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SOKENDAI Lecture Series "Mathematics for Physics " Mathematical Tools for Nonlinear Phenomena (Licture Series-I)

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SOKENDAI Lecture Series " Mathematics for Physics"

Mathematical Tools for Nonlinear Phenomena

(Lecture Series-I)

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Abstract

Mathematical tools dealing with nonlinear phenomena are numerous and varied and have received much attention of many researchers during the last 100 years. Marvelous results obtained in plasma physics for past few years have actually enhanced its prestige and importance of nonlinear physics considerably in the eyes of plasma fusion community. In this series of lectures, an attempt giving an introductory presentation of a variety of complementary methods and viewpoints that may be used in the study of broad spectrum of nonlinear phenomena is presented. The organization of this series of lectures consists of the three perspectives such as (1) mathematical tools of nonlinear phenomena, (2) WKB methods and related topics, and (3) mathematical topics associated with bifurcation phenomena. First, the mathematical tools of nonlinear phenomena (lecture-1) are presented in this article.

Keyword: Typical nonlinear equations, analytical tools, non-secular perturbation method, multi-time expansion method, reductive perturbation method, Hopf-Cole transformation, Bäcklund transformation, Konno-Sanuki transformation, Steepest descent method, Van del pol equation, Mathieu equation, K-dV equation, Sine-Gordon equation, Nonlinear Schrödinger equation, 1D and 2D-Solitons, Convective Cells

Contents

1. Brief Introduction of Typical Nonlinear Equations

In this chapter, we briefly represent an introduction of couple of typical nonlinear differential equations and viewpoints that can be used in the study of a broad spectrum of nonlinear dynamic systems.

1.1 Self excited(autonomous) oscillation(自励振動)

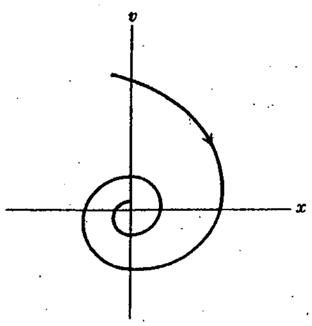
We here have the following two cases, namely, ″ positive resistivity″($\varepsilon > 0$) and ″ negative resistivity” ($\varepsilon < 0$) associated with the autonomous oscillation, which is described by the following differential equation

$$\ddot{x} + \varepsilon\dot{x} + x = 0, \quad \dot{x} \equiv dx/dt$$
$$\dot{x} = v, \quad \dot{v} = -\varepsilon\dot{x} - x, \qquad (1\text{-}1\text{-}1)$$

It should be noted that the case of $\varepsilon > 0$ corresponds to the positive resistivity, leading “ attractor problem” and the case of $\varepsilon < 0$ is related to the negative resistivity associated with “ no attractor problem but the excitation one. In order to study these dynamical processes, the diagram method in phase space is often used.

Diagram Method (v, x)

$$\frac{dv}{dx} = -\frac{\varepsilon v + x}{v}, \qquad (1\text{-}1\text{-}2)$$



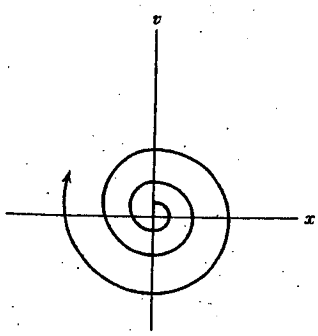


Fig. 1a Resistive, attractor ($\varepsilon > 0$) Fig. 1b Negative resistive, excitation ($\varepsilon < 0$)

1.2 Van del Pol equation

The Van del Pol equation is a well-known nonlinear differential equation, describing the excitation circuit of electric oscillation in vacuum Tube (Ref. 1 and Ref. 2), which is given as

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0, \quad (\varepsilon > 0), \qquad (1\text{-}2\text{-}1)$$

where the second term in (1-2-1), $-\varepsilon(1 - x^2)$ may give the negative resistivity in case of $x^2 < 1$ and it leads the " excitation" of oscillation. As the solution grows in time, the resistive term hardly affects the excitation as x-> 1 and it finally saturate.
When $\varepsilon < 1$, we solve Van del Pol equation by using **Mathematica**.

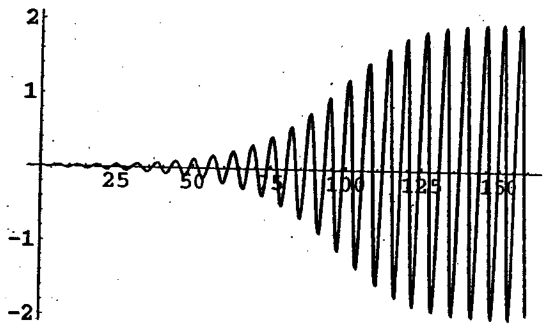


Fig. 2a Solution of Van del Pol equation, where x=0.01, $\ddot{x} = 0$, $\varepsilon = 0.1$ at t=0

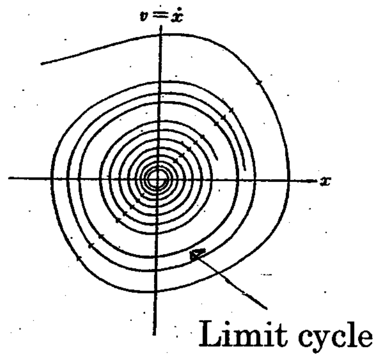


Fig. 2b Limit cycle in attractor problem

Note!

Averaged work due to $-\varepsilon(1 - x^2)\dot{x}$ term should be zero near " limit cycle" , namely,

$$\int_0^{T=2\pi} (1 - x^2)\dot{x}dx = \int_0^T (1 - x^2)\dot{x}^2 dt \cong 0. \qquad (1\text{-}2\text{-}2)$$

If $x = a\cos(t)$ on limit cycle, we obtain

$$\int_0^{2\pi} (1 - a^2\cos^2 t)a^2 \sin^2 t dt = \pi a^2 \left(1 - \frac{a^2}{4}\right) = 0, \quad (1\text{-}2\text{-}3)$$

which gives the limit cycle with $a = \pm 2$.

1.3 Examples of autonomous oscillation or relaxation oscillation

A solution of Van del Pol equation in case of $\varepsilon \gg 1$ can be solved by **Mathematica** in the same way of Fig. 2a. The solution and the phase diagram for a typical example are illustrated for given parameters and initial condition in Fig. 3a and Fig. 3b, respectively.

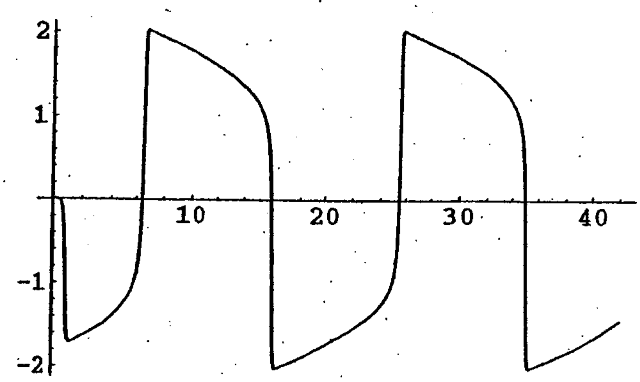


Fig. 3a Solution of Van del Pol equation for $\varepsilon = 10$

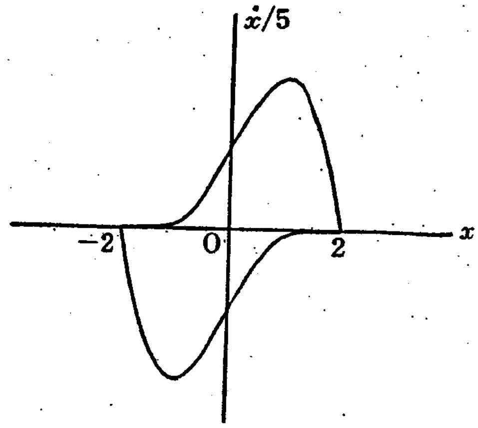


Fig. 3b Limit cycle in phase diagram for $\varepsilon = 10$

Experimental observation associated with autonomous oscillation

Bifurcation nature of the electrostatic potential of toroidal helical plasmas have been studied experimentally in detail by the Heavy Ion Beam Probing (so called HIBP) measurements in the Compact Helical System(CHS) experiment. These observations reveal a similar dynamics to the autonomous oscillation mentioned in this chapter. Self-sustained oscillation (electric pulsation) is observed in ECR+NBI heating phase as shown in Fig. 4. Pulsating behavior of central potential in the low-density region and the line-averaged electron density measured with HCN interferometer are represented by solid line and by dashed line in Fig. (a), respectively. Spatial structural change of the potential before and after transition is also shown in Fig. 4(b). Here, ρ is the normalized minor radius. As is shown in Fig. 4, the bifurcation nature causes a stationary self-excited oscillation in potential termed potential pulsation. The phenomenon shows a number of variation and is repetitive back and forth transitions between bifurcated states.

