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On the bicoherence analysis of plasma turbulence

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Abstract

The bicoherence of fluctuations in a system of drift waves and zonal flows is discussed. In strong drift-wave turbulence, where broad-band fluctuations are excited, the bicoherence is examined. A Langevin equation formalism of turbulent interactions allows us to relate the bicoherence coefficient to the projection of nonlinear force onto the test mode. The dependence of the summed bicoherence on the amplitude of zonal flows is clarified. The importance of observing biphase is also stressed. The results provide a basis for measurement of nonlinear interaction in a system of drift waves and zonal flow.

Keywords: bicoherence, biphase, drift wave turbulence, zonal flows, geodesic acoustic modes, data analysis

I. Introduction

Plasma turbulence has been subject to intensive study in last decades [1-3]. This is because the turbulent transport is a key in realizing the controlled thermonuclear fusion, and is because the plasma turbulence plays a key role in structure formation. Such efforts in understanding the structure formation in laboratory as well as natural plasmas are explained in, e.g. [4-6].

In addition to progress in theoretical understanding of plasma turbulence, efforts have also been focused to the direct measurement of the elementary nonlinear interactions. The identification of mesoscale structures (such as zonal flow [4] and geodesic acoustic modes, GAMs [7]) and their interaction with ambient turbulence is a highlight of the experimental study of plasma turbulence. The identification of a mesoscale zonal flow has been in progress [8], and the efforts in measurement the nonlinear interactions are also on-going. One routine method in measuring the nonlinear interactions among the fluctuating quantities is the bicoherence method [9, 10]. This allows us to measure the strength and spectra of triplet correlations. The application of this method to plasma turbulence has been widely discussed [11-19]. Very recently, the bicoherence method is applied to the experimental study of GAMs and background turbulence [19]. Although the bicoherence method is routinely applied to the plasma physics experiments, the interpretation of the bicoherence data has not been thoroughly considered. The progress of modelling plasma turbulence, and, in particular, the importance of the nonlinear interaction between the mesoscale structure and broad band turbulence has stimulated the efforts to understand the measurement of the bicoherence of signals [15]. More detailed study in understanding bicoherence data is required.

In this article, we discuss the bicoherence of plasma turbulence in the presence of broad band drift wave turbulence. The assumption of a large-degree-of freedom has given a formulation of Langevin equation of a dressed-test mode [20]. Based on this picture, the bicoherence of fluctuating fields is formulated as a projection of the nonlinear force onto the dressed-test mode. Bicoherence coefficients are evaluated in terms of the spectrum of the fluctuating field, the coefficient of nonlinear interaction, and the autocorrelation time of the fluctuations. Two cases are investigated. The first is the case where a large number of unstable modes are excited and are in a stationary state due to the mutual nonlinear interactions. The second example is when the zonal flow and GAMs exist in broad band fluctuations. Properties of bicoherence data are explained. A clear contrast of biphase between these two cases are demonstrated. A brief discussion on the statistical convergence is also presented. This study provides an interpretation of the bicoherence signal in understanding the nonlinear interaction process.

II. Response of Test Wave which is Target of the Experimental Study

An example of the dynamical equations of fluctuation fields, in the range of drift wave frequency of strongly magnetized plasmas, is expressed as [2]. Among many issues in the nonlinear processes of drift wave turbulence, the importance of the $E \times B$ nonlinearity and the phase relations between different fluctuating quantities (such as the density and electric field) have been recognized. The former is essential in nonlinear stabilizing process of drift waves and in driving zonal flows from drift wave fluctuations. The latter is the key for driving turbulence and turbulent transport. The details of the theories covering both mechanisms are explained in [6]. Despite the importance of cross-correlation function between different fluctuating fields, focus is made on the $E \times B$ nonlinearity in this article, and the fluctuating fields are represented by a scalar variable g (such as electrostatic potential). This simplification is accepted as the first step, because this nonlinearity has essential role in the interaction of the drift wave and zonal flow. One can use a one-field model such as Hasegawa-Mima equation [21]. The nonlinear dynamical equation may be written in a form

$$\frac{\partial}{\partial t} g + \left(-\gamma + iL_0\right) g = \sum N\, gg \ . \tag{1}$$

where γ is a linear growth rate, L_0 represents the linear frequency, and N denotes the coefficient of nonlinear interaction. N may include operators, as is explicitly shown in §3.3.

In this chapter, we discuss a response of a test mode against a nonlinear interaction between a particular pair of modes in turbulent fluctuations which are composed of a large number of excited modes. This response is a basis for clarifying the relation between the bicoherence and nonlinearity in dynamical equations,

The nonlinear terms for drift wave turbulence are modelled as

$$\sum N\, gg = -\nu_T\, g + \tilde{S} \tag{2}$$

where ν_T is the nonlinear damping rate of the target mode and $\tilde{S}$ is a random fluctuating force (noise) [2, 20]. It has been shown that the nonlinear term can be separated into the memory term (coherent term) which is coherent to g and into the fluctuating force (incoherent term), the projection of which onto g vanishes in a long time average [22]. The spectral functions satisfy the relation [2,3,20]

$$\left\langle g^2 \right\rangle = \frac{\left\langle \tilde{S}\, \tilde{S} \right\rangle}{2\left(\nu_T - \gamma\right)} \ . \tag{3}$$

The eddy-damping rate ν_T is a function of the turbulence spectrum as is explained in [2, 3].

The response of a test mode against a nonlinear effect from a particular pair of modes is deduced by use of Eq.(2). The fluctuation spectrum is expressed, in general, by the space-time Fourier decomposition, e. g., a power spectrum $I(k, \omega)$. However, such a complete dataset is not easily accessible, because experimental data are usually taken by one-point (or few-points) measurements. The bicoherence analysis of experimental data has often been performed on the temporal Fourier spectrum. Such studies have relevance for cases where the condition of the wavenumber matching is approximately satisfied if the frequency matching condition is fulfilled. In studies where only frequency spectrum is used, the effective nonlinear interaction is analyzed, in which matching conditions of wavenumbers are treated as an average. Although limited in accuracy, this simplified data analysis has a relevance in investigating the interactions between drift waves and zonal flows as a first step. Considering these experimental situations, we introduce Fourier components as

$$g(t) = \sum_p g_p \exp(-ipt) . \tag{4}$$

We choose one particular frequency ω for a test wave g_ω. An imposition of the test mode g_ω affects the p-Fourier component g_p via the nonlinear term $N_{p,\omega} g_{p-\omega} g_\omega$. (Note again that the matching conditions of wavenumbers are included (as an average) in calculating the nonlinear coupling coefficient $N_{p,\omega}$.) A response of g_p to the imposition of the nonlinear term g_ω is evaluated as follows [1]. We separate one term $N_{p,\omega} g_{p-\omega} g_\omega$ from the total nonlinear terms $\sum N \tilde{g}\tilde{g}$, and express the rest in terms of the nonlinear damping term and fluctuating force as

$$\sum N \tilde{g}\tilde{g} - N_{p,\omega} g_{p-\omega} g_\omega e^{-ipt} = -\nu'_T g + \tilde{S}' . \tag{5}$$

according to the same theoretical argument that is used in deriving Eq.(2). The response of g_p against the imposition of g_ω is written as

$$\frac{\partial}{\partial t} g_p + \left(\nu'_T - \gamma + i(L_0 - p)\right) g_p = N_p g_{p-\omega} g_\omega + \tilde{S}'_p . \tag{6}$$

