Equilibrium and Stability of High-beta Toroidal Plasmas with Toroidal and Poloidal Flow in Reduced Magnetohydrodynamic Models

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Abstract. Effects of flow, finite ion temperature and pressure anisotropy on equilibrium and stability of a high-beta toroidal plasma are studied in the framework of reduced magnetohydrodynamics (MHD). A set of reduced equilibrium equations for high-beta tokamaks with toroidal and poloidal flow comparable to the poloidal sound velocity is derived in a unified form of single-fluid and Hall MHD models and a two-fluid MHD model with ion finite Larmor radius (FLR) terms. Pressure anisotropy is introduced with equations for the parallel heat flux which are closed by a fluid closure model. It is solved analytically for the single-fluid model and the solutions shows complicated characteristics in the region around the poloidal sound velocity due to pressure anisotropy and the parallel heat flux. Numerical solutions are found by using the finite element method for the two-fluid model with FLR effects in the case of isotropic, adiabatic pressure and indicate the following features of two-fluid equilibria: the isosurfaces of the magnetic flux, the pressure and the ion stream function do not coincide with each other, and the solutions depend on the sign of the radial electric field. Reduced single-fluid MHD equations with time evolution that are consistent with the above equilibria are also derived in order to study their stability. They conserve the energy up to the order required by the equilibria.

1. Introduction

Flows in magnetically confined plasmas may play an important role for the formation of steep structure where the scale lengths of microscopic effects cannot be neglected. In plasma flows driven by neutral beam injection, pressure anisotropy is relevant. Microscopic effects on equilibria in the presence of flow and pressure anisotropy have been studied with two-fluid or Hall magnetohydrodynamic (MHD) models [1]. However, a consistent treatment of hot ions in a two-fluid framework must include the ion finite Larmor radius (FLR) effects.

In toroidally confined plasmas, both of the poloidal and toroidal components of flow are important. The single-fluid MHD equations for equilibria with flow reduce to the so-called generalized Grad-Shafranov (GS) equation and the Bernoulli law with five free functions of the magnetic flux in axisymmetric systems [2]. These systems of a nonlinear partial differential equation (PDE) and a nonlinear algebraic equation can be solved numerically by iteration schemes when the PDE is elliptic in the whole region. However, the generalized GS equation can be either elliptic, hyperbolic or singular depending on the magnitude of the poloidal flow velocity relative to the velocities of MHD waves.

Recently, a reduced set of single-fluid MHD equilibrium equations has been derived for high-beta tokamaks with flow comparable to the poloidal sound velocity [3]. The ordering for this system eliminates the Alfvén singularity and the second hyperbolic region where the poloidal flow velocity exceeds the phase velocity of the fast magnetosonic wave and degenerate the first hyperbolic region near the velocity of the slow magnetosonic wave to give rise to the elliptic PDEs with the poloidal-sonic singularity. This model has been extended to two-fluid equilibria with ion FLR effects for isotropic and adiabatic diagonal pressure of ions and electrons [4].

We study the effects of flow, finite ion temperature and pressure anisotropy on equilibrium and stability of a high-beta toroidal plasma are studied in the framework of reduced magnetohydrodynamics (MHD). A set of reduced equilibrium equations for high-beta tokamaks with toroidal and poloidal flow comparable to the poloidal sound velocity has been formulated from two-fluid MHD equations with ion FLR terms. We include pressure anisotropy associated with the parallel heat flux that was neglected in the previous formulation [4]. We have found an analytic solution for the single-fluid model by extending that for single fluid equilibrium equations with adiabatic and isotropic pressure [5]. Numerical solutions are found by using the finite element method for the two-fluid model with FLR effects in the case of isotropic, adiabatic pressure. We have derived reduced single-fluid MHD equations with time evolution that are consistent with the above equilibria in order to study their stability and shown that the energy is conserved up to the order required by the equilibria.

2. Equilibria with flow

2.1. Equations for steady state

The equations for two-fluid equilibria with hot ions and pressure anisotropy are given from fluid moment equations for collisionless, magnetized plasma [6] as

$$\nabla\cdot(n\mathbf{v})=0,\quad \nabla\times\mathbf{E}=0,\quad \mu_0\mathbf{j}=\nabla\times\mathbf{B},\tag{1}$$

$$m_i n\mathbf{v}\cdot\nabla\mathbf{v}=\mathbf{j}\times\mathbf{B}-\sum_{s=i,e}\left[\nabla p_{s\perp}+\mathbf{B}\cdot\nabla\left(\frac{p_{s\parallel}-p_{s\perp}}{B^2}\mathbf{B}\right)\right]-\lambda_i\nabla\cdot\Pi_i^{gv},\tag{2}$$