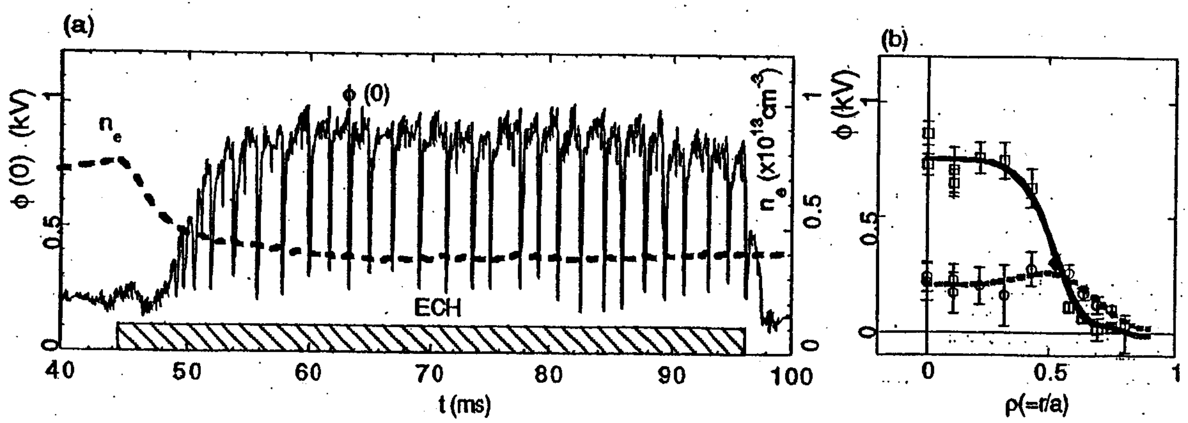


Fig. 4 Self-sustained oscillation (electric pulsation) (a) Pulsating behavior of central potential(solid line) and the line-averaged electron density(dashed line). (b) Spatial structural change of the potential before and after transition. (from Ref. 28)

1.4 Other typical equations

We here consider the following two typical equations, namely, **Rayleigh equation and Lineard equation.**

Rayleigh Equation

Putting $x=\sqrt{3}\dot{y}$ in the Van del Pol equation (1-2-1), we have

$$\dddot{y}-\varepsilon(1-3\dot{y}^2)\ddot{y}+\dot{y}=0,\ \ddot{y}-\varepsilon(1-\dot{y}^2)\dot{y}+y=c \quad (1\text{-}4\text{-}1)$$

or

$$\ddot{x}-\varepsilon(1-\dot{x}^2)\dot{x}+x=0,\ y-c=x. \quad (1\text{-}4\text{-}2)$$

This is called as the Rayleigh equation.

Lineard Equation

The general form of the Rayleigh equation is called as the Lineard equation, which is given by

$$\ddot{x}+\phi(\dot{x})+x=0. \quad (1\text{-}4\text{-}3)$$

This equation can be written as following two coupled equations

$$\frac{dx}{dt}=v,\ \frac{dv}{dt}=-[x+\phi(v)] \quad , \quad (1\text{-}4\text{-}3)$$

which yields the characteristic curve

$$\frac{dv}{dx} = -\frac{x + \phi(v)}{v} \quad . \qquad (1\text{-}4\text{-}4)$$

This curve is called as the Lineard characteristic curve. Illustrating this curve in phase space (x, v), we can study the solution of (1-4-3).

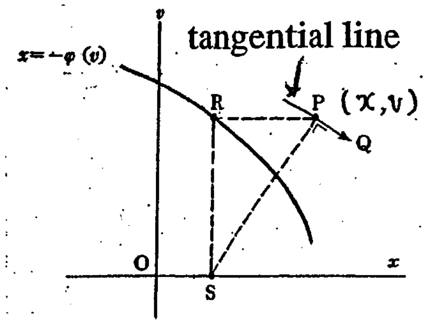


Fig. 5 Example of Lineard characteristic curve is plotted.

Two typical cases of the characteristic curves for the damping oscillation (1-1-1) and Rayleigh oscillation (1-4-3) are plotted in Fig. 6.

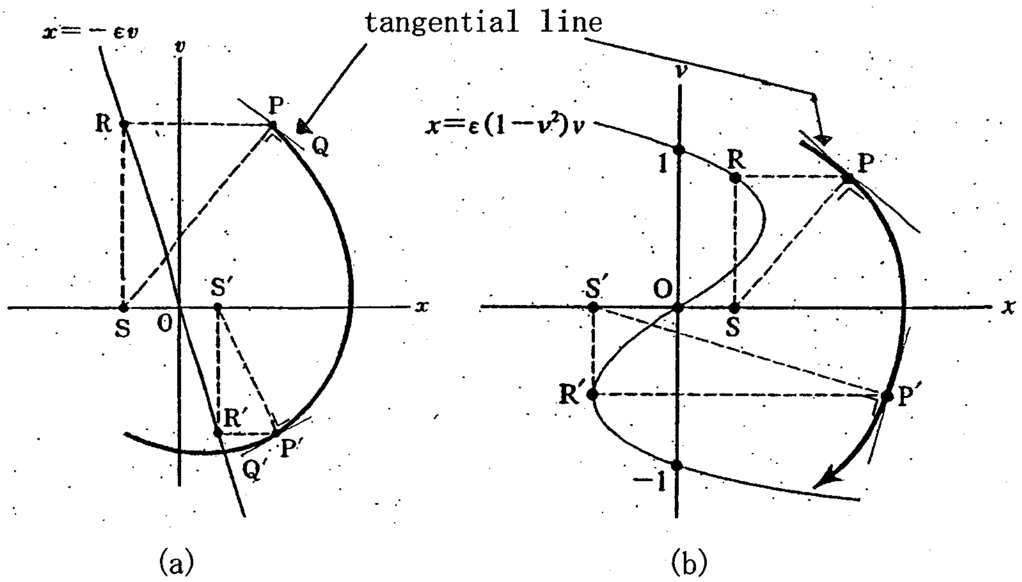


Fig. 6 Characteristic curves, (a) damping oscillation and (b) Rayleigh Oscillation.

Coffee Break

Autonomous (Self-Excited) Oscillation

Rattling phenomena when open a badly door

Rattling oscillation when drag a desk

Violin (particularly, not so good violinist)

Camera stroboscopic lamp

Small electric bulb on Christmas tree (switch on and off)

Leaves of tree along breeze

Billow due to breezing on the surface of lake

Note: quoted from Prof. M. Toda (Ref.8),

誰が風を見たでしょうか
誰も風を見はしない
それでも木の葉をゆるがせて
風はそっと通り過ぎていく

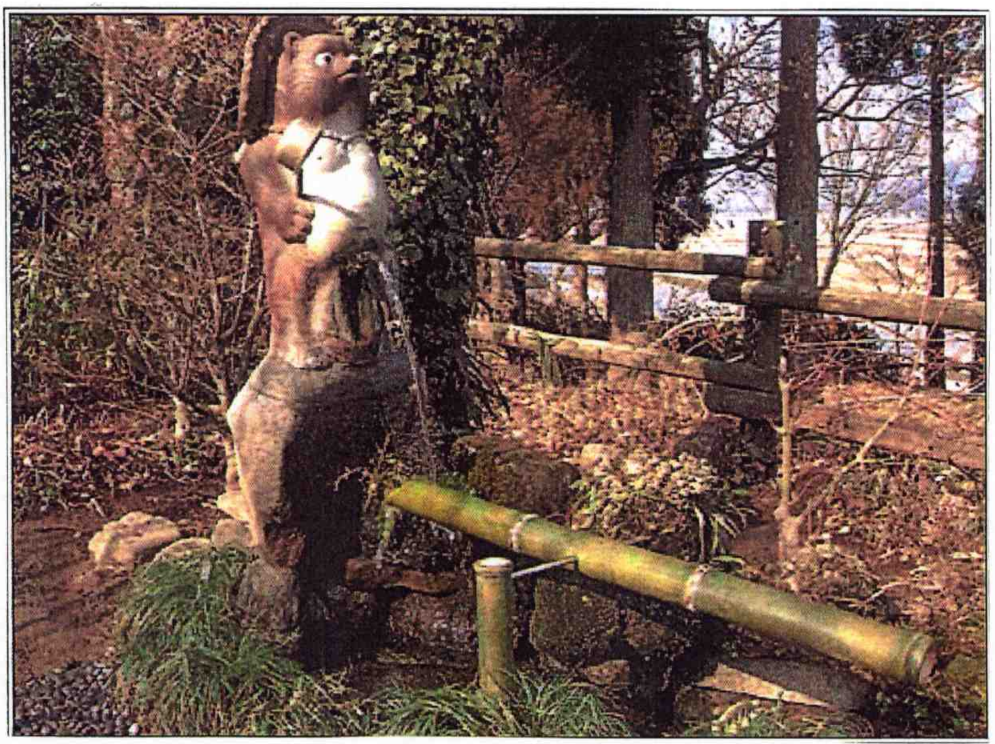


Fig.7 "Shishiodoshi"(鹿威し)(http://www2u.biglobe.ne.jp/~g-grass/sisiodoshi/sisiodoshi.html