This process has been employed in statistical theories (see, e.g., [1,2].) The meaning of this equation is that, in calculating the dynamics of g_p, the nonlinear effects except $N_{p,\omega} g_{p-\omega} g_\omega$ are combined into $\nu'_T g$ and $\tilde{S}'$. That is, the LHS of Eq.(6) denotes the response of the "dressed mode". Because of a large degrees of freedom of fluctuations excited in the plasma of interest, we employ the test wave approximation

$$\nu_T^{'} \simeq \nu_T \tag{7a}$$

and

$$\tilde{S}^{'} \simeq \tilde{S} \ . \tag{7b}$$

The meaning of Eq.(7) is that $\sum N\,\tilde{g}\tilde{g}$ and $\sum N\,\tilde{g}\tilde{g} - N_{p,\omega} g_{p-\omega} g_{\omega} e^{-ipt}$ are approximately equal to each other because of the large numbers of modes are excited in broad-band turbulence. The concept of the dressed mode and the validity of the approximation is discussed in [1]. Equation (6) is solved as

$$g_p = \exp\left(-\hat{\nu}_p\, t\right)\int_{-\infty}^{t} \mathrm{d}t' \exp\left(\hat{\nu}_p\, t'\right) N_{p,\,\omega} g_{p-\omega} g_{\omega} + \tilde{g}_p \ , \tag{8a}$$

and

$$\tilde{g}_p = \exp\left(-\hat{\nu}_p\, t\right)\int_{-\infty}^{t} \mathrm{d}t' \exp\left(\hat{\nu}_p\, t'\right) \tilde{S}_p^{'} \ , \tag{8b}$$

where $\hat{\nu}_p = \nu_{T,\,p} - \gamma_p + i\left(L_0 - p\right)$ with the help of Eq.(7a). The first term in the RHS of Eq.(8a) represents the response against the imposition of the test mode, and $\tilde{g}_p$ represents the response against the noise excitation through nonlinear interaction with other modes. (Equation (8b) is a Brownian motion if $\tilde{S}$ is Gaussian white noise.) Because Eq.(7b) holds for the broad band turbulence, an approximate relation

$$\left\langle \left| g_p \right|^2 \right\rangle \simeq \left\langle \left| \tilde{g}_p \right|^2 \right\rangle \tag{9}$$

holds.

The integrand $g_{p-\omega} g_{\omega}$ in Eq.(8a) loses the coherence as $t - t^{'}$ becomes longer than the autocorrelation time, $\tau_{a,\,p} = \min\left(\tau_{c,\,\omega}, \tau_{c,\,p-\omega}\right)$, where $\tau_{c,\,p}$ is the autocorrelation time of the fluctuation $\tau_{c,\,p}^{-1} = \nu_{T,\,p} - \gamma_p$. By noting this fact, the integral in Eq.(8a) is evaluated as

$$\exp\left(-\hat{\nu}_p\, t\right)\int_{-\infty}^{t} \mathrm{d}t' \exp\left(\hat{\nu}_p\, t'\right) N_{p,\,\omega} g_{p-\omega} g_{\omega} \simeq \tau_p\, N_{p,\,\omega} g_{p-\omega} g_{\omega} \tag{10}$$

with

$$\tau_p = \hat{\nu}_p^{-1}\left\{1 - \exp\left(-\hat{\nu}_p\,\tau_{a,\,p}\right)\right\} \tag{11}$$

That is,

$$g_p \simeq \tau_p\, N_{p,\,\omega} g_{p-\omega} g_\omega + \tilde{g}_p \ . \tag{12}$$

A similar argument applies to $g_{p-\omega}$, and we have

$$g_{p-\omega} \simeq \tau_{p-\omega}\, N_{p-\omega,\,\omega} g_p g_\omega^* + \tilde{g}_{p-\omega} \tag{13}$$

where the relation $g_{-\omega}^* = g_\omega$ is used.

Equations (12) and (13) show the responses of the Fourier components g_p and $g_{p-\omega}$ against the imposition of the test mode g_ω . The amplitude g_p is separated into $\tau_p\, N_{p,\,\omega} g_{p-\omega} g_\omega$ and $\tilde{g}_p$; the former is the result of the nonlinear interaction $g_{p-\omega} g_\omega$, and the latter, $\tilde{g}_p$, is statistically independent from the former.

III. Bicoherence Analysis

The bispectrum estimator $\hat{B}(\omega,\, p)$, the squared bicoherence $\hat{b}^2(\omega,\, p)$, and the summed-bicoherence $\sum \hat{b}^2$ are defined as

$$\hat{B}(\omega,\, p) = \left\langle g_p^* g_{p-\omega} g_\omega \right\rangle \ , \tag{14}$$

$$\hat{b}^2(\omega,\, p) = \frac{\left|\hat{B}(\omega,\, p)\right|^2}{\left\langle\left|g_p g_{p-\omega}\right|^2\right\rangle\left\langle\left|g_\omega\right|^2\right\rangle} \ , \tag{15}$$

and

$$\sum \hat{b}^2(\omega) = \sum_p \hat{b}^2(\omega,\, p) \ . \tag{16}$$

We see that this bispectrum estimator is in proportion to the projection of the response g_p to the nonlinear force $N_p g_\omega g_{p-\omega}$. Relations between the bicoherence and nonlinear interactions are discussed in this chapter.

A. Case of broad band turbulence

We first study the case where fluctuations are composed of broad band spectrum as is shown in Fig.1(a). In this case all of three components, g_p , $g_{p-\omega}$ and g_ω , follow similar relations like Eqs.(12) and (13). We have

$$g_\omega \simeq \tau_\omega N_{\omega,p} g_p g_{p-\omega}^* + \tilde{g}_\omega \ . \tag{17}$$

From Eqs.(12), (13) and (17), the bicoherence is expressed in terms of the nonlinear terms. The derivation is given in the Appendix A and the results are summarized here.