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}=\frac{\lambda_H}{ne}\left[\mathbf{j}\times\mathbf{B}-\nabla p_{e\perp}-\mathbf{B}\cdot\nabla\left(\frac{p_{e\parallel}-p_{e\perp}}{B^2}\mathbf{B}\right)\right],\tag{3}$$

$$\frac{3}{2}\mathbf{v}\cdot\nabla p_i+\frac{5}{2}p_i\nabla\cdot\mathbf{v}+\left(p_{i\parallel}-p_{i\perp}\right)\left\{\mathbf{b}\cdot(\mathbf{b}\cdot\nabla\mathbf{v})-\frac{1}{3}\nabla\cdot\mathbf{v}\right\}+\lambda_{i\parallel}\nabla\cdot\left(q_{i\parallel}\mathbf{b}\right)+\lambda_i\nabla\cdot\mathbf{q}_{i\perp}=0,\tag{4}$$

$$\frac{1}{2}\mathbf{v}\cdot\nabla p_{i\parallel}+\frac{1}{2}p_{i\parallel}\nabla\cdot\mathbf{v}+p_{i\parallel}\mathbf{b}\cdot(\mathbf{b}\cdot\nabla\mathbf{v})+\lambda_{i\parallel}\nabla\cdot\left(q_{iB\parallel}\mathbf{b}\right)+\lambda_i\nabla\cdot\mathbf{q}_{iB\perp}-2\lambda_i\mathbf{q}_{iB\perp}\cdot(\mathbf{b}\cdot\nabla\mathbf{b})=0,\tag{5}$$

$$\begin{aligned}&\frac{3}{2}\left(\mathbf{v}-\frac{\lambda_H}{en}\mathbf{j}\right)\cdot\nabla p_e+\frac{5}{2}p_e\nabla\cdot\left(\mathbf{v}-\frac{\lambda_H}{en}\mathbf{j}\right)\\&+\left(p_{e\parallel}-p_{e\perp}\right)\left\{\mathbf{b}\cdot\left[\mathbf{b}\cdot\nabla\left(\mathbf{v}-\frac{\lambda_H}{en}\mathbf{j}\right)\right]-\frac{1}{3}\nabla\cdot\left(\mathbf{v}-\frac{\lambda_H}{en}\mathbf{j}\right)\right\}+\nabla\cdot\left(q_{e\parallel}\mathbf{b}\right)+\lambda_i\nabla\cdot\mathbf{q}_{e\perp}=0,\end{aligned}\tag{6}$$

$$\begin{aligned}&\frac{1}{2}\left(\mathbf{v}-\frac{\lambda_H}{en}\mathbf{j}\right)\cdot\nabla p_{e\parallel}+\frac{1}{2}p_{e\parallel}\nabla\cdot\left(\mathbf{v}-\frac{\lambda_H}{en}\mathbf{j}\right)+p_{e\parallel}\mathbf{b}\cdot\left[\mathbf{b}\cdot\nabla\left(\mathbf{v}-\frac{\lambda_H}{en}\mathbf{j}\right)\right]\\&+\nabla\cdot\left(q_{e\parallel}\mathbf{b}\right)+\lambda_i\nabla\cdot\mathbf{q}_{e\perp}-2\lambda_i\mathbf{q}_{eB\perp}\cdot(\mathbf{b}\cdot\nabla\mathbf{b})=0,\end{aligned}\tag{7}$$

where m_i is the ion mass, n is the density, $\mathbf{v}$ is the ion flow velocity, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, $\mathbf{j}$ is the current density, $p_{\{i,e\}\{\parallel,\perp\}}$ are the ion and electron parallel and perpendicular pressures, $p_{\{i,e\}}\equiv(p_{\{i,e\}\parallel}+2p_{\{i,e\}\perp})/3$ is the total pressures, Π_i^{gv} is the ion gyroviscous tensor, $B\equiv|\mathbf{B}|$, $\mathbf{b}\equiv\mathbf{B}/B$, and $\mathbf{q}_{\{i,e\}\{\parallel,\mathrm{B}\parallel\}}$ and $\mathbf{q}_{\{i,e\}\{\perp,\mathrm{B}\perp\}}$ are the parallel and perpendicular heat fluxes defined as

$$\mathbf{q}_{s\parallel}\equiv\mathbf{q}_{sB\parallel}+\mathbf{q}_{sT\parallel},\quad \mathbf{q}_{s\perp}\equiv\mathbf{q}_{sB\perp}+\mathbf{q}_{sT\perp}\quad (s=i,e),\tag{8}$$

$$\mathbf{q}_{s\parallel}\equiv\frac{m_i}{2}\int\left(v-\bar{v}\right)^2\left(\mathbf{v}_\parallel-\bar{\mathbf{v}}_\parallel\right)fd^3\mathbf{v},\quad \mathbf{q}_{sB\parallel}\equiv\frac{m_s}{2}\int\left(v_\parallel-\bar{v}_\parallel\right)^2\left(\mathbf{v}_\parallel-\bar{\mathbf{v}}_\parallel\right)fd^3\mathbf{v},\tag{9}$$

$$\mathbf{q}_{sB\perp}\equiv\frac{m_s}{2}\int\left(v_\parallel-\bar{v}_\parallel\right)^2\left(\mathbf{v}_\perp-\bar{\mathbf{v}}_\perp\right)fd^3\mathbf{v},\quad \mathbf{q}_{sT\perp}\equiv\frac{m_s}{2}\int\left(v_\perp-\bar{v}_\perp\right)^2\left(\mathbf{v}_\perp-\bar{\mathbf{v}}_\perp\right)fd^3\mathbf{v}.\tag{10}$$