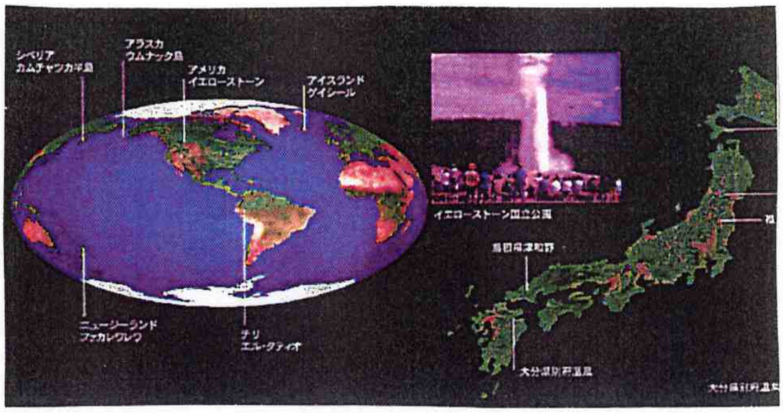


Fig. 8 Intermittent Hot Spring (間歇泉)

(http://www.town.shikabe.hokkaido.jp/kanketusen)

Fig. 9 Intermittent Hot Spring (Ireland)

(from http://ja.wikipedia.org/wiki/間欠泉)

Fig. 10 Intermittent Hot Spring(Onikoube(鬼首)Miyagi prefecture)

(from http://www.towninf.co.jp/p/04/121539)

1.5 Logistic equation

We here define N as a population or species. If the velocity (N') Is proportional to N, we have

$$\frac{dN}{dt} = kN, \quad ----> N = N_0 e^{kt} \ , \qquad (1\text{-}5\text{-}1)$$

If k<0, it decays and it grows in time when k>0.

If the time is discrete, (1-5-1) is replaced by the following discrete equation

$$N_{n+1} - N_n = k\tau N_0, \quad N_{n+1} = N_n(1 + k\tau) = (1 + k\tau)^2 N_{n-1} = ...$$
$$N_n = (1 + k\tau)^n N_0 \ , \qquad (1\text{-}5\text{-}2)$$

If we introduce $\tau = t/n$ and use the relation

$$(1 + k\tau)^n = (1 + \frac{kt}{n})^n = \{(1 + \frac{kt}{n})^{n/kt}\}^{kt}, \qquad (1\text{-}5\text{-}3)$$

$$\lim_{n->\infty}(1 + \frac{a}{n})^{n/a} = e, \quad \lim_{n->\infty}(1 + k\tau)^n = e^{kt}, \qquad (1\text{-}5\text{-}4)$$

we finally have the solution

$$N = N_0 e^{kt}. \qquad (1\text{-}5\text{-}5)$$

It should be noted that the solution (1-5-2) reduces to solution (1-5-1) by taking the continuous limit.

Digression

"Logistic" means a physical distribution(物流、兵站).

The Logistic(兵站)means the transport of the materials to the front line(物資を前線に送る).

Q. Is any nonlinear equation always not solvable?
A. Not always.

As the population (N) increases, it tends to saturate in time and the following model equation is employed,

$$\frac{dN}{dt}=\alpha N(1-\beta N),\quad \alpha,\beta \text{ are constants,} \tag{1-5-6}$$

which is a nonlinear equation for N. Since (1-5-6) can be written as

$$\frac{1}{N^2}\frac{dN}{dt}=\alpha(\frac{1}{N}-\beta),\quad \frac{d}{dt}(\frac{1}{N}-\beta)=-\alpha(\frac{1}{N}-\beta)\ , \tag{1-5-7}$$

we can solve (1-5-7) and have the solution

$$\frac{1}{N}-\beta=(\frac{1}{N_0}-\beta)e^{-\alpha t},\quad \text{--->}\ N=\frac{1}{\beta+(1/N_0-\beta)e^{-\alpha t}}\ . \tag{1-5-8}$$

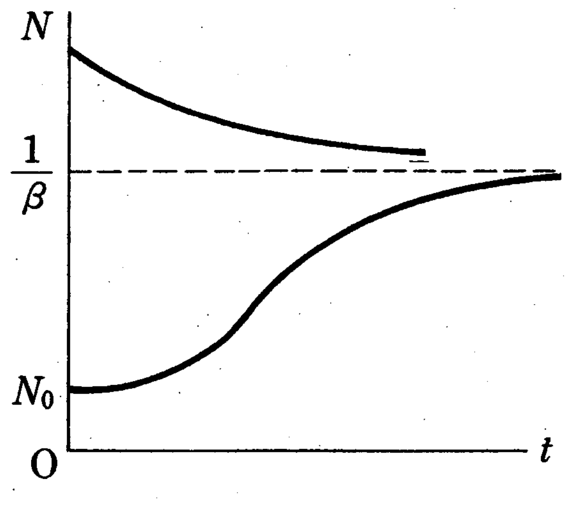


Fig. 11 Logistic curve

Linearization of (1-5-6)

If we linearize (1-5-6) in the following

$$\frac{d}{dt}(\frac{1}{N}-\beta)=-\alpha(\frac{1}{N}-\beta),\ (\frac{1}{N_{n+1}}-\beta)-(\frac{1}{N_n}-\beta)=-\alpha\tau(\frac{1}{N_n}-\beta),$$

$$N_{n+1}-N_n=\alpha\tau N_{n+1}(1-\beta N_n)\ ,\ \frac{1}{N_{n+1}}=\frac{1}{N_n}(1-\alpha\tau)-\alpha\beta\tau,$$

we can solve it and have the solution

$$\frac{1}{N_n}-\beta=(1-\alpha\tau)^n(\frac{1}{N_0}-\beta),\ N_n=\frac{1}{\beta+(\frac{1}{N_0}-\beta)(1-\alpha\tau)^n},\quad (1\text{-}5\text{-}9)$$

Taking the limit $n->\infty$, then the solution (1-5-9) reduces to

$$N_n=\frac{1}{\beta+(\frac{1}{N_0}-\beta)e^{-\alpha t}}\quad . \quad (1\text{-}5\text{-}10)$$

Comment on discrete equation of (1-5-6)

The discrete equation of (1-5-6) is represented as

$$N_{n+1}-N_n=\alpha N_n(1-\beta N_n). \quad (1\text{-}5\text{-}11)$$

It should be noted that this equation could not be lenearized in general. For $\beta<<1$, the solution is similar to (1-5-8). But, the solution N becomes Chaotic when β exceeds some critical value (note: this case will be discussed later in Lecture 3)

It is generally pointed out that when the population increases, it tends to saturate due to self-controlling effect.

Examples

Mouse and suicide in the river, # relation between human population and Agriculture Revolution or Industrial Revolution

These are beyond description on the basis of ″ Logistic Equation″.

1.6 Two variables Lotka-Volterra equation

The two variable Lotka-Volterra equation is represented by the following two coupled equations,

$$\frac{dx}{dt} = \varepsilon_1 x - k_1 xy \ , \ \frac{dy}{dt} = -\varepsilon_2 y + k_2 xy . \qquad (1\text{-}6\text{-}1)$$

The General Form of Lotka-Volterra Eq. is also given as

$$\frac{dN_j}{dt} = \varepsilon_j N_j + \frac{1}{\beta_j}\sum_{k=1}^{n} \alpha_{jk} N_j N_k, (j = 12,.....n), \quad \alpha_{jk} = -\alpha_{kj} . \qquad (1\text{-}6\text{-}2)$$

The time evolution of the solutions (x and y) of (1-6-1) is plotted in Fig. 12. It turns out from Fig. 12 that the solutions x and y show periodic motion with slightly shifted phase and the species (x) increases in time and simultaneously the species (y) decreases. This type of mutual motions is repeated periodically.

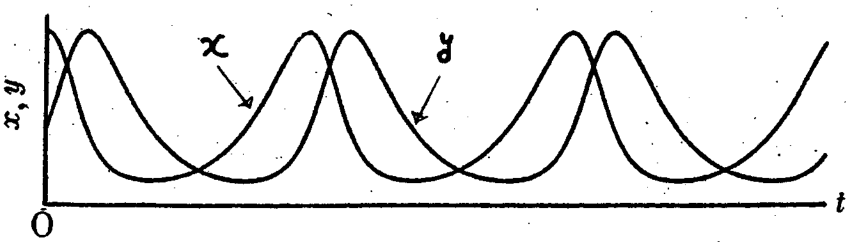


Fig. 12 typical solution of two variable Lotka-Voltrra equation

It should be noted that (1-6-2) has the following conservation law

$$G = \sum_{j=1}^{n} r_j (e^{x_j} - x_j), \ x_j = \log\left(\frac{N_j}{\overline{N}_j}\right), \ r_j = \beta_j \overline{N}_j, \qquad (1\text{-}6\text{-}3)$$

where $\overline{N}_j$ is the stationary solution. Since this equation has only the energy conservation, it is a non-integral system and the motion of this equation becomes chaotic and/or ergodic.