1. Summary of results

<u>*Bispectrum indicator*</u>

The bicoherence indicator , which is the third order correlation function, is expressed in terms of the second-order correlation functions and the nonlinear coupling coefficient N as

$$\hat{B}(\omega, p) = \tau_p N_{p,\omega}^* \left\langle \left| g_{p-\omega} \right|^2 \left| g_\omega \right|^2 \right\rangle \\ + \tau_{p-\omega} N_{p-\omega,p} \left\langle \left| g_p \right|^2 \left| g_\omega \right|^2 \right\rangle + \tau_\omega N_{\omega,p} \left\langle \left| g_{p-\omega} \right|^2 \left| g_p \right|^2 \right\rangle \ . \tag{18}$$

In order to have more explicit interpretations, we employ an estimate for the RHS of Eq.(18). Three terms of spectral functions in the RHS of Eq.(18), $\left\langle \left| g_{p-\omega} \right|^2 \left| g_\omega \right|^2 \right\rangle$, $\left\langle \left| g_p \right|^2 \left| g_\omega \right|^2 \right\rangle$, and $\left\langle \left| g_{p-\omega} \right|^2 \left| g_p \right|^2 \right\rangle$, depend on p and ω but have similar magnitude for broad band fluctuations. One can have simplified evaluation as

$$\hat{B}(\omega, p) \simeq \left(\tau_p N_{p,\omega}^* + \tau_{p-\omega} N_{p-\omega,p} + \tau_\omega N_{\omega,p} \right) \left\langle \left| g_{p-\omega} \right|^2 \left| g_p \right|^2 \right\rangle \ . \tag{19}$$

<u>*Squared bicoherence*</u>

Substitution of Eq.(19) into Eq.(15) gives the squared bicoherence. In order to have comparisons with experimental observations, a crude estimate,

$$\frac{\left\langle \left| g_{p-\omega} \right|^2 \left| g_p \right|^2 \right\rangle}{\left\langle \left| g_\omega \right|^2 \right\rangle} \simeq \left\langle \left| g_p \right|^2 \right\rangle \ . \tag{20}$$

is employed here. This approximation is employed when g_ω , g_p and $g_{p-\omega}$ belongs to the broad-band spectrum. As is shown in the Appendix A, the squared bicoherence is given, with the help of Eq.(20), as

$$\hat{b}^2(p,\omega) = \left| \tau_p N^*_{p,\omega} + \tau_{p-\omega} N_{p-\omega,p} + \tau_\omega N_{\omega,p} \right|^2 \left\langle \left| g_p \right|^2 \right\rangle . \tag{21}$$

If phases between $\tau_p N^*_{p,\omega}$, $\tau_{p-\omega} N^*_{p-\omega,p}$ and $\tau_\omega N^*_{\omega,p}$ are randomly distributed, one has

$$\begin{aligned} &\left| \tau_p N^*_{p,\omega} + \tau_{p-\omega} N_{p-\omega,p} + \tau_\omega N_{\omega,p} \right|^2 \\ &\sim \left| \tau_p N^*_{p,\omega} \right|^2 + \left| \tau_{p-\omega} N_{p-\omega,p} \right|^2 + \left| \tau_\omega N_{\omega,p} \right|^2 \sim 3 \left| \tau_p N^*_{p,\omega} \right|^2 . \end{aligned} \tag{22}$$

The randomness of phases between $\tau_p N^*_{p,\omega}$, $\tau_{p-\omega} N^*_{p-\omega,p}$ and $\tau_\omega N^*_{\omega,p}$ are discussed for the case of drift wave turbulence in §3.3.

Summed bicoherence

Equations (21) and (22) provide the expression for the summed bicoherence

$$\sum \hat{b}^2(\omega) \sim \sum_p 3 \left| \tau_p N^*_{p,\omega} \right|^2 \left\langle \left| g_p \right|^2 \right\rangle \sim 3 \left| \tau_p N^*_{p,\omega} \right|^2 |g|^2 , \tag{23}$$

with

$$|g|^2 = \sum_p \left\langle \left| g_p \right|^2 \right\rangle . \tag{24}$$

2. Interpretation

Equation (18) shows that the magnitude of $\hat{B}(\omega, p)$ is an indicator of the nonlinear force. The bispectrum estimator is composed of the terms which are proportional to the projection of the nonlinear term $g_{p-\omega} g_\omega$ onto the response of g_p to the nonlinear force $N_{p,\omega} g_{p-\omega} g_\omega$. Thus, the bispectrum indicator provides the evaluation of nonlinear interaction in observed data. The squared bicoherence shows the magnitude of the three-mode interaction.

In addition, Eq.(19) shows that the phase of $\hat{B}(\omega, p)$, the biphase, is directly related to the phase of the nonlinear coefficient $N^*_{p,\omega}$. The biphase indicates the phase of $\tau_p N^*_{p,\omega}$. That is, the biphase shows the relation between the nonlinear force and the test mode. Thus, the magnitude as well as the biphase give information about aspects of the nonlinear interactions. For instance, the measurement of the phase of $\hat{B}(\omega, p)$ gives the phase of N once the real frequency and the decorrelation rate are measured.

The interpretation of Eq.(23) is as follows. The term $|N_{p,\omega}||g|$ represents a nonlinear force (in a normalized unit in a dimension of the 'frequency'), and $\tau_p |N_{p,\omega}||g|$ indicates the competition between this nonlinear force and the effective correlation time τ_p. Equation (23) is rewritten as

$$|N_{p,\omega}| \simeq \frac{1}{\sqrt{3}\tau_p |g|}\sqrt{\sum \hat{b}^2(\omega)} \ . \tag{25}$$

The RHS is composed of three terms, $|g|$, τ_p and $\sum \hat{b}^2(\omega)$. The fluctuation level $|g|$ is measurable, and the correlation time τ_p is evaluated by the autocorrelation time $\tau_{c,p}$, which is measured from fluctuation data. Thus, once the summed bicoherence $\sum \hat{b}^2(\omega)$ is measured, the magnitude of the nonlinear coupling coefficient $|N_{p,\omega}|$ is evaluated.

B. Case of a sharp peak within a broad band fluctuations

When the drift wave fluctuations coexist with the mesoscale fluctuation, such as zonal flow or geodesic acoustic modes (GAMs), the interaction between the modes in the sharp peak and broad band fluctuations attracts attentions. Here, the suffix ω indicates the mode which belongs to the sharp peak of the spectrum, and $(p, p-\omega)$ denotes the broad band background turbulence. (See Fig.1(b).) The test mode in a sharp peak is denoted by ω here.

1. Response of a test mode

The amplitude of the modes in a sharp peak is considered to be strongly influenced by a self-nonlinear interaction, not solely determined by the fluctuating force from broad band turbulence. In the case of zonal flow dynamics, the negative eddy-viscosity-like effect by the drift wave turbulence destabilizes the zonal flows, contrary to the case of drift waves for which Eq.(2) is used. Self-interaction is effective for the saturation of the zonal flow [6]. We introduce the amplitude of the sharp spectral mode, $g_{\omega,0}$, which is assumed to be determined by the self-nonlinear effects and by the excitation by turbulence force $\sum_{q \neq p} N g_q g^*_{q-\omega}$ (i.e., the $g_p g^*_{p-\omega}$ term is subtracted). Imposing the nonlinear interaction term $g_p g^*_{p-\omega}$ on the test mode, and one has the response of g_ω after the similar procedure that gives Eq.(12). Thus,

$$g_\omega \simeq \tau_\omega N_{\omega,\, p} g_p g^*_{p-\omega} + g_{\omega,\, 0} \ , \tag{26}$$

where the first term in the RHS represents the response against the beat interaction $N_{\omega,\, p} g_p g^*_{p-\omega}$, and τ_ω is calculated after Eq.(11). The autocorrelation time of the test mode $\tau_{c,\, \omega}$ is much longer than those of background turbulence, $\tau_{c,\, p-\omega}$, so that τ_ω in Eq.(26) is replaced by the autocorrelation time of background fluctuations $\tau_{c,\, p-\omega}$. That is, one has an expression

$$g_\omega \simeq \tau_{c,\, p-\omega} N_{\omega,\, p} g_p g^*_{p-\omega} + g_{\omega,\, 0} \ . \tag{27}$$

In other words, g_ω is composed of a component $g_{\omega,\, 0}$ (which is independent of $g_p g^*_{p-\omega}$) and a fluctuating component owing to the kick $g_p g^*_{p-\omega}$.