The electron mass m_e is neglected because $m_e\ll m_i$. The electron gyroviscosity is also neglected since $\rho_e\ll\rho_i$. We have introduced the artificial indices λ_H, λ_i and $\lambda_{i\parallel}$ that label the non-ideal terms: (λ_H,λ_i)=(0,0) for single-fluid (ideal) MHD, (1,0) for two-fluid MHD with electron diamagnetic effects but zero ion Larmor radius (Hall MHD) and (1,1) for two-fluids with finite ion Larmor radius, and $\lambda_{i||}$ =0,1 represents the ion pressure with and without parallel heat flux respectively. The fluid moment equations must be closed by a certain closure model. Here, we adopt the following fluid closure model for the fourth-order moments in the equations for the parallel heat fluxes,

$$\begin{aligned}&\int d^3\mathbf{v}\left(v_i-\overline{v}_i\right)\left(v_j-\overline{v}_j\right)\left(v_k-\overline{v}_k\right)\left(v_l-\overline{v}_l\right) f\left(\mathbf{x},\mathbf{v},t\right)\\ &=\frac{1}{n}\Big[\int d^3\mathbf{v}\left(v_{[i}-\overline{v}_{[i}\right)\left(v_j-\overline{v}_j\right) f\left(\mathbf{x},\mathbf{v},t\right)\Big]\Big[\int d^3\mathbf{v}\left(v_k-\overline{v}_k\right)\left(v_{l]}-\overline{v}_{l]}\right) f\left(\mathbf{x},\mathbf{v},t\right)\Big],\end{aligned} \tag{11}$$

where the square brackets around indices represent the minimal sum over permutations of uncontracted indices needed to yield completely symmetric tensors. It is noted that the equations for the parallel heat flux equations for mass-less electrons turn to the equations for the parallel and perpendicular electron pressures [1],

$$\mathbf{B}\cdot\nabla\left(p_{e||}/n\right)=0,\quad \mathbf{B}\cdot\nabla\Big[\left(p_{e||}/p_{e\perp}-1\right)B\Big]=0, \tag{12}$$

and the parallel electron heat flux are calculated from (6) and (7) by substituting (12).

In order to obtain a simple, closed expression for ion FLR terms, asymptotic expansions in terms of the small parameter $\delta\sim\rho_i/a$, where ρ_i is the ion Larmor radius and a is the macroscopic scale length, are used. With a slow dynamics ordering, $v\sim\delta v_{thi}$ where v and v_{thi} are the flow and ion thermal velocities respectively, the ion FLR terms [6][7] are much simplified in the reduced models for large-aspect-ratio, high-beta tokamaks [8][9] after relating δ to the inverse aspect ratio expansion parameter $\varepsilon\equiv a/R_0\ll 1$ where ε is the inverse aspect ratio and a and R_0 are the characteristic scale lengths of the minor and major radii respectively [10] [11]. With the slow dynamics ordering, one finds

$$m_i n v^2\sim\|\Pi_i^{gv}\|\sim\delta^2 p_{\{i,e\}},\quad q_{\{i,e\}}\sim v p_{\{i,e\}}\sim\delta v_{thi} p_{\{i,e\}}. \tag{13}$$

The high-beta tokamak orderings are

$$B_p\sim\varepsilon B_0,\quad p_{\{i,e\}\{||,\perp\}}\sim\varepsilon\left(B_0^2/\mu_0\right),\quad |\nabla_{||}|\sim 1/R_0,\quad |\nabla_\perp|\sim 1/a. \tag{14}$$

We assume strong pressure anisotropy, $|p_{||}-p_\perp|\sim p$. Poloidal-sonic flow is introduced with $\varepsilon\sim\delta$. Under these orderings, the FLR terms [6][7] are approximated up to the order required to include ion kinetic energy (13) to obtain the following expressions

$$\mathbf{q}_{sB\perp}\simeq\frac{\mathbf{b}}{e_s B}\times\left[\frac{1}{2}p_{s\perp}\nabla\left(\frac{p_{s||}}{n}\right)+\frac{p_{s||}\left(p_{s||}-p_{s\perp}\right)}{n}\left(\mathbf{b}\cdot\nabla\mathbf{b}\right)\right], \tag{15}$$

$$\mathbf{q}_{sT\perp}\simeq\frac{\mathbf{b}}{e_s B}\times\left[2p_{s\perp}\nabla\left(\frac{p_{s\perp}}{n}\right)\right], \tag{16}$$

$$\nabla\cdot\Pi_i^{gv}\equiv\nabla\cdot\left(\sum_{N=1}^{5}\Pi_i^{gvN}\right), \tag{17}$$