Examples of these phenomena

A plankton feeding fish in the Adriatic Sea in the Balkan Peninsula,

Relation between Rabbit and fox in Canadian grass-covered plain.

Tea Time

2. Asymptotic Methods in Nonlinear Oscillation

2.1 Secular terms

When one discuss the topic associated with the famous " three body problem " during the initial stage of celestial mechanism, as **Bogoliubov and Mitropolsky** pointed out in the text book [Ref. 2], it is well known that the difficulty arises that it is hardly to use the conventional expansion method in powers of a smallness parameter to get the suitable results during sufficient long interval of time. The results based on these expansion techniques may contain so-called the **secular terms** like

$$t^m sin(\alpha t) \text{ and } t^m \cos(\alpha t) \ ,$$

with time "t" appearing outside the sine and cosine symbol.

To demonstrate this difficulty, we consider a following trivial example based on following simplified equation

$$\frac{dx}{dt} = -\varepsilon t \ , \qquad (2\text{-}1\text{-}1)$$

which gives the solution, $x = Ce^{-\varepsilon t}$. If one applies the conventional expansion method to Eq. (2-1-1), we obtain

$$x = C(1 - \varepsilon t + \frac{\varepsilon^2 t^2}{2} - ...) \ . \qquad (2\text{-}1\text{-}2)$$

It should be noted that the formulae are applicable only when $\varepsilon t << 1$ and we have no appreciable change during this time interval. One consider a solution of a nonlinear equation containing a smallness parameter ε in the form,

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f(x, \frac{dx}{dt}). \qquad (2\text{-}1\text{-}3)$$

This is a general form of differential equation with a perturbation.

If the solution of (2-1-3) is sought in the form, $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + ...$, it can be easily seen that the use of the simple expansion method leads the appearance of **secular terms** mentioned above. Asymptotic solution of (2-1-3) would be discussed in detail later.

In order to demonstrate the problem associated with secular terms, let us next consider, as an example, the non-attenuating oscillation of some mass m attracted towards an equilibrium position by an elastic force, which is described by

$$m\frac{d^2x}{dt^2}+\alpha x+\gamma x^3=0. \qquad (2\text{-}1\text{-}4)$$

If we put $\alpha/m=\omega^2$, $\gamma/m=\varepsilon$, $x=x_0+\varepsilon x_1+.....$and apply an expansion method to solve (2-1-4), we have a solution with a second order approximation as

$$x=a\cos(\omega t+\theta)-\frac{3\varepsilon}{8\omega}a^3 t sin(\omega t+\theta)+\frac{\varepsilon a^3}{32\omega^2}\cos 3(\omega t+\theta), \qquad (2\text{-}1\text{-}5)$$

which contains the following secular term

$$-\frac{3\varepsilon}{8\omega}a^3 t sin(\omega t+\theta).$$

The solution (2-1-5) contradicts with an accurate solution, which can be described by an elliptic function in the form

$$x=x_{\max} cn\left\{\frac{2K}{\pi}\varphi\right\}, \qquad (2\text{-}1\text{-}6)$$

where cn,K denote the elliptic cosine function(be discussed in Appendix) and the complete elliptic integration of the first kind, respectively.

2.2 Non-secular perturbation method

For simplicity, we now the consider oscillations close to linear oscillations. It should be noted that the asymptotic method discussed by Bogoliubov and Mitropolsky [Ref. 2] is one of the powerful methods to study various kind of oscillations.

Asymptotic solution

We again discuss the differential equation (2-1-3)

$$\frac{d^2x}{dt^2}+\omega^2 x=\varepsilon f(x,\frac{dx}{dt}), \qquad (2\text{-}2\text{-}1)$$

where ε is the small positive parameter and the right hand side of (2-2-1) corresponds to a perturbation term including the friction term. When $\varepsilon=0$, we have purely harmonic oscillation as

$$x = a\cos\varphi, \tag{2-2-2}$$

with a constant amplitude (a) and a phase angle (φ),

$$\frac{da}{dt} = 0, \ \frac{d\varphi}{dt} = \omega, \ (\varphi = \omega t + \vartheta),$$

where the amplitude a and the phase ϑ are time independent constants, depending on the initial conditions. When $\varepsilon \neq 0$, the perturbation may result in the appearance of harmonics in the solution of (2-2-1) and consequently the instantaneous frequency ($d\varphi/dt$) depends on the amplitude a.

The general solution of (2-2-1) is given in the following form

$$x = a\cos\varphi + \varepsilon u_1(a,\varphi) + \varepsilon^2 u_2(a,\varphi) + \ldots\ldots \tag{2-2-3}$$

Here $u_1(a,\varphi), u_2(a,\varphi)$ etc. are periodic function of the angle φ with a period 2π and quantities a and φ are function of time, which are given as

$$\frac{da}{dt} = \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \ldots., \ \frac{d\varphi}{dt} = \omega + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \ldots, \tag{2-2-4}$$

It should be noted that $u_1(a,\varphi), u_2(a,\varphi) \cdots$ $A_j(a,)$ and $B_j(a)$ [j=1, 2⋯] have to be chosen to serve as a solution of (2-2-1).

For the **first approximation**, we have

$$x = a\cos\varphi, \ \frac{da}{dt} = \varepsilon A_1(a), \ \frac{d\varphi}{dt} = \omega + \varepsilon B_1(a), \tag{2-2-5}$$

and we have the following solution for the **second approximation**,

$$x = a\cos\varphi + \varepsilon u_1(a,\varphi), \ \frac{da}{dt} = \varepsilon A_1(a) + \varepsilon^2 A_2(a), \ \frac{d\varphi}{dt} = \omega + \varepsilon B_1(a) + \varepsilon^2 B_2(a). \tag{2-2-6}$$

Here, $A_j(a)$ and $B_j(a)$ (j=1, 2) are given as

$$\begin{aligned} A_1(a) &= -\frac{1}{2\pi\omega}\int_0^{2\pi} f(a\cos\varphi, -a\omega sin\varphi) sin\varphi d\varphi \\ B_1(a) &= -\frac{1}{2\pi a\omega}\int_0^{2\pi} f(a\cos\varphi, -a\omega sin\varphi)\cos\varphi d\varphi \end{aligned} \tag{2-2-7}$$

and

$$u_1(a,\varphi) = \frac{g_0(a)}{\omega^2} - \frac{1}{\omega^2}\sum_{n=2}^{\infty}\frac{g_n(a)\cos(n\varphi) + h_n(a)sin(n\varphi)}{n^2 - 1}, \tag{2-2-8}$$

with the abbreviations of

$$g_n(a)=\frac{1}{2\pi}\int_0^{2\pi} f(a\cos\varphi,-a\omega sin\varphi)\cos(n\varphi)d\varphi$$
$$h_n(a)=\frac{1}{2\pi}\int_0^{2\pi} f(a\cos\varphi,-a\omega sin\varphi)sin(n\varphi)d\varphi \quad , \qquad (2\text{-}2\text{-}9)$$

where the explicit formula for $A_2(a)$ and $B_2(a)$ in (2-2-6) is relatively complicated and are given by Eq. (1.30) in Ref. 2.

2.3 Autonomous oscillation

We here study a case with weak nonlinear friction as an example of (2-2-1), which is described by

$$\frac{d^2x}{dt^2}+\omega^2 x=\varepsilon f(x)\frac{dx}{dt}. \qquad (2\text{-}3\text{-}1)$$

Introducing the following function and its Fourier series in the form,

$$F^*(x)=\int_0^x f(x)dx,\ \ F^*(a\cos\varphi)=\sum F_n{}^*(a)sin(n\varphi),$$

and differentiating it with respect to φ, we get

$$f(a\cos\varphi)a\omega sin\varphi=\sum_{n=0}^{\infty} F_n{}^*(a)sin(n\varphi).$$

Substitution of this relation into (2-2-6) gives $A_1(a)=F_1^*(a)/2,\ B_1(a)=0$. Finally, we have the first approximation as

$$x=a\cos\varphi,\ \ \frac{da}{dt}=\frac{\varepsilon}{2}F_1^*(a),\ \ \frac{d\varphi}{dt}=\omega, \qquad (2\text{-}3\text{-}2)$$

and the second approximation is given as

$$x=a\cos\varphi+\frac{\varepsilon}{\omega}\sum_{n=2}^{\infty}\frac{nF_n{}^*(a)sin(n\varphi)}{n^2-1}, \qquad (2\text{-}3\text{-}3)$$

$$\frac{da}{dt}=\frac{\varepsilon}{2}F_1^*(a),\ \ \frac{d\varphi}{dt}=\omega+\varepsilon^2 B_2(a), \qquad (2\text{-}3\text{-}4)$$

where

$$B_2(a) = -\frac{1}{8a\omega}F_1^*(a)\frac{dF_1^*(a)}{da} - \frac{1}{2\omega a^2}\sum_{n=2}^{\infty}\frac{n^2 F_n^{*2}(a)}{n^2-1}. \qquad (2\text{-}3\text{-}5)$$

It should be noted that we have to take account of the second order term in the phase equation (2-2-13) provided the first order term in (2-2-12) is included.