2. Bicoherence

The bicoherence is given from Eqs.(12), (13) and (27) as is explained in the Appendix B. The result is summarized here.

<u>*Bicoherence indicator*</u>

The Bicoherence indicator is evaluated as

$$\hat{B}(\omega,\, p) = \left\langle g^*_p g_{p-\omega} g_\omega \right\rangle = \left(\tau_p N^*_{p,\, \omega} \left| g_{p-\omega} \right|^2 + \tau_{p-\omega} N_{p-\omega,\, p} \left| g_p \right|^2 \right) \left| g_{\omega,\, 0} \right|^2$$
$$+ \tau_{c,\, p-\omega} N_{\omega,\, p} \left| g_{p-\omega} \right|^2 \left| g_p \right|^2 \ . \tag{28}$$

The first term (with parenthesis) in the RHS of Eq.(28) is due to the modulation of the background fluctuation by the imposition of the test mode (e. g., zonal flow). The last term in the RHS comes from the influence on the test mode by back-ground fluctuations. As is explained in the next section, the phases of $\tau_p N^*_{p,\, \omega}$ and $\tau_{p-\omega} N_{p-\omega,\, p}$ are close to each other for the interaction between the zonal flow and drift wave fluctuations [6]. Based on the estimate $\tau_p N^*_{p,\, \omega} \left| g_{p-\omega} \right|^2 \simeq \tau_{p-\omega} N_{p-\omega,\, p} \left| g_p \right|^2$ for components $g_{p-\omega}$ and g_p , which belong to the broad-band spectrum, a simplified form of $\hat{B}$ may be used for convenience as

$$\hat{B}(\omega,\, p) \simeq 2\tau_{p-\omega} N^*_{p-\omega,\, p} \left| g_p \right|^2 \left| g_{\omega,\, 0} \right|^2 + \tau_{c,\, p-\omega} N_{\omega,\, p} \left| g_{p-\omega} \right|^2 \left| g_p \right|^2 \tag{29}$$

The term which is proportional to $\left| g_{\omega, 0} \right|^2$ in the Bicoherence indicator has been pointed out in [15]. The second term is the contribution of the broad band turbulence, and $\hat{B}(\omega, p)$ at ω does not vanish even in the limit of $\left| g_{\omega, 0} \right|^2 \to 0$.

The squared bicoherence

A simplified expression for the squared bicoherence is given in Appendix B as

$$\hat{b}^2(\omega, p) \simeq 4\tau_{p-\omega}^2 \left| N_{p-\omega, p} \right|^2 \left| g_{\omega, 0} \right|^2 + 4\tau_{p-\omega}\tau_{c, p-\omega} \,\mathrm{Re}\left(N_{p-\omega, p} N_{\omega, p}^{*} \right) \left| g_p \right|^2$$
$$+ \left| \tau_{c, p-\omega} N_{\omega, p} \right|^2 \left| g_{p-\omega} \right|^2 \left| g_p \right|^2 \left\langle \left| g_\omega \right|^2 \right\rangle^{-1} . \qquad (30)$$

In obtaining Eq.(30), the approximate relations $\left\langle \left| g_p g_{p-\omega} \right|^2 \right\rangle \simeq \left\langle \left| g_{p-\omega} \right|^2 \right\rangle \left\langle \left| g_p \right|^2 \right\rangle$ and $\left\langle \left| g_{p-\omega} \right|^2 \right\rangle \simeq \left\langle \left| g_p \right|^2 \right\rangle$ are used, because $g_{p-\omega}$ and g_p belong to the broad-band spectrum. In addition, $\left| g_{\omega, 0} \right|^2 \simeq \left\langle \left| g_\omega \right|^2 \right\rangle$ is employed as is in Eq.(9).

Summed bicoherence

The summed bicoherence coefficient is then expressed as

$$\sum \hat{b}^2(\omega) = 4\, M\, \overline{\tau_{p-\omega}^2 \left| N_{p-\omega, p} \right|^2} \left| g_\omega \right|^2 + 4\overline{\tau_{p-\omega}\tau_{c, p-\omega} \,\mathrm{Re}\left(N_{p-\omega, p} N_{\omega, p}^{*} \right)} \left| g \right|^2$$
$$+ \Delta g_\omega^2 \left\langle \left| g_\omega \right|^2 \right\rangle^{-1} , \qquad (31)$$

where M is the number of Fourier component, $M = \sum_p 1$, the over-bar $-$ is an average over the Fourier component, $M\, \overline{\tau_{p-\omega}^2 \left| N_{p-\omega, p} \right|^2} = \sum \tau_{p-\omega}^2 \left| N_{p-\omega, p} \right|^2$, $\left| g^2 \right|$ is the level of background turbulence as is given in Eq.(24), and Δg_ω^2 is the variance of the amplitude of g_ω, owing to the kicks from background fluctuations,

$$\sum_p \left(\tau_{c, p-\omega} N_{\omega, p} \right)^2 \left| g_{p-\omega} \right|^2 \left| g_p \right|^2 = \Delta g_\omega^2 . \qquad (32)$$

(Note that the variation Δg_ω^2 is defined in a time scale which is longer than $\tau_{c, p}$ but is shorter than the time scale that $g_{\omega, 0}$ varies.)

3. Interpretation

It should be emphasized that the phase of the Bicoherence indicator $\hat{B}(\omega, p)$ in Eq.(29) can be different from that for the case of broad-band turbulence. For instance, when ω is chosen to be the frequency of zonal flows (zonal flow, GAMs), the phase of the $\hat{B}(\omega, p)$ weakly depends on p. This is particularly noticeable when one study the coupling between the drift wave and zonal flows.