$$\nabla\cdot\Pi_i^{gv1}\simeq-m_i n\mathbf{v}_{*i}\cdot\nabla\mathbf{v}-\nabla\chi_v-\nabla\times\left[\frac{m_i p_{i\perp}}{2eB^2}(\nabla\cdot\mathbf{v})\mathbf{B}\right], \tag{18}$$

$$\nabla\cdot\Pi_i^{gv2}\simeq-\nabla\chi_q-\nabla\times\left[\frac{m_i}{4eB^2}(\nabla\cdot\mathbf{q}_{iT\perp})\mathbf{B}\right], \tag{19}$$

$$\nabla\cdot\Pi_i^{gv3}\simeq\nabla\times\left\{\mathbf{B}\times\left[\frac{m_i}{eB^2}\left(\mathbf{c}+\mathbf{d}\right)\right]\right\}, \tag{20}$$

$$\nabla\cdot\Pi_i^{gv4}\simeq\nabla\cdot\Pi_i^{gv5}\simeq 0, \tag{21}$$

$$\mathbf{c} \simeq -\left(\frac{p_{i\|}-p_{i\perp}}{B}\right)\nabla\times\left(\frac{1}{en}\nabla p_{i\perp}\right),\quad \mathbf{d}\simeq\nabla\times\left[\left(2\mathbf{q}_{iB\perp}-\frac{1}{2}\mathbf{q}_{iT\perp}\right)\times\mathbf{b}\right], \tag{22}$$

$$\mathbf{v}_{*i}\equiv-\frac{1}{en}\nabla\times\left(\frac{p_i}{B^2}\mathbf{B}\right),\quad \chi_v\equiv\frac{m_i p_{i\perp}}{2eB^2}\mathbf{B}\cdot(\nabla\times\mathbf{v}),\quad \chi_q\simeq\frac{m_i}{4eB^2}\mathbf{B}\cdot\left(\nabla\times\mathbf{q}_{i\perp}\right), \tag{23}$$

$$\begin{aligned}&\mathbf{v}\cdot\nabla q_{i\|}+\left(2q_{i\|}-q_{iB\|}\right)\nabla\cdot\mathbf{v}+\frac{p_{i\|}}{m_i}\mathbf{b}\cdot\nabla\left(\frac{2p_{i\perp}+3p_{i\|}}{2n}\right)\\&-\frac{p_{i\perp}\left(p_{i\|}-p_{i\perp}\right)}{m_i nB}\mathbf{b}\cdot\nabla B+\frac{\lambda_i}{eB}\nabla\left(\frac{2p_{i\perp}+3p_{i\|}}{2n}\right)\cdot\left[\mathbf{b}\times\left(p_{i\perp}\nabla v_\|+\nabla q_{iT\|}\right)\right]\\&+\frac{\lambda_i p_{i\perp}}{eB}\left[\nabla\left(\frac{p_{i\perp}}{n}\right)\cdot\mathbf{b}\times\nabla v_\|+\nabla\left(\frac{1}{n}\right)\cdot\mathbf{b}\times\nabla q_{T\|}\right]+\mathbf{q}_{i\perp}\cdot\nabla v_\|+2\mathbf{q}_{iB\perp}\cdot\nabla v_\|\simeq 0,\end{aligned} \tag{24}$$

$$\begin{aligned}&\mathbf{v}\cdot\nabla q_{iB\|}+q_{iB\|}\nabla\cdot\mathbf{v}+\frac{3p_{i\|}}{2m_i}\mathbf{b}\cdot\nabla\left(\frac{p_{i\|}}{n}\right)\\&+\frac{3}{2}\frac{\lambda_i}{eB}\nabla\left(\frac{p_{i\|}}{n}\right)\cdot\left[\mathbf{b}\times\left(p_{i\perp}\nabla v_\|+\nabla q_{iT\|}\right)\right]+3\lambda_i\mathbf{q}_{iB\perp}\cdot\nabla v_\|\simeq 0.\end{aligned} \tag{25}$$

2.2. Reduced Equilibrium Equations

Here, we shall consider the corresponding toroidal axisymmetric equilibria, where, in cylindrical coordinates (R,φ,Z), the magnetic field **B** and the electric field **E** can be written as

$$\mathbf{B}=\nabla\psi(R,Z)\times\nabla\varphi+I(R,Z)\nabla\varphi,\quad \mathbf{E}\equiv-\nabla\Phi. \tag{26}$$

The asymptotic expansion is defined in terms of the inverse aspect ratio ε. The variables are expanded as $f=f_0+f_1+f_2+f_3+\ldots$, where $f_1\sim\varepsilon f_0$, $f_2\sim\varepsilon^2 f_0$ and $f_3\sim\varepsilon^3 f_0$. The leading order force balance gives

$$p_{i\perp 1}+p_{e\perp 1}+B_0 I_1/\mu_0 R_0=\text{const}, \tag{27}$$

where $B_0\equiv I_0/R_0$. The following leading order quantities are shown to be arbitrary functions of the first-order magnetic flux ψ_1,

$$\Phi_1=\Phi_1(\psi_1),\quad n_0=n_0(\psi_1),\quad p_{s\{\|,\perp\}1}=p_{s\{\|,\perp\}1}(\psi_1)\quad I_1=I_1(\psi_1). \tag{28}$$