Van-der-Pol Equation

As an example of application of the non-secular perturbation method mentioned before to nonlinear oscillations, we here investigate the Van-der-Pol equation, which is given by

$$\frac{d^2x}{dt^2} - \varepsilon(1-x^2)\frac{dx}{dt} + x = 0. \qquad (2\text{-}3\text{-}6)$$

If we put $\omega^2 = 1,\ f(x) = 1 - x^2$ in (2-3-1), Eq. (2-2-6) reduces to Van-der-Pol equation (2-3-6). Since we have in this case

$$F^*(a\cos\varphi) = a\left(1-\frac{a^2}{4}\right)\cos\varphi - \frac{a^3}{12}\cos(3\varphi),$$

the first approximated solution is given as

$$x = a\cos\varphi,\quad \frac{da}{dt} = \frac{\varepsilon a}{2}\left(1-\frac{a^2}{4}\right),\quad \frac{d\varphi}{dt} = 1, \qquad (2\text{-}3\text{-}7)$$

which finally gives the solution in the form

$$x = \frac{a_0\exp(\frac{1}{2}\varepsilon t)}{\sqrt{1+\frac{1}{4}a_0^{\,2}(\exp(\varepsilon t)-1)}}\cos(\omega t + \vartheta). \qquad (2\text{-}3\text{-}8)$$

It turns out from (2-3-8) that small the value of the amplitude may be, it will increase monotonically and attain the limiting value of 2 and oscillations with increasing amplitude are automatically excited, namely, the system is a self-excited oscillation (**limit cycle**). As was shown in (2-3-8), $a(t)$ always tends to the limit cycle 2 when $t \to \infty$.

Note: "Forced Van-der-Pol equation" -frequency entrainment

When a periodic force is applied to a self-exciting system associated with Van-der-Pol equation, the oscillator would give up its independent

mode of oscillation and acquire the frequency of the applied oscillating force. This phenomenon is known as "**entrainment**".
(See Ref. 1, textbook by E. Atlee Jackson: "Perspectives of nonlinear dynamics")

Typical examples:

1) **Two clocks on the same wall. Van der Pol** observed (1927) that the coupling mechanism tends to keep synchronous time, provided their independent frequencies are not too far apart.
2) **Locking of two organ pipes or turning forks with nearly the same independent frequencies. Rayleigh** observed this phenomenon.

2.4 Non-secular perturbation method

Parametric Resonances

We now consider the problems associated with the following topics,
1) Boundary of resonance domain, 2) Pass over from off-resonance zone to resonance domain.

Hill equation:

$$\frac{d^2x}{dt^2} + \omega^2[1 - AF(t)]x = 0, \qquad (2\text{-}4\text{-}1)$$

where F (t) is assumed to be periodic.

When the periodic function F (t) has a form, $F(t) = P_0\cos(\nu t)$, the Hill equation reduces to **Mathieu's equation**

$$\frac{d^2x}{dt^2} + \omega^2[1 - h\cos(\nu t)]x = 0, \qquad (2\text{-}4\text{-}2)$$

where $h = AP_0$. We next discuss an approximate solution and determine the zone of stability in the simplest case of Mathieu's equation (2-4-2) assuming that $h \ll 1$. As an example, we study only the fundamental demultiplication resonance for (2-4-2). Assuming that $\omega \cong \nu/2$, we construct approximate solutions corresponding to the resonance case.

For the first approximation, we have

$$x = a\cos\left(\frac{\nu}{2}t + \vartheta\right), \qquad (2\text{-}4\text{-}5)$$

where

$$\frac{da}{dt} = -\frac{ah\omega^2}{2\nu} sin(2\vartheta), \quad \frac{d\vartheta}{dt} = \omega - \frac{\nu}{2} - \frac{ah\omega^2}{2\nu}\cos(2\vartheta). \qquad (2\text{-}4\text{-}6)$$

Introducing new variables u and v in the form, $u = a\cos\vartheta, v = asin\vartheta$, (2-4-6) reduces to the following a set of two linear equations with constant coefficients,

$$\frac{du}{dt} = [-\frac{h\omega^2}{2\nu} - \left(\omega - \frac{\nu}{2}\right)]v, \quad \frac{dv}{dt} = [-\frac{h\omega^2}{2\nu} + \left(\omega - \frac{\nu}{2}\right)u. \qquad (2\text{-}4\text{-}7)$$

The nature of the solution of (2-4-7) or (2-4-6) depends on the roots of the characteristic equation

$$\lambda^2 - \frac{h^2\omega^4}{4\nu^2} + \left(\omega - \frac{\nu}{2}\right)^2 = 0. \qquad (2\text{-}4\text{-}8)$$

The general solution of (2-4-7) is given as

$$u = C_1 e^{\lambda t} + C_2 e^{-\lambda t}$$

$$v = C_1 \frac{-\frac{h\omega^2}{2\nu} + \left(\omega - \frac{\nu}{2}\right)}{\lambda} e^{\lambda t} + C_2 \frac{\frac{h\omega^2}{2\nu} - \left(\omega - \frac{\nu}{2}\right)}{\lambda} e^{-\lambda t}, \qquad (2\text{-}4\text{-}9)$$

where $C_j (j = 1,2)$ are arbitrary constants determined by the initial conditions. We note that the amplitude a is a bounded function provided λ is imaginary. The condition for λ to be real is

$$\frac{h\omega^2}{2\nu} > \left|\omega - \frac{\nu}{2}\right|, \quad \text{or} \quad \frac{h\omega}{4} > \left|\omega - \frac{\nu}{2}\right|,$$

where we used the relation, $\nu = 2\omega + o(h)$. If the frequency of the external force is in the interval

$$2\omega\left(1 - \frac{h}{4}\right) < \nu < 2\omega\left(1 + \frac{h}{4}\right), \qquad (2\text{-}4\text{-}10)$$

the fundamental demultiplication resonance will appear. The inequalities represent the zone of instability within which the solution becomes unstable and oscillations will be excited automatically.

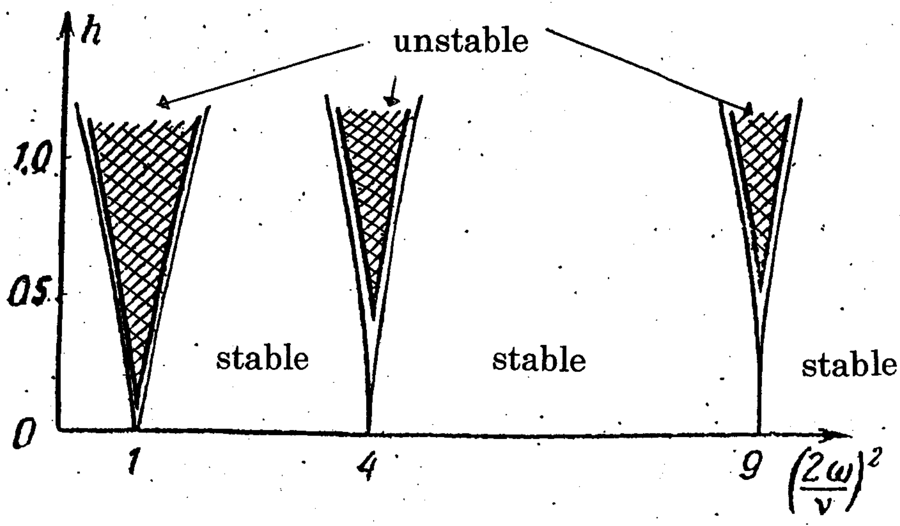


Fig. 13 Zone of instabilities for the fundamental, first and second demultiplication resonances is roughly plotted.

3. Application of Non-Secular Perturbation

3.1 Stability boundary of Mathieu equation based on second order approximation near cyclotron resonance

The motivation of the present discussion is to derive the **formula for Ponderomotive force near cyclotron resonance.**

Ponderomotive force has been proposed as a practical method associated with the following topics

#) RF plugging of open ended devices,

#) RF stabilization of MHD modes,

#) Nonlinear effects such as parametric processes.

We first introduce a topic associated with the Rf-stabilization of an axisymmetric tandem mirror by the ponderomative force in Phaedrus tandem mirror in Wisconsin. Radial ponderomotive force due to the RF electric field opposes the centrifugal force by the field -line curvature to ensure interchange stability. This is indicated by the sensitive dependence on the sign of the difference between the RF frequency and the ion-cyclotron frequency.