The result Eq.(31) is interpreted as follows. (i) First, the summed bicoherence $\sum \hat{b}^2(\omega)$ has a sharp and broad components: The first term in the RHS indicates a peak in the summed bicoherence, and the second and third terms a broad distribution in a wide frequency region. That is, the peak in the summed bicoherence appears at the peak of the power spectrum. (ii) Second, the magnitude of the peak in the summed bicoherence is in proportion to the magnitude of the mode, the nonlinear interaction coefficient, $\left|N_{p-\omega, p}\right|^2$, the autocorrelation time of the background fluctuations, and by the number of Fourier components, M, which are used in the data analysis. The first term in the RHS of Eq.(31), which comes from the modulation of background drift wave fluctuations by imposed zonal flows, is proportional to M. This is because the majority of the drift waves responds to the imposed quasi-coherent oscillation in a similar way. As a result of this, the summed bicoherence becomes larger as the number of Fourier components increases. (iii) Third, the detection this first term of Eq.(31) is possible, in the data analysis, as follows: When the peak in the summed bicoherence $\sum \hat{b}^2(\omega)$ is obtained, (a) the dependence of $\sum \hat{b}^2(\omega)$ on the amplitude of the sharp mode $\left|g_\omega\right|^2$ must be studied, and (b) the peak height of $\sum \hat{b}^2(\omega)$ must be investigated by observing the effects of the choice of M. (iv) Fourth, the second term is the contribution of the broad band fluctuations, and is given by the same response as Eq.(23). The difference in the numerical coefficients 3 and 4 is due to (i) the difference in the number of combinations, and to (ii) the difference in the phase difference among nonlinear coefficients. (v) Fifth, the last term is a small correction, when the self-nonlinear effects for g_ω is strong.

Some further comment may be made on the peak of the summed bicoherence. When the peak is apparent in $\sum \hat{b}^2(\omega)$, it is approximated as

$$\sum \hat{b}^2(\omega) \sim 4\, M\, \overline{\tau_{p-\omega}^2 \left|N_{p-\omega, p}\right|^2} \left|g_\omega\right|^2 \tag{33}$$

in the vicinity of the peak of $\sum \hat{b}^2(\omega)$. The Fourier decomposition is usually made as discretizing the frequency rage as $p = n\Delta\omega$, where $\Delta\omega$ is the width of the frequency, and $n = 0 \pm 1, \pm 2, \cdots \pm M$. When the half-width of the test mode at frequency ω is narrower than $\Delta\omega$, then the peak in the Fourier series $\left|g_\omega\right|^2$ does not depend on the choice of $\Delta\omega$. In this case, if one performs a convergence study such as increasing M and decreasing $\Delta\omega$, the peak value of $\sum \hat{b}^2(\omega)$ is in proportion to M. If $\Delta\omega$ is smaller than the half-width of the test mode, then $M|g_\omega|^2$ converges to a finite number. Then $\sum \hat{b}^2(\omega)$ also converges.

Equation (33) shows that the magnitude of the nonlinear coefficient $|N|$ is measured by observing the total bicoherence $\sum \hat{b}^2(\omega)$ together with the spectral variables $|g_\omega|^2$ and τ_p. The information of the phase of N is also obtained from the biphase.

IV. Explicit Forms

An example is discussed for drift wave fluctuations. A normalized electrostatic potential

$$n \equiv \frac{\tilde{n}}{n_0}\frac{L_n}{\rho_s}\,,\ \phi \equiv \frac{e\tilde{\phi}}{T_e}\frac{L_n}{\rho_s} \tag{34}$$

is introduced, where $\tilde{n}$ is the density perturbation, n_0 is the average density, $\tilde{\phi}$ is the electrostatic potential fluctuation, ρ_s is the ion gyroradius at electron temperature, and L_n is the density gradient scale length. (The normalized variables n and ϕ are of the order unity in a stationary drift wave turbulence [1].) The Hasegawa-Mima model gives the response of drift wave in the presence of zonal flow (pure zonal flow or GAMs) as [7]

$$\frac{\partial}{\partial t}\phi_d + \frac{i\omega_*}{1+k_\perp^2\rho_s^2}\phi_d - \frac{c_s\rho_s^4}{L_n}\left[\phi_d, \Delta_\perp\phi_d\right] = \frac{c_s}{L_n}\frac{q_x k_y k_\perp^2\rho_s^4}{1+k_\perp^2\rho_s^2}\phi_Z\phi_d\ , \tag{35}$$

where the suffix d and z indicate drift waves and zonal flow, respectively, q_x is the radial wavenumber of zonal flow, and $\boldsymbol{k}$ denotes the wavevector of drift waves. The second and third terms in the LHS of Eq.(35) stand for the linear response and nonlinear self-interaction of drift waves, respectively. The RHS represents the coupling between the zonal flow and drift waves.

The interaction of drift waves has the coupling coefficient

$$N \simeq \frac{c_s}{L_n}\frac{k_x k_y k_\perp^2\rho_s^4}{1+k_\perp^2\rho_s^2}\ . \tag{36}$$

This coefficient has a magnitude like

$$\left|N\right| \sim \frac{c_s}{2L_n}\frac{k_\perp^4\rho_s^4}{1+k_\perp^2\rho_s^2} . \tag{37}$$

It should be noticed that the phase of Eq.(36) can take a value in a wide range. This is because the sign of the wave number in the poloidal direction k_y is determined by the diamagnetic drift direction, but the wave number in the radial direction k_x can have a wide variety (including complex values) for drift waves. The interaction between the zonal flow and drift waves has the coefficient

$$N = \frac{c_s}{L_n}\frac{q_x k_y k_\perp^2\rho_s^4}{1+k_\perp^2\rho_s^2} . \tag{38}$$

In this form, one sees that k_x is not included and is replaced by q_x. The sign of k_y is dominated by the propagation of drift waves relative to the diamagnetic drift velocity. Therefore, the coefficient N keeps a same phase for components of drift wave fluctuations.

The decorrelation time of drift waves through self-nonlinear interaction has been evaluated as

$$\tau_p^{-1} \sim h\left(k_\perp\rho_s\right)\omega_*\phi , \tag{39}$$

in the strong turbulence limit, where $h\left(k_\perp\rho_s\right)$ stands for a numerical coefficient of the order of unity. For the case of Eq.(23), one has

$$\sum \hat{b}^2(\omega) \sim 3\left(\frac{1}{h\left(k_\perp\rho_s\right)}\frac{k_x k_\perp^2\rho_s^3}{1+k_\perp^2\rho_s^2}\right)^2 . \tag{40}$$

That is, the summed bicoherence has a weak dependence on the drift wave amplitude so long as the wavenumbers are unaltered. Equation (36) shows that the biphase of $\hat{B}$ spreads over the range of 0 and 2π .