The second order poloidal momentum balance gives

$$p_{i\perp 2}+p_{e\perp 2}+B_0 I_2/\mu_0 R_0-\left(x/R_0\right)\sum_{s=i,e}(p_{s\|1}-p_{s\perp 1})\equiv g_*\left(\psi_1\right). \tag{29}$$

The second order quantities are written in the following forms

$$p_{s\{\|,\perp\}2}-p'_{s\{\|,\perp\}1}\psi_2=\left(x/R_0\right)C_{s\{\|,\perp\}}\left(\psi_1\right)+P_{s\{\|,\perp\}2*}\left(\psi_1\right), \tag{30}$$

$$\Phi_2-\Phi'_1\psi_2=\left(x/R_0\right)C_\Phi\left(\psi_1\right)+\Phi_{2*}\left(\psi_1\right), \tag{31}$$

$$v_\|=\left(x/R_0\right)C_{v\|}\left(\psi_1\right)+v_{\|*}\left(\psi_1\right), \tag{32}$$

$$n_1-n'_0\psi_2=\left(x/R_0\right)C_n\left(\psi_1\right)+n_{1*}\left(\psi_1\right), \tag{33}$$

$$q_{s\{\|,B\|\}}=\left(x/R_0\right)C_{sq\{\|,B\|\}}\left(\psi_1\right)+q_{s\{\|,B\|\}*}\left(\psi_1\right), \tag{34}$$

$$I_2-I'_1\psi_2=\left(\frac{x}{R_0}\right)C_I\left(\psi_1\right)+\frac{\mu_0 R_0}{B_0}\left[g_*\left(\psi_1\right)-\sum_{s=i,e}P_{s\perp 2*}\left(\psi_1\right)\right], \tag{35}$$

where $x\equiv R-R_0$. The coefficients in the first terms on the right-hand sides of (30) - (35), $C_{\ldots}(\psi_1)$, are obtained by solving the equations for the higher-order quantities as functionals of the lowest order quantities, (28), while those in the second terms, denoted by '*', are arbitrary functions of ψ_1. The third order poloidal momentum balance gives

$$\frac{B_0 I_3}{\mu_0 R_0} + \sum_{s=i,e} \left\{ p_{s\perp 3} - \left(\frac{x}{R_0}\right)\left(p_{s\|2} - p_{s\perp 2}\right) - \frac{1}{2}\left(\frac{x}{R_0}\right)^2 \left[p_{s\|1} - p_{s\perp 1} - \left(C_{s\|} + C_{s\perp}\right)\right]\right\} + \frac{I_1}{\mu_0 R_0^2}\left(I_2 - I_1'\psi_2\right) - g_*'\psi_2 + F\frac{|\nabla \psi_1|^2}{2\mu_0} - \lambda_i(\chi_v + \chi_q) \equiv E_*(\psi_1), \tag{36}$$

where E_* is an arbitrary function of ψ_1. The set of reduced equilibrium equations consists of the first two orders of the Grad-Shafranov (GS) equation of which the first order is same as that for static equilibria,

$$\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial Z^2}\right)\psi_1 = -\mu_0 R_0^2 \left[\left(\frac{x}{R_0}\right)\sum_{s=i,e}(p_{s\|1}' + p_{s\perp 1}') + g_*'\right] - \left(\frac{I_1^2}{2}\right)', \tag{37}$$

and the second order is given by

$$\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial Z^2}\right)\psi_2 + \left[\mu_0 R_0^2 \left(\frac{x}{R_0}\right)\sum_{s=i,e}\left(p_{s\perp 1}'' + p_{s\|1}''\right) + \mu_0 R_0^2 g_*'' + \left(\frac{I_1^2}{2}\right)''\right]\psi_2 = \frac{1}{R}\frac{\partial \psi_1}{\partial R} + F\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial Z^2}\right)\psi_1 + F'\frac{|\nabla\psi_1|^2}{2} - \mu_0 R_0^2\left[E_*' + \left(\frac{x}{R_0}\right)\sum_{s=i,e}\left(P_{s\perp 2*}' + P_{s\|2*}'\right) + \frac{1}{2}\left(\frac{x}{R_0}\right)^2 \sum_{s=i,e}\left(p_{s\perp 1}' + p_{s\|1}' + C_{s\perp 1}' + C_{s\|1}'\right)\right], \tag{38}$$

where

$$F(\psi_1) \equiv \left(\frac{B_0^2}{\mu_0}\right)^{-1}\left[m_i n_0 R_0^2 \left(\Phi_1' + \frac{\lambda_H - \lambda_i}{en_0} p_{i\perp 1}'\right)\left(\Phi_1' + \frac{\lambda_H}{en_0} p_{i\perp 1}'\right) + \sum_{s=i,e}(p_{s\|1} - p_{s\perp 1})\right] \tag{39}$$

includes the $E\times B$ and the ion diamagnetic poloidal flows with the gyroviscous cancellation and the pressure anisotropy. The ion stream function Ψ is defined as

$$n\mathbf{v} = \nabla\Psi \times \nabla\varphi + nRv_\varphi \nabla\varphi. \tag{40}$$