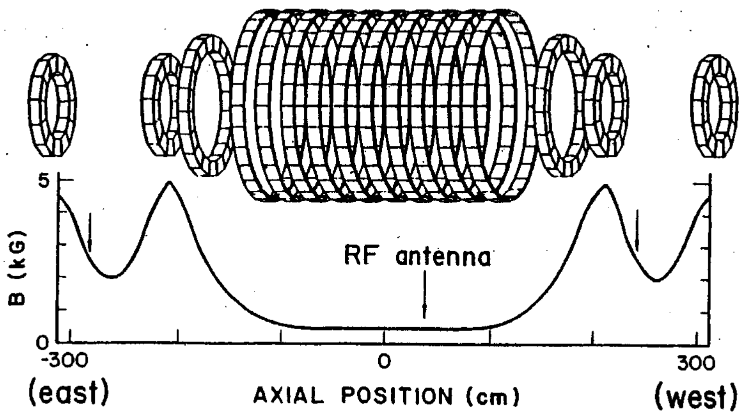


Fig. 14 Axisymmetric coil sets and axial field strength profile in Phaedrus tandem mirror in Wisconsin (quoted from Ref. 13)

Strong dependence of stability on $\omega - \omega_{ci}$, where ω_{ci} is the ion cyclotron frequency, in the central cell is illustrated in Fig. 15. For $\omega > \omega_{ci}$, the fluctuation level is reduced remarkably while for $\omega < \omega_{ci}$ amplitude oscillations are observed. Why the results are so sensitive to the small change of B-field (a few Gauss)? This unsophisticated question may lead the reconsideration of the formula of ponderomotive force near resonance.

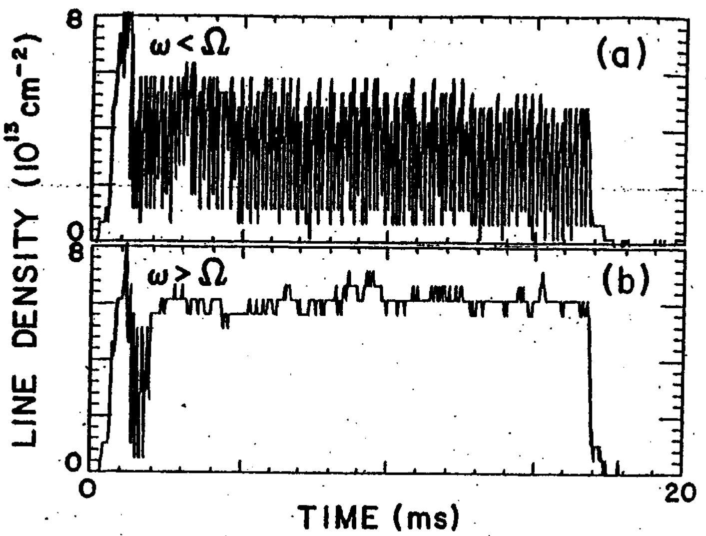


Fig. 15 Central cell line density for single cell operation in Phaedrus Tandem mirror (from Ref. 13)

We consider the conventional expression of ponderomotive force under "**local**" and "**adiabatic**" **approximations**, which is given as

$$\phi_p = \frac{q^2}{4m} \frac{E_\perp^{\ 2}}{(\omega^2 - \omega_c^{\ 2})} \ , \qquad (3\text{-}1\text{-}1)$$

where ω is the frequency of incident RF wave and ω_c is the cyclotron frequency. It should be noted that the expression (3-1-1) is singular as $\omega -> \omega_c$ (Ref. 14).

3.2 Extended ponderomotive expression including " nonlocality" and "nonadiabaticity"

We here note that the adiabaticity is characterized by the relation, $\left|\frac{\omega}{\omega_{ci}} - 1\right| >> \frac{\rho_i}{\Delta}$, where Δ is the scale length of localized RF field and the locality is also characterized by $\xi_i << \Delta$, ξ_i is the excursion length of particle motion in the localized RF field.

In Ref. 15, B. M. Lamb et al. studied a physically correct expression of P. F. by taking account of the finite transit time of particles through axial localized RF-Field. Some results are shown in Fig. 16

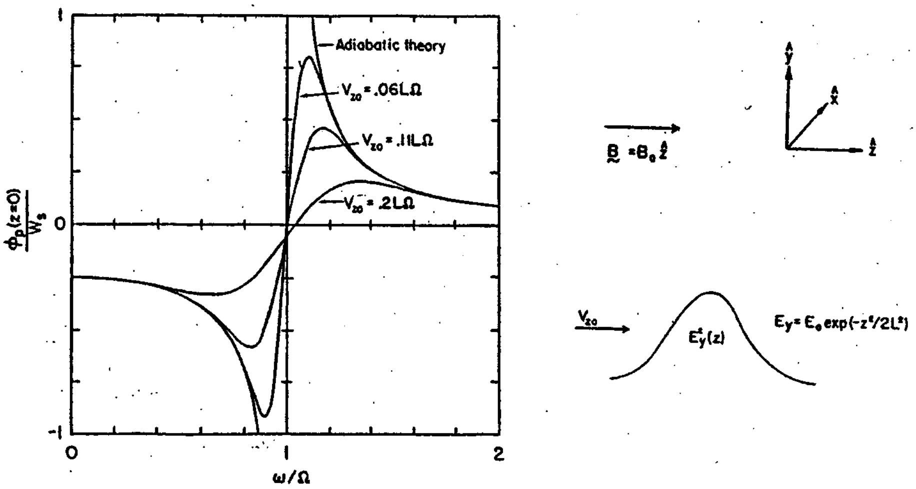


Fig. 16 Frequency dependence of ponderomotive potential ϕ_p for several values of $V_{z0}/L\omega_c$. Also shown is the singular results of the adiabatic theory. (quoted from Ref. 15)

We should note that the conventional expression is still valid provided $\left|\frac{\omega}{\omega_c}-1\right| >> \frac{\rho}{\Delta}$. The maximum locates at $\frac{\omega}{\omega_c} \cong 1+\frac{\pi}{2}\frac{V_{z0}}{\Delta\omega_c}$. These results are illustrated in Fig. 16

In Ref. 19, Hatori and Washimi discussed the compact and general adiabatic expression for ponderomotive scalar and vector potentials based on **"Lie-Transformation Formula"** and extended the formula by taking account of nonlocal effects with finite ξ/Δ.

$$\phi^{av}(r) = \int_0^1 d\varepsilon < \vec{r} + \varepsilon\vec{\xi}(r,t) > sin(\omega t), \qquad (3\text{-}2\text{-}1)$$

where $\vec{E}(\vec{r},t) = -\nabla\phi(x)sin(\omega t)$ and $< A > = \frac{1}{2\pi}\int_{-T}^{T} A(t)dt,\ T = 2\pi n/\omega$. We have to get an explicit solution of ξ for given RF electric field.

To demonstrate the usefulness of the non-secular perturbation method, let consider the derivation of ponderomotive force near ion cyclotron resonance. For simplicity, we apply the following electric field and configuration which are shown in Fig. 17.

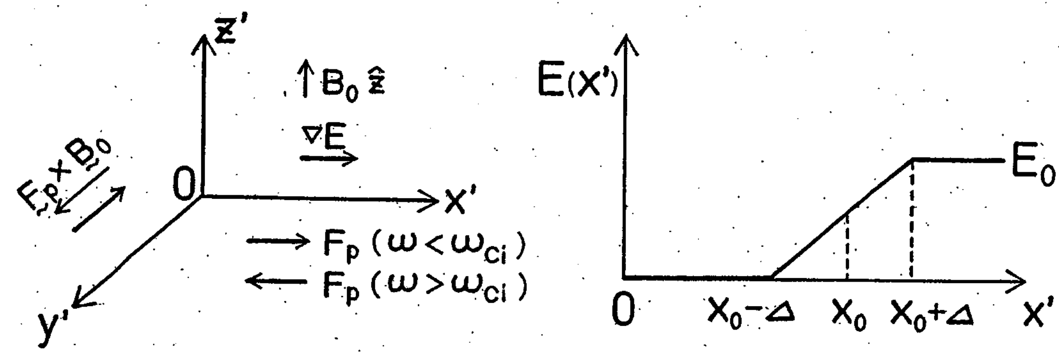


Fig. 17 RF electric field profile and configuration (Ref. 16)

Hereafter we assume relatively small RF electric field with $E \approx O(\varepsilon)$ and we define the coordinate, $\hat{\xi} = x_G + \xi(t;\varepsilon)$ with $x_G = \rho_i \cos\omega t$, where x_G is the coordinate associated with cyclotron motion. We can determine $\xi(t;\varepsilon)$ from the following equation

$$\frac{d^2\hat{\xi}}{dt^2}+\omega_c{}^2[1-\varepsilon\gamma\cos(\omega t)]\hat{\xi}=f(t),\ f(t)=\varepsilon\omega_E{}^2(\bar{x}-x_0+\Delta)\cos(\omega t), \quad (3\text{-}2\text{-}2)$$

with abbreviations, $\omega_E{}^2=\frac{qE_0}{2\Delta m}$, $\gamma=\frac{\omega_E{}^2}{\omega_c{}^2}$, where $\bar{x}$ is the guiding center coordinate. In order to solve (3-2-2), we first discuss an approximate solution for (3-2-2) by

$$\frac{d^2\hat{\xi}_0}{dt^2}+\omega_c{}^2[1-\varepsilon\gamma\cos(\omega t)]\hat{\xi}_0=0. \quad (3\text{-}2\text{-}3)$$

