis. The PDF and ensemble average of \( E \) were obtained. The probability of L/H transition was obtained, and a life-time of each state was calculated. The phase boundary of two states was given by the equal-probability condition for the H-and L-states. Owing to the suppression effect on turbulent noise by the \( E \), shear, the boundary was found to deviate from the Maxwell's construction rule. Implications to experiments are as follows: First, the appearance of H-mode in plasma parameters must be judged by the ensemble averages of statistical models which have a noise source, not by a value of deterministic model. (See Fig.1(a).) Due to the noise, each transition occurs being scattered around the ensemble average. This must be noticed in the future comparison of experimental database with theories. Second, the ensemble average \( \langle X \rangle \) does not show a hysteresis.
against global parameters $X_{nc}$, even though a deterministic model predicts the hysteresis. (Ensemble average of heat flux is also obtained. This is discussed in a full paper [19].) Third, the observation of hysteresis in experiments critically depends on the speed of global parameter change: this is another feature which characterizes the non-equilibrium properties. Fourth, the probabilistic onsets may change the occurrence of dithering between H- and L-states; In conjunction with it, coupled dynamics with global pressure gradient are discussed in [19]. In this article, the model of Eq.(3) was taken to show a typical example of probabilistic transition. Other mechanisms have been known to influence L-H transitions. [9] The inclusion of zonal flow excitation in statistical theory [14] or the coupling of dynamics of different scale lengths [20] must be investigated for quantitative analysis of tokamak plasmas, and are left for future studies.

Authors wish to acknowledge Dr. M. Yagi, Dr. Y. Miura, Prof. A. Fukuyama and Prof. A. Yoshizawa for useful discussions. This work is partly supported by the Grant-in-Aid for Scientific Research of MEXT Japan and by the collaboration programmes of National Institute for Fusion Science and of the Research Institute for Applied Mechanics of Kyushu University.

REFERENCE