The asymptotic expansion yields

$$\Psi_1'(\psi_1) = -\frac{R_0}{B_0} n_0 \left(\Phi_1' + \frac{\lambda_H}{en_0} p_{i\perp 1}'\right), \quad \Psi_2 = \Psi_1'\psi_2 + \left(\frac{x}{R_0}\right) C_{\Psi*}(\psi_1) + \Psi_{2*}(\psi_1). \tag{41}$$

2.3. Analytic solution for the single fluid model

The reduced GS equations (37) and (38) can be solved for the single fluid model (λ_H,λ_i, $\lambda_{i\|}$)=(0,0,1) as for the case of adiabatic, isotropic pressure [5]. The solution is found when the pressures and the square of the poloidal Alfvén Mach number are linear with the first-order magnetic flux,

$$p_{\{i,e\}\{\|,\perp\}1} = \varepsilon\left(B_0^2/\mu_0\right) p_{\{i,e\}\{\|,\perp\}1c}\overline{\psi}_1, \quad M_{Ap}^2 \equiv \mu_0 m_i n_0 R_0^2 \Phi_1'^2 / B_0^2 = \varepsilon M_{Apc}^2 \overline{\psi}_1. \tag{42}$$

Figure 1 show profiles of an analytical solution; (a) the magnetic structure is modified by the flow due to the centrifugal force and through the Bernoulli law, (b) the pressure isosurfaces depart from magnetic flux surfaces due to the poloidal flow and (c) anisotropic pressure profiles are self-consistently determined in the presence of flow. Figure 2 shows the shift of the magnetic axis from the geometric axis as a function of the square of the poloidal Mach number. There are three singular points where the poloidal flow velocity equals the phase velocities of either slow magnetosonic or two ion acoustic waves,

$$M_{Apc}^2 = \left(6p_{i\|1c} + p_{e\|1c} \pm \sqrt{24p_{i\|1c}^2 + p_{e\|1c}^2}\right)\Big/2, \quad p_{i\|1c}, \tag{43}$$

which arise from the heat flux equations [12]. This result indicates a qualitative difference from the previous one obtained with adiabatic pressure [5], where the only one singular point corresponding to the slow magnetosonic wave exists.

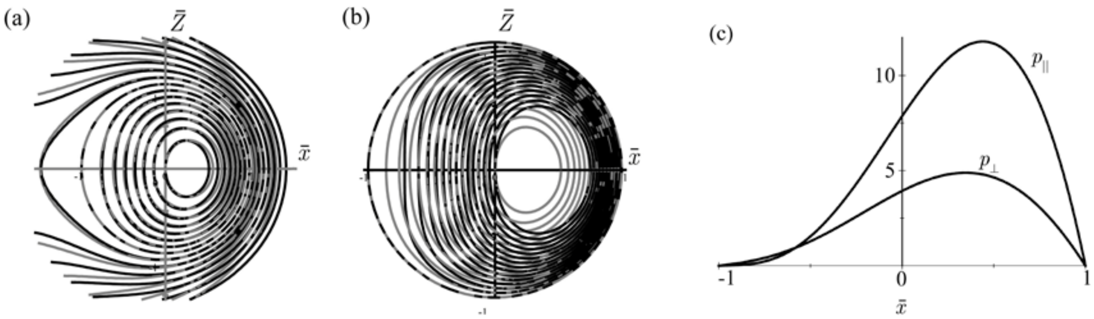

FIG. 1. Analytical solution for single-fluid equilibrium with flow: (a) the magnetic flux surfaces (black) compared with its static case (gray), (b) the isosurfaces of the average pressure ($p_{||}$+ $p_{\perp}$)/2 (black) and the magnetic flux surfaces (gray) and (c) radial profiles of $p_{||}$ and $p_{\perp}$ in the midplane.

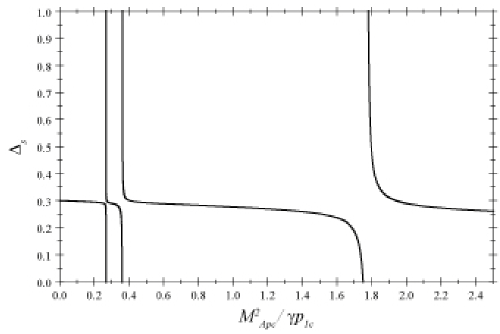

FIG. 2. The shift of the magnetic axis from the geometric axis as a function of the square of the poloidal Mach number.

2.4. Numerical results

We have solved the equilibrium equations for two-fluid equilibria with flow and FLR effects (λ_H,λ_i)=(1,1) numerically by using the finite element method for the case of isotropic, adiabatic ion pressure [4]. Figure 3 shows the following features of two-fluid equilibria: (i) the isosurfaces of the magnetic flux ψ, the pressure p and the ion stream function Ψ do not coincide with each other [Fig. 3 (a) - (c)], (ii) the solutions depend on the sign of the $E\times B$ flow to the ion diamagnetic flow [Fig. 3 (d) - (f)].