Although we discussed the first approximate solution of (2-4-2) same as (3-2-3), we here study an approximate solution of the second order because we are interested in the expression for ponderomotive force in the first cyclotron resonance ($\omega\cong\omega_c$). Following the formula in Ref. 1, the second order approximate solution of (3-2-3) is given as

$$\hat{\xi}_0(t)=a\cos(\omega t+\vartheta)-\frac{\varepsilon\gamma a\omega_c{}^2}{2\omega(\omega+2\omega)}\cos(2\omega t+\vartheta)-\frac{\varepsilon\gamma a\omega_c{}^2}{2\omega(\omega-2\omega)}\cos(\vartheta), \quad (3\text{-}2\text{-}4)$$

where the amplitude a and the phase ϑ are determined from the set of equations of second approximation

$$\begin{aligned}\frac{da}{dt}&=-\frac{\varepsilon^2\gamma^2a\omega_c{}^4}{8\omega^2(2\omega_c-\omega)}sin(2\vartheta)\\ \frac{d\vartheta}{dt}&=\omega_c-\omega-\frac{\varepsilon^2\gamma^2\omega_c{}^4}{4\omega(4\omega_c{}^2-\omega^2)}-\frac{\varepsilon^2\gamma^2\omega_c{}^4}{8\omega^2(2\omega_c-\omega)}\cos(2\vartheta)\end{aligned} \quad . \quad (3\text{-}2\text{-}5)$$

Using the same process to drive the solutions (2-4-3) to (2-4-7) for the first approximated solution, we can solve (3-2-5) and obtain the second approximated solution for (3-2-5). Also, we can solve (3-2-2) based on this solution (3-2-4) with (3-2-5). Here, we restrict our discussion to evaluate the zone of instability near the first resonance ($\omega\cong\omega_c$) up to the second order with respect to γ, which is given by the inequality

$$1+\frac{\varepsilon^2\gamma^2\omega_c{}^4(3\omega-2\omega_c)}{4\omega^3(4\omega_c{}^2-\omega^2)}<\left(\frac{\omega_c}{\omega}\right)^2<1+\frac{\varepsilon^2\gamma^2\omega_c{}^4(\omega+2\omega_c)}{4\omega^3(4\omega_c{}^2-\omega^2)} \quad . \quad (3\text{-}2\text{-}6)$$

Dependence of the RF strength parameter γ on the zone of instability of the second order near the first resonance is shown in Fig. 18. The particle orbit becomes unstable within two boundaries for fixed parameter $\varepsilon\gamma$ and the orbit is oscillatory outside the boundaries.

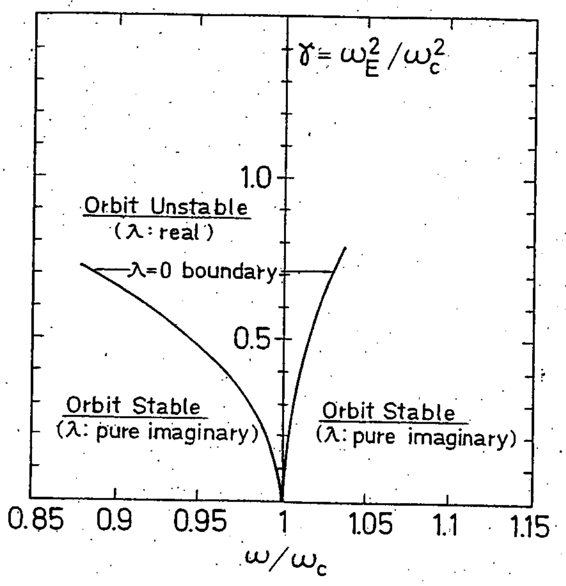


Fig. 18 Dependence of the RF strength parameter γ on the zone of instability versus normalized frequency ω/ω_{ci} (Ref. 16)

After solving (3-2-2) and substituting its solution into the following expression for ponderomative force similar to (3-2-1) as

$$F_p = \frac{qE_0}{2\Delta m}\int_0^1 d\varepsilon < \xi(t;\varepsilon) sin(\omega t), \qquad (3\text{-}2\text{-}7)$$

we obtain the expression for ponderomotive force near the first ion cyclotron resonance after lengthy calculation. We drop the explicit expression for the final solution of (3-2-7) in this lecture note because the expression is relatively complicated form and the detailed derivation is out of scope of this lecture. It should be noted that the expression (3-2-7) reduces to the adiabatic expression in case of cold limit ($\rho_i -> 0$). Detailed derivation of the expression of poderomotive force based on (3-2-7) and comparison between the adiabatic expression and the nonlocal and non-adiabatic expression has been made in Ref. 16, IPPJ-736 (1985) (see Eq. (46) with (47)-(50)).

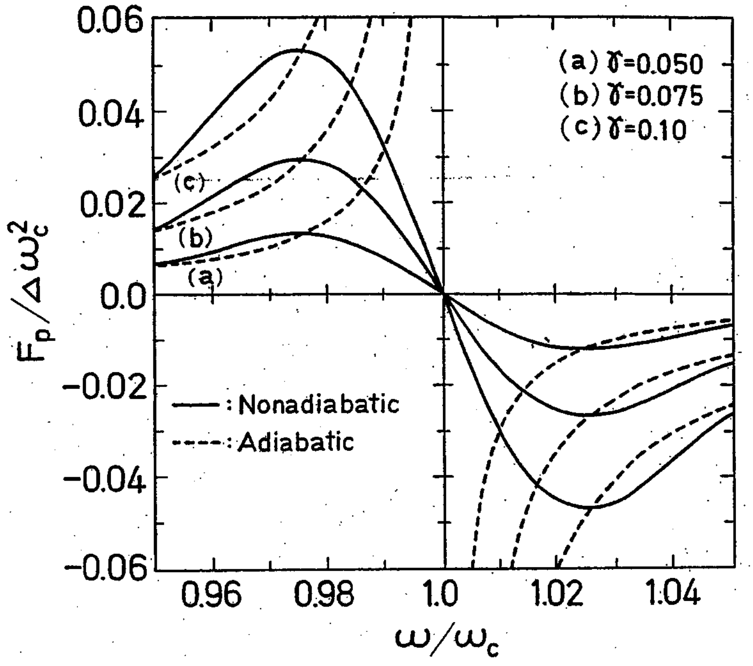


Fig. 19 Frequency dependence of the normalized ponderomotive force $F_p/\Delta\omega_c^2$ for several values of $\varepsilon\gamma$ are illustrated. Also shown are the singular results based on the adiabatic approximation. Parameters, n=20, $\rho_i/\Delta = 0.5$, $\bar{x} = x_0$ are used in the expression (3-2-7). (Ref. 16)

3. 3 Extended expression for ponderomotive force near cyclotron resonance

The ponderomotive force in magnetized plasmas is derived by the **renormalization theory** of wave-particle interaction based on the Vlasov equation. (Ref. 17: M. Kono and H. Sanuki, J. Plasma Physics 38(1987) 43.)

The significant feature of this expression is the non-singular behavior at resonance and this is related to the onset of the diffusive motion of particles due to the orbit instability near resonances.

Sanuki and Hatori discussed the ponderomoive expression based on Lie-Transformation (see, Ref. 16 and (3-2-7) in this lecture note), showing that the particle motion is described by the **inhomogeneous Mathieu equation** and the **orbit instability** are responsible for suppression of the divergence in the adiabatic approximation. It should be noted that the result is sensitive to the period used for averaging with respective to time (period is characterized by the parameter "n" in the time average formula) and the average over the particle distribution was not carried out in this discussion.

We here briefly introduce the expression for ponderomotive force for waves with $k_{//} = 0$, propagating perpendicularly to a uniform magnetic field. When the frequency is close to the mth-order cyclotron frequency $m\omega_c$, A numbers of higher order resonances appear. However, the transition of particles from one cyclotron orbit to another through the electric field may be less important and the contribution from the self-interaction is the most important whenever the amplitude is not large enough for the **onset of global chaos.** However, we may have an **orbit instability**, which drives the particles into diffusive motion when the amplitude exceeds a certain value. We have to take into account of these situations for deriving the expression near resonances. Although we cannot explain whole derivations of the formula in this lecture, we briefly introduce the final result.

The expression for ponderomotive force at the resonance point, $\omega = m\omega_c$ is given as

$$\vec{F}_\alpha(x,t) = -\nabla \frac{\omega_{p\alpha}{}^2 \omega^2}{\omega_c{}^2} \frac{|E_0(x,t)|^2}{4\pi} \left[\frac{J_{m0}(k_{0\perp}\rho)}{k_{0\perp}\rho} \right]^2 \frac{1}{(2\omega + \Delta\omega)^2 + \lambda^2}, \quad (3\text{-}3\text{-}1)$$

with

$$\Delta\omega = \frac{1}{2}|B|^2 \frac{m\omega_c}{4(m\omega_c)^2 - \omega^2} |E_0|^2, \quad (3\text{-}3\text{-}2)$$

$$\lambda^2 = \frac{1}{16}|A|^2|B|^2 \left[\frac{1}{\omega^2} + \frac{4(m\omega_c)^2 + 5\omega^2}{\left[4(m\omega_c)^2 - \omega^2\right]^2} \right] |E_0|^4 - \left[\frac{(m\omega_c)^2 - \omega^2}{2m\omega_c} + \Delta\omega \right]^2, \quad (3\text{-}3\text{-}3)$$

where we dropped the explicit definitions of the coefficients A and B (see, paper by Kono and Sanuki (1987)). From (3-3-3), λ is purely imaginary for small amplitude, giving the frequency shift but becomes real, implying **orbit instability** which may drive the particles into diffusive motion when the amplitude exceeds a certain value.