In the case of the GAMs and drift waves, Eqs.(31) and (38) gives the expression for Eq.(31) (where the first and second terms are kept) as

$$\sum \hat{b}^2(\omega) \sim 4M\left(\frac{1}{h\left(k_\perp\rho_s\right)}\frac{q_x k_\perp^2\rho_s^3}{1+k_\perp^2\rho_s^2}\right)^2\frac{\phi_Z^2}{\phi_d^2} + 4\left(\frac{1}{h\left(k_\perp\rho_s\right)}\frac{k_x k_\perp^2\rho_s^3}{1+k_\perp^2\rho_s^2}\right)^2 , \tag{41}$$

so long as the frequency width for decomposing the Fourier series is wider than the half width of the GAMs peak. The wavenumber q_x for zonal flows is smaller than k_x for drift waves. However, the dependence on M possibly gives a larger value of the summed bicoherence. From Eqs.(40) and (41), the total bicoherence at the frequency of zonal flows and that at the drift wave range of frequencies are compared as

$$\frac{\sum \hat{b}^2(\mathrm{GAMs})}{\sum \hat{b}^2(\mathrm{drift})} \sim \frac{4M}{3}\left(\frac{q_x}{k_x}\right)^2 \frac{\phi_Z^2}{\phi_d^2} + \frac{4}{3} \ . \tag{42}$$

It is also noted that the total bicoherence of Eq.(41) is dependent on the local gradient of the zonal flow, $q_x = \left|\phi_Z^{-1} \nabla \phi_Z\right|$.

It is also useful to compare Eq.(42) with the estimate of the theory. In the predator-prey model, one has the ratio of the zonal flow amplitude and the fluctuation amplitude of drift waves as

$$\frac{\phi_Z^2}{\phi_d^2} = \left(\frac{k_x}{q_x}\right)^4 \frac{\gamma_L - \gamma_{nd}}{\gamma_{damp}} \ . \tag{43}$$

where γ_L and γ_{nd} are the linear growth rate and nonlinear damping rate (via drift wave-drift wave interactions) of drift waves, respectively, and γ_{damp} is the (collisional) damping rate of zonal flow. (See section 2 of [6] for more details.) Substituting Eq.(43) into Eq.(42), one has

$$\frac{\sum \hat{b}^2(\mathrm{GAMs})}{\sum \hat{b}^2(\mathrm{drift})} \sim \frac{4M}{3}\left(\frac{k_x}{q_x}\right)^2 \frac{\gamma_L - \gamma_{nd}}{\gamma_{damp}} + \frac{4}{3} \ . \tag{44}$$

The zonal flow is excited when $\gamma_L > \gamma_{damp}$ holds, so that the first term on the RHS is usually much greater than unity when the zonal flows are excited.

V. Summary

In this article, we discussed the bicoherence spectrum for drift wave turbulence in strongly magnetized plasma. The case without zonal flows and that with zonal flows were analyzed. In the presence of a broad band turbulence, the nonlinear interactions are theoretically formulated in a form of the Langevin equation, and the bicoherence spectrum was shown to indicates the projection of the nonlinear force onto the test mode. Based on this formalism, the magnitude of bispectrum was investigated for the drift wave -zonal flow systems. It was shown that the total bicoherence for the zonal flows (zonal flow and GAMs) increases as the amplitude of the zonal flows increase. Comparison between the bispectral data for zonal flows and for drift waves were also given. These findings generalized the result in ref. [15].

Explicit formula for bispectrum are summarized in the text, such as Eqs.(19), (21), and (23) for the interaction of broad-band fluctuations, and Eqs.(29) and (31) for the interaction of a sharp peak with broad band fluctuations. In these expressions, the bicoherence is expressed in terms of the coefficient of the nonlinear interaction and quantities which are given by quadratic spectral functions. Therefore, by measuring the fluctuation spectrum, autocorrelation time, and bispectral functions, the nonlinear interaction of each three-wave coupling is quantitatively estimated from experimental data. Thus, the study of bicoherence will provide a fruitful understanding of nonlinear interactions in turbulent plasmas.

It might be useful to add a few comment on the statistical variance, which is caused by finite number of realizations. The statistical error for the bicoherence indicator is estimated as

$$|\varepsilon| \simeq \frac{1}{\sqrt{N_R}} \left| \overline{g_p} \right|^3 \tag{45}$$

where $\overline{g_p}$ is a typical value of Fourier amplitude in the broad band spectrum and N_R is the number of realizations employed in the analysis. The variance for the total bicoherence is given as

$$\varepsilon_b = \frac{M}{N_R} \tag{46}$$

where M is the number of Fourier components. In order to have a statistically-admissible estimates, the bicoherence indicator and total bicoherence must be larger than Eq.(45) and Eq.(46), respectively. Equations (40) and (41), combined with Eq.(46), provide the necessary number of realizations N_R .

By observing the dependence of the total bicoherence on the amplitude of zonal flows, one can directly measure the nonlinear interaction of zonal flows and background drift waves directly. The dependence on the number of Fourier component was also clarified. The other issue is the phase of the bispectrum estimator. The importance of observing the biphase was also demonstrated. When ω is chosen at the frequency of zonal flows, the phase of the bispectrum estimator $\hat{B}(\omega, p)$ has a weak dependence on p . These properties will be used in the experimental study of turbulence [24]. It should be noticed that in the regime of the Dimits upshift, where the majority of fluctuation energy is converted into the zonal flows, the ratio Eq.(44) becomes very large.

It should be noted that the analysis in this article is valid for cases where the condition of the wavenumber matching is approximately satisfied if the frequency matching condition is fulfilled. This means that the coefficient of nonlinear interaction $N_{p,\,\omega}$ is an effective value, in which averaging over the wavenumber space is included. Experimental estimates of $N_{p,\,\omega}$, e.g., Eqs.(25) or (33) provide effective values. This shortcoming is due to limitations that only few-points measurements are usually available. It is necessary to measure the space-time Fourier decomposition, e. g., a power spectrum $I(k,\,\omega)$ and more complete bicoherence studies are necessary in order to establish better understanding of the system of drift waves and zonal flows.

The result in this article is limited to a single-field model, and the crossphases between multiple fluctuating fields (e.g., $\tilde{n}$, $\tilde{\phi}$, $\tilde{T}$, etc.) are not considered. This simplification is relevant as the first step, because the $\boldsymbol{v} \cdot \nabla \boldsymbol{v}$ nonlinearity has the essential role in the interaction of the drift wave and zonal flow. The result here is applied to the study of coupling between the zonal flow and drift waves. Nevertheless, the other nonlinear interactions (e.g., $\boldsymbol{v} \cdot \nabla p$ and other nonlinear terms) can also be influential in quantitative determination of the turbulence level. Experimental studies on cross bicoherence analysis may be possible in near future, and theoretical interpretation for them is required as well. Such analysis on multiple fields is left for future studies. It is noted that one point measurement has limitation in measuring the absolute value of nonlinear interactions. When the coherence lengths of triplet modes g_p, $g_{p-\omega}$ and g_ω are different, additional care is necessary.