3. Reduced MHD Equations for Stability of Poloidal-sonic Flow

In this section, we show the reduced single-fluid MHD equations for the stability of high-beta tokamak equilibria with poloidal-sonic flow. We modify the reduced equations found by Strauss [9] to apply for high-beta plasmas with poloidal-sonic dynamics with non-constant density. The flow velocity and the magnetic field are written as

$$\mathbf{v} \equiv \left(1 + x/R_0\right)\nabla U \times \left(\mathbf{B}/B\right) + v_{\|}\left(\mathbf{B}/B\right), \quad \mathbf{B} \equiv \mathbf{H} \times \nabla\varphi + I\nabla\varphi, \tag{44}$$

$$\boldsymbol{H} \equiv \nabla\psi - \frac{1}{\xi}\frac{\partial^2 F}{\partial\varphi\partial\Theta}\nabla\xi\left(R,Z\right) + \xi\frac{\partial^2 F}{\partial\varphi\partial\xi}\nabla\Theta\left(R,Z\right). \tag{45}$$

The set of reduced equations is obtained as

$$\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial F}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial^2 F}{\partial\Theta^2}=\frac{\mu_0 p}{\xi J B_0},\quad J\equiv R_0\left(\nabla\xi\times\nabla\Theta\right)\cdot\nabla\varphi \tag{46}$$

$$\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\widehat{\Phi}}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial^2\widehat{\Phi}}{\partial\Theta^2}=\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\mu_0 p}{B_0}\frac{\partial U}{\partial\xi}\right)+\frac{1}{\xi}\frac{\partial}{\partial\Theta}\left(\frac{\mu_0 p}{B_0}\frac{\partial U}{\partial\Theta}\right), \tag{47}$$

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+R_0\left(\nabla U\times\nabla\varphi\right)\cdot\nabla\right]\left[\nabla_\perp\cdot\left(\rho R_0\nabla_\perp U\right)\right]+\left\{\rho,\frac{R_0^2\left|\nabla_\perp U\right|^2}{2}\right\}-\left\{R^2-R_0^2,p\right\}\\&+\left[\mathbf{H}\times\nabla\varphi+B_0R_0\left(1-\frac{p}{B_0^2/\mu_0}\right)\nabla\varphi\right]\cdot\nabla\left(Rj_\varphi\right)+\mu_0\frac{j_\varphi}{B_0}\frac{\partial p}{\partial\varphi}+\frac{1}{B_0R_0}\frac{\partial\mathbf{H}}{\partial\varphi}\cdot\nabla p=0,\end{aligned} \tag{48}$$

$$\frac{\partial\psi}{\partial t}=\frac{R^2}{R_0}\nabla\varphi\cdot\left(\nabla U\times\boldsymbol{H}\right)+B_0\frac{\partial U}{\partial\varphi}-\frac{\partial\widehat{\Phi}}{\partial\varphi}, \tag{49}$$

$$\frac{\partial \mathrm{H}}{\partial t}=\nabla\left\{\frac{R^2}{R_0}\left[\boldsymbol{H}\times\nabla\varphi+B_0R_0\left(1-\frac{p}{B_0^2/\mu_0}\right)\nabla\varphi\right]\right\}+\frac{\mu_0}{B_0}\left(\frac{\partial U}{\partial\varphi}\nabla p-\frac{\partial p}{\partial\varphi}\nabla U\right), \tag{50}$$

$$\rho\left[\frac{\partial}{\partial t}+R_0\left(\nabla U\times\nabla\varphi\right)\cdot\nabla\right]v_\parallel+\frac{R}{B_0R_0}\left[\left(1+\frac{p}{B_0^2/\mu_0}\right)\boldsymbol{H}\times\nabla\varphi+B_0R_0\nabla\varphi\right]\cdot\nabla p=0, \tag{51}$$

$$\begin{aligned}&\frac{\partial p}{\partial t}+\frac{v_\parallel R}{B_0R_0}\left[\left(1+\frac{p}{B_0^2/\mu_0}\right)\mathbf{H}\times\nabla\varphi+B_0R_0\nabla\varphi\right]\cdot\nabla p\\&+\left(R/BR_0\right)\left[I\nabla U\times\nabla\varphi+\left(\nabla U\cdot\nabla\varphi\right)\mathbf{H}-\left(\mathbf{H}\cdot\nabla U\right)\nabla\varphi\right]\cdot\nabla p\\&+\gamma p\left[\mathbf{H}\times\nabla\varphi+B_0R_0\left(1-\frac{p}{B_0^2/\mu_0}\right)\nabla\varphi\right]\cdot\nabla\left[\frac{v_\parallel R}{B_0R_0}\left(1+\frac{p}{B_0^2/\mu_0}\right)\right]\\&+\gamma p\nabla\left(R/BR_0\right)\cdot\left[I\nabla U\times\nabla\varphi+\left(\nabla U\cdot\nabla\varphi\right)\mathbf{H}-\left(\mathbf{H}\cdot\nabla U\right)\nabla\varphi\right]\\&+\frac{\gamma pR}{BR_0}\left[\left(\nabla U\cdot\nabla\varphi\right)\nabla\cdot\mathbf{H}+\mathbf{H}\cdot\nabla\left(\nabla U\cdot\nabla\varphi\right)-\nabla\varphi\cdot\nabla\left(\mathbf{H}\cdot\nabla U\right)+\nabla I\cdot\left(\nabla U\times\nabla\varphi\right)\right]=0,\end{aligned} \tag{52}$$