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Appendix A Bicoherence in a case of broad band turbulence

We first study the case where fluctuations are composed of broad band spectrum as is shown in Fig.1(a). For this case, the triplet average of three components, g_p, $g_{p-\omega}$ and g_ω is discussed in this appendix. From Eqs.(12), (13) and (17), one has

$$g_p^* g_{p-\omega} g_\omega = \tilde{g}_p^* \tilde{g}_{p-\omega} \tilde{g}_\omega + \tau_p N_{p,\,\omega}^* g_{p-\omega} g_\omega \tilde{g}_{p-\omega}^* \tilde{g}_\omega^* + \tau_{p-\omega} N_{p-\omega,\,p} g_p g_\omega^* \tilde{g}_p^* \tilde{g}_\omega + \tau_\omega N_{\omega,\,p} g_{p-\omega}^* g_p^* \tilde{g}_p \tilde{g}_{p-\omega} \,. \tag{A1}$$

In the lowest order of $\tau_p N^*_{p,\,\omega}$, one has $g_q \tilde{g}^*_q \simeq \left| g_q \right|^2$ (for $q = p,\ \omega,\ p - \omega$), so that Eq.(A1) is rewritten as

$$g^*_p g_{p-\omega} g_\omega = \tilde{g}^*_p \tilde{g}_{p-\omega} \tilde{g}_\omega + \tau_p N^*_{p,\,\omega} \left| g_{p-\omega} \right|^2 \left| g_\omega \right|^2$$
$$+ \tau_{p-\omega} N_{p-\omega,\,p} \left| g_p \right|^2 \left| g_\omega \right|^2 + \tau_\omega N_{\omega,\,p} \left| g_{p-\omega} \right|^2 \left| g_p \right|^2 \qquad \text{(A2)}$$

in the lowest order of $\tau_p N^*_{p,\,\omega}$. The first term is mutually uncorrelated,

$$\left\langle \tilde{g}^*_p \tilde{g}_{p-\omega} \tilde{g}_\omega \right\rangle = 0 \ , \qquad \text{(A3)}$$

in the limit where $\tilde{S}$ is taken as a noise [2,22]. One has

$$\hat{B}(\omega,\ q) = \left\langle g^*_p g_{p-\omega} g_\omega \right\rangle = \tau_p N^*_{p,\,\omega} \left\langle \left| g_{p-\omega} \right|^2 \left| g_\omega \right|^2 \right\rangle$$
$$+ \tau_{p-\omega} N_{p-\omega,\,p} \left\langle \left| g_p \right|^2 \left| g_\omega \right|^2 \right\rangle + \tau_\omega N_{\omega,\,p} \left\langle \left| g_{p-\omega} \right|^2 \left| g_p \right|^2 \right\rangle \ . \qquad \text{(A4)}$$

In order to have more explicit interpretations, we employ an estimate for the RHS of Eq.(A4). Three terms of spectral functions in the RHS of Eq.(A4), $\left\langle \left| g_{p-\omega} \right|^2 \left| g_\omega \right|^2 \right\rangle$, $\left\langle \left| g_p \right|^2 \left| g_\omega \right|^2 \right\rangle$, and $\left\langle \left| g_{p-\omega} \right|^2 \left| g_p \right|^2 \right\rangle$, depend on p and ω but have similar magnitude for broad band fluctuations. One can have simplified evaluation as

$$\hat{B}(\omega,\ q) \simeq \left(\tau_p N^*_{p,\,\omega} + \tau_{p-\omega} N_{p-\omega,\,p} + \tau_\omega N_{\omega,\,p} \right) \left\langle \left| g_{p-\omega} \right|^2 \left| g_p \right|^2 \right\rangle \ . \qquad \text{(A5)}$$

The squared bicoherence is defined as Eq.(15). Substitution of Eq.(A5) into Eq.(15) gives

$$\hat{b}^2 = \frac{\left| \tau_p N^*_{p,\,\omega} + \tau_{p-\omega} N_{p-\omega,\,p} + \tau_\omega N_{\omega,\,p} \right|^2 \left\langle \left| g_{p-\omega} \right|^2 \left| g_p \right|^2 \right\rangle}{\left\langle \left| g_\omega \right|^2 \right\rangle} \ . \qquad \text{(A6)}$$

A crude estimate,

$$\frac{\left\langle \left| g_{p-\omega} \right|^2 \left| g_p \right|^2 \right\rangle}{\left\langle \left| g_\omega \right|^2 \right\rangle} \simeq \left\langle \left| g_p \right|^2 \right\rangle . \tag{A7}$$

is employed here, because all of $g_{p-\omega}$, g_p and g_ω belong to the broad-band spectrum. This allows direct comparisons with experimental observations. This approximation gives

$$\hat{b}^2(p, \omega) = \left| \tau_p N_{p,\omega}^* + \tau_{p-\omega} N_{p-\omega,\,p} + \tau_\omega N_{\omega,\,p} \right|^2 \left\langle \left| g_p \right|^2 \right\rangle . \tag{A8}$$

If phases between $\tau_p N_{p,\omega}^*$, $\tau_{p-\omega} N_{p-\omega,\,p}^*$ and $\tau_\omega N_{\omega,\,p}^*$ are randomly distributed, one has

$$\begin{aligned} &\left| \tau_p N_{p,\omega}^* + \tau_{p-\omega} N_{p-\omega,\,p} + \tau_\omega N_{\omega,\,p} \right|^2 \\ &\sim \left| \tau_p N_{p,\omega}^* \right|^2 + \left| \tau_{p-\omega} N_{p-\omega,\,p} \right|^2 + \left| \tau_\omega N_{\omega,\,p} \right|^2 \sim 3 \left| \tau_p N_{p,\omega}^* \right|^2 . \end{aligned} \tag{A9}$$

The summed-bicoherence $\sum \hat{b}^2$ is defined as Eq.(16). Equations (A8) and (A9) provide the relation

$$\sum \hat{b}^2(\omega) \sim \sum_p 3 \left| \tau_p N_{p,\omega}^* \right|^2 \left\langle \left| g_p \right|^2 \right\rangle \sim 3 \left| \tau_p N_{p,\omega}^* \right|^2 \left| g \right|^2 , \tag{A10}$$

with $\left| g \right|^2 = \sum_p \left\langle \left| g_p \right|^2 \right\rangle$.

Appendix B Bicoherence in a case of a sharp peak within a broad band turbulence

The triplet product is given from Eqs.(12), (13) and (27) as

$$\begin{aligned} g_p^* g_{p-\omega} g_\omega = \tilde{g}_p^* \tilde{g}_{p-\omega} g_{\omega,\,0} &+ \tau_p N_{p,\omega}^* \left| g_{p-\omega} \right|^2 \left| g_{\omega,\,0} \right|^2 \\ &+ \tau_{p-\omega} N_{p-\omega,\,p} \left| g_p \right|^2 \left| g_{\omega,\,0} \right|^2 + \tau_{c,\,p-\omega} N_{\omega,\,p} \left| g_{p-\omega} \right|^2 \left| g_p \right|^2 . \end{aligned} \tag{B1}$$

where the relation $g_q \tilde{g}_q^* \simeq \left| g_q \right|^2$ is also used. The average of the first term $\left\langle \tilde{g}_p^* \tilde{g}_{p-\omega} g_{\omega,\,0} \right\rangle$ is considered to vanish because $\tilde{g}_p^*$ and $\tilde{g}_{p-\omega}$ are responses to independent noises. Thus one has the evaluation of the Bicoherence indicator by use of the lowest-order correlation as

$$\hat{B}(\omega, p) = \left\langle g_p^* g_{p-\omega} g_\omega \right\rangle = \left(\tau_p N_{p,\omega}^* \left| g_{p-\omega} \right|^2 + \tau_{p-\omega} N_{p-\omega,p} \left| g_p \right|^2 \right) \left| g_{\omega,0} \right|^2 + \tau_{c,p-\omega} N_{\omega,p} \left| g_{p-\omega} \right|^2 \left| g_p \right|^2 . \tag{B2}$$