$$\begin{aligned}&\frac{1}{R^2}\frac{\partial I}{\partial t}+\left(\nabla U\times\nabla\varphi\right)\cdot\nabla\left(\frac{I^2}{R_0RB}\right)+\frac{I}{R_0RB}\left(\nabla U\cdot\nabla\varphi\right)\nabla\cdot\mathbf{H}+\mathbf{H}\cdot\nabla\left[\frac{I}{R_0RB}\left(\nabla U\cdot\nabla\varphi\right)\right]\\&+\left(\mathbf{H}\times\nabla\varphi\right)\cdot\nabla\left(\frac{\mathbf{H}\cdot\nabla U}{R_0RB}\right)+\frac{\mu_0}{R_0RB}\left(\mathbf{H}\cdot\nabla U\right)\left(\nabla\varphi\cdot\nabla p\right)=0,\end{aligned} \tag{53}$$

where $\widehat{\Phi}\equiv\Phi+B_0U$. The above equations satisfy the energy conservation up to the order required by the equilibrium in the previous section as

$$\frac{1}{2}\frac{\partial}{\partial t}\int d^3\mathbf{x}\left\{\left[\rho\left(\left|\nabla U\right|^2+v_\parallel^2\right)+\frac{1}{\mu_0R^2}\left(\left|\mathbf{H}\right|^2+I^2\right)+\frac{2p}{\gamma-1}\right]\right\}=0, \tag{54}$$

which can be shown by asymptotic expansions in terms of ε.

4. Concluding Remarks

We have derived a set of reduced equilibrium equations for high-beta tokamaks with toroidal and poloidal flow comparable to the poloidal sound velocity and pressure anisotropy is derived in a unified form of single-fluid and Hall MHD models and a two-fluid MHD model with ion finite Larmor radius (FLR) terms. We have found an analytic solution for the single-fluid model and have shown complicated characteristics in the region around the poloidal sound velocity due to pressure anisotropy and the parallel heat flux. We have found numerical solutions by using the finite element method for the two-fluid model with FLR effects in the case of isotropic, adiabatic pressure and have shown the following features of two-fluid equilibria: the isosurfaces of the magnetic flux, the pressure and the ion stream function do not coincide with each other, and the solutions depend on the sign of the radial electric field. We have derived reduced single-fluid MHD equations with time evolution that are consistent with the above equilibria in order to study their stability and have shown that the energy is conserved up to the order required by the equilibria.

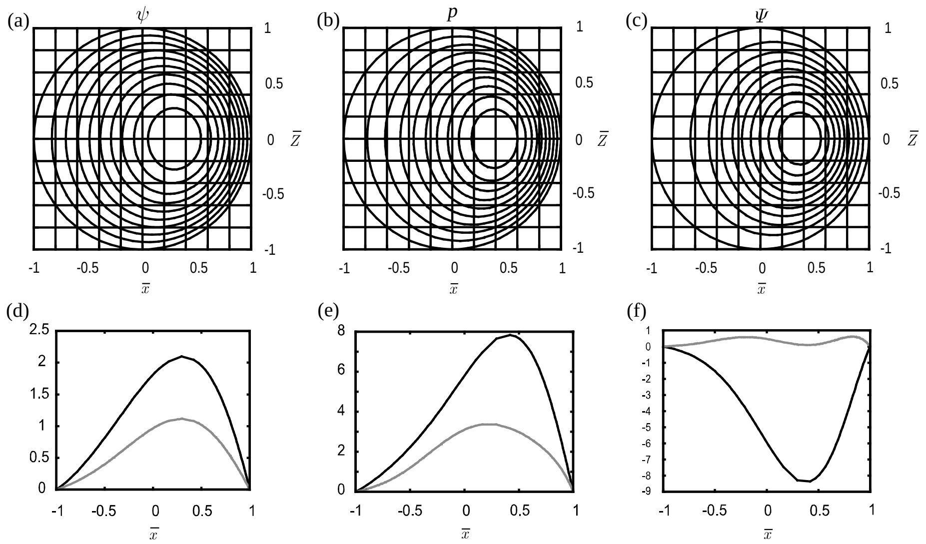

FIG. 3. Isosurfaces of (a) the magnetic flux ψ, (b) the pressure p and (c) the ion stream function Ψ, and profiles of the normalized values of (d) ψ (e) p and (f) Ψ in the midplane, where black (gray) lines are for the case where the signs of $E\times B$ and the ion diamagnetic flows are the same (opposite).

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