The first term (with parenthesis) in the RHS of Eq.(B2) is due to the modulation of the background fluctuation by the imposition of the test mode (e. g., zonal flow), and the last term in the RHS comes from the influence on the test mode by back-ground fluctuations. As is explained in §III, the phases of $\tau_p N_{p,\omega}^*$ and $\tau_{p-\omega} N_{p-\omega,p}^*$ are common for the interaction between the zonal flow and drift wave fluctuations. Based on the estimate $\tau_p N_{p,\omega}^* \left| g_{p-\omega} \right|^2 \simeq \tau_{p-\omega} N_{p-\omega,p}^* \left| g_p \right|^2$, a simplified form of $\hat{B}$ may be used for convenience as

$$\hat{B}(\omega, p) \simeq 2\tau_{p-\omega} N_{p-\omega,p}^* \left| g_p \right|^2 \left| g_{\omega,0} \right|^2 + \tau_{c,p-\omega} N_{\omega,p} \left| g_{p-\omega} \right|^2 \left| g_p \right|^2 \tag{B3}$$

The term which is proportional to $\left| g_{\omega,0} \right|^2$ in the Bicoherence indicator has been pointed out in [15]. The second term is the contribution of the broad band turbulence, and $\hat{B}(\omega, p)$ at ω does not vanish even in the limit of $\left| g_{\omega,0} \right|^2 \to 0$.

The squared bicoherence is calculated from Eq.(B3) as

$$\hat{b}^2(\omega, p) = \frac{4\tau_{p-\omega}^2 \left| N_{p-\omega,p} \right|^2 \left| g_p \right|^4 \left| g_{\omega,0} \right|^4}{\left\langle \left| g_p g_{p-\omega} \right|^2 \right\rangle \left\langle \left| g_\omega \right|^2 \right\rangle} + \frac{4\tau_{p-\omega} \tau_{c,p-\omega} \operatorname{Re}\left(N_{p-\omega,p} N_{\omega,p}^* \right) \left| g_{p-\omega} \right|^2 \left| g_p \right|^4 \left| g_{\omega,0} \right|^2}{\left\langle \left| g_p g_{p-\omega} \right|^2 \right\rangle \left\langle \left| g_\omega \right|^2 \right\rangle} + \frac{\left| \tau_{c,p-\omega} N_{\omega,p}^* \right|^2 \left| g_{p-\omega} \right|^4 \left| g_p \right|^4}{\left\langle \left| g_p g_{p-\omega} \right|^2 \right\rangle \left\langle \left| g_\omega \right|^2 \right\rangle} \tag{B4}$$

For further transparency of argument, the approximate relations $\left\langle \left| g_p g_{p-\omega} \right|^2 \right\rangle \simeq \left\langle \left| g_{p-\omega} \right|^2 \right\rangle \left\langle \left| g_p \right|^2 \right\rangle$ and $\left\langle \left| g_{p-\omega} \right|^2 \right\rangle \simeq \left\langle \left| g_p \right|^2 \right\rangle$ are used, because $g_{p-\omega}$ and g_p belong to the broad-band spectrum. In addition, an approximation

$\left|g_{\omega,0}\right|^2 \simeq \left\langle \left|g_\omega\right|^2 \right\rangle$ is employed as is in Eq.(9). By the help of these approximations, the first term in Eq.(B4) is estimated as

$$\frac{4\tau_{p-\omega}^2 \left|N_{p-\omega,p}\right|^2 \left|g_p\right|^4 \left|g_{\omega,0}\right|^4}{\left\langle \left|g_p g_{p-\omega}\right|^2 \right\rangle \left\langle \left|g_\omega\right|^2 \right\rangle} \simeq 4\tau_{p-\omega}^2 \left|N_{p-\omega,p}\right|^2 \left|g_{\omega,0}\right|^2 , \tag{B5}$$

and is dependent only weakly on the choice of p . The second term in Eq.(B4) may be approximated as

$$\frac{4\tau_{p-\omega}\tau_{c,p-\omega} \operatorname{Re}\left(N_{p-\omega,p} N_{\omega,p}^*\right) \left|g_{p-\omega}\right|^2 \left|g_p\right|^4 \left|g_{\omega,0}\right|^2}{\left\langle \left|g_p g_{p-\omega}\right|^2 \right\rangle \left\langle \left|g_\omega\right|^2 \right\rangle}$$
$$\simeq 4\tau_{p-\omega}\tau_{c,p-\omega} \operatorname{Re}\left(N_{p-\omega,p} N_{\omega,p}^*\right) \left|g_p\right|^2 . \tag{B6}$$

The last term in Eq.(B4) is rewritten as

$$\frac{\left|\tau_{c,p-\omega} N_{\omega,p}\right|^2 \left|g_{p-\omega}\right|^2 \left|g_p\right|^2}{\left\langle \left|g_\omega\right|^2 \right\rangle} . \tag{B7}$$

Thus, simplified expressions for the squared bicoherence and total bicoherence are given as Eqs.(30) and (31).

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Fig.1

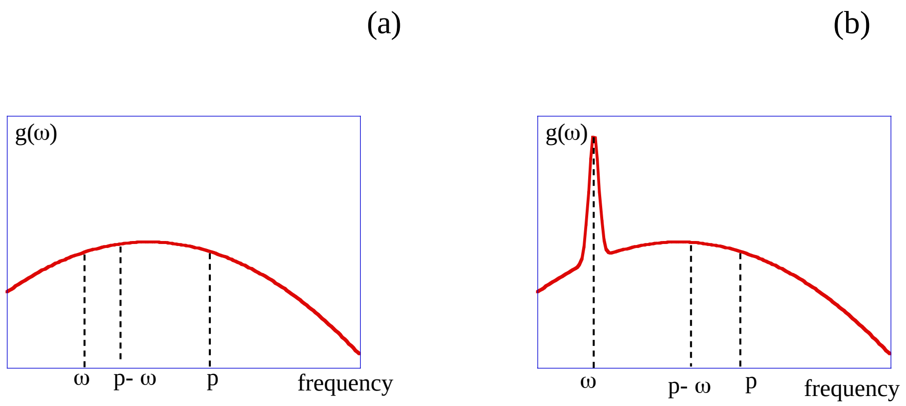

Fig.1 Schematic drawing of the spectrum. A broad band spectrum (a) and that with a shahp peak (b).