On the other hand, if all the harmonics of the cyclotron motion are out of resonance, we reach the expression known as the adiabatic ponderomotive force:

$$\vec{F}_{\alpha}(x,t) = -\nabla \sum_{m>0} \frac{\omega_{p\alpha}^{2}}{\omega^{2} - (m\omega_{c})^{2}} \frac{|E_{0}(x,t)|^{2}}{4\pi} \left[\frac{J_{m}(k_{0\perp}\rho)}{k_{0\perp}\rho} \right]^{2}, \qquad (3\text{-}3\text{-}4)$$

which is the extended expression based on particle distribution function even under the adiabatic approximation. Here, α means the particle spices and $J_m(k_{\perp}\rho)$ is the m-th order of Bessel function.

In order to understand the physics behind (3-3-1) - (3-3-4), we finally consider the motion of a particle interacting with an electrostatic wave propagating perpendicularly to the magnetic field. This is the case just under consideration above. If the magnetic field and electric field are given as

$$\vec{B} = (0,0,B_0),\ \vec{E} = (0, E\cos(ky - \omega t - \vartheta), 0),$$

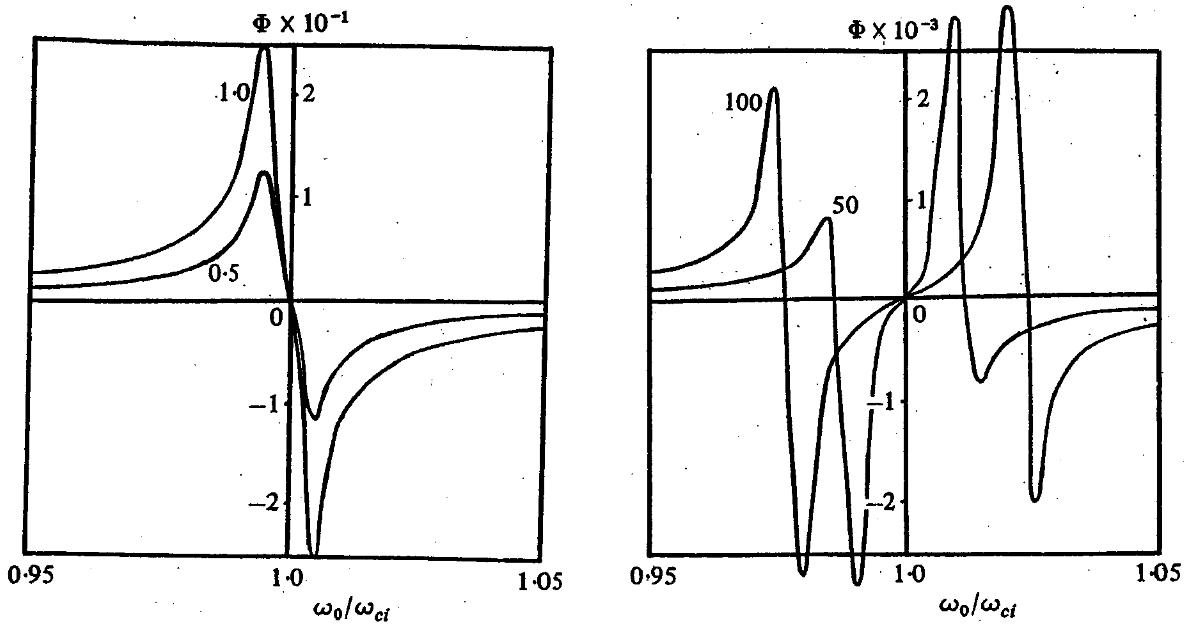
the Hamiltonian of this system is given by

$$H = \frac{1}{2}\left[\left(\frac{dy}{dt} \right)^{2} + y^{2} \right] - \varepsilon sin(y - vt - \vartheta), \qquad (3\text{-}3\text{-}5)$$

where $\varepsilon = eEk/m\omega^2$, $v = \omega/\omega_c$ and (3-3-5) represents the motion of 1D harmonic oscillator perturbed by a sinusoidal wave. In this system, the frequency shift due to the sinusoidal wave is given by $\Delta\omega = O(\varepsilon^{\frac{1}{2}})$. It should be noted that the overlapping of adjacent orbits of the harmonic oscillator occurs provided $\Delta\omega(E) > 1$, which yields the threshold for the amplitude (E_c) satisfying the relation

$$\frac{|E_c|^2}{4\pi nT} \approx \frac{1}{16}\left(\frac{\omega_{ci}}{\omega_{pi}} \right)^{2} \frac{1}{(k\rho)^2}. \qquad (3\text{-}3\text{-}6)$$

In case of $E < E_c$, the particles are confined in the vicinity of the original orbits although the particles becomes somewhat diffusive (**local stochastic instability**). On the other hand, for $E > E_c$, the broadening of the orbits exceeds the distance between adjacent orbits, resulting in global diffusion in phase space (so called **global stochastic instability**).



(a) (b)

Fig. 20 Ponderomotive potential versus normalized frequency for fixed amplitude. The results for W=0.1 and 0.5 for fixed $k_\perp \rho = 0.1$ are in Fig. 20 (a) and the results for W=50 and 100 for fixed $k_\perp \rho = 0.5$ in Fig. 20 (b). (Ref. 17)

Ponderomotive potential, $F_\alpha = -\nabla \phi_\alpha$ against ω/ω_c is plotted for the fixed amplitudes, $W = \left(\frac{\omega_{pi}}{\omega_c}\right)^2 \frac{|E_0|^2}{4\pi n_0 T}$ and $k_\perp \rho$. The results for W=0.1 and 0.5 for fixed $k_\perp \rho = 0.1$ are illustrated in Fig. 20 (a) and the results for W=50 and 100 for fixed $k_\perp \rho = 0.5$ in Fig. 20 (b), respectively. These results show that the stochastic instability is responsible for the removable of the singularity, which appears in the adiabatic approximation. The results shown in Fig. 20 (b) corresponds to the case where $\Delta\omega$ and λ are large enough to have several zeros of denominator of (3-3-1), indicating that the approximation of retaining the contribution of self-interactions becomes invalid. Detailed discussion is beyond the scope of this lecture. The derivation of (3-3-6) was discussed in Ref. 20.

3. Analyses of Wave Equations

4.1 Linear and nonlinear wave equations

Example 1. Simple Wave equation

$$\frac{\partial^2 u}{\partial t^2} = {c_0}^2 \frac{\partial^2 u}{\partial x^2}. \qquad (4\text{-}1\text{-}1)$$

The solution is described by the following d'Alembert Solution,

$$u = f(x - c_0 t) + g(x + c_0 t). \qquad (4\text{-}1\text{-}2)$$

Example 2. Linear and non-dispersive wave

$$u_t + c_0 u_x = 0, \ \ u = f(x - c_0 t), \qquad (4\text{-}1\text{-}3)$$

where the solution is given as

$$u = a\sin(kx - \omega t) \ \ , \omega/k = c_0 \,(\text{const.}). \qquad (4\text{-}1\text{-}4)$$

This wave is called the linear and nondispersive wave.

Example 3. Linear and dispersive wave

$$u_t + c_0 u_x + \alpha u_{xxx} = 0. \qquad (4\text{-}1\text{-}5)$$

If we assume the solution in the form,

$$u = a\sin(kx - \omega t)$$

the phase and group velocities are given as

$$c = \frac{\omega}{k} = c_0 - \alpha k^2, \ (\text{phase velocity}), \ \ c_g = \frac{d\omega}{dk}, \quad (\text{group velocity}).$$

This is the linear and dispersive wave.

Example 4. Simple nonlinear wave

$$u_t + u u_x = 0, \quad u = f(x - ut). \qquad (4\text{-}1\text{-}6)$$

The characteristics of differential equation is $dx/dt = u$. Therefore, the part of large amplitude moves faster than that with smaller amplitude and the wave steepning and breaking phenomena may come up. These typical solutions are illustrated in Fig. 21.

$$\phi_t = \nu\phi_{xx}. \qquad (4\text{-}2\text{-}2)$$

It should be noted that the nonlinear transformation just eliminates the nonlinear term and Burgers equation finally reduces to a "heat type of equation".

How to solve a nonlinear differential equation depends on how to find a nonlinear transformation.

This transforms the problem into an initial value problem. Namely,

$$u = F(x) \quad \text{at } t=0, \qquad (4\text{-}2\text{-}3)$$

$$\phi = \Phi(x) = \exp\left[-\frac{1}{2\nu}\int_0^x F(\eta)d\eta\right], \quad \text{at } t=0. \qquad (4\text{-}2\text{-}4)$$

The solution of Burgers Equation is given as the following Heat Equation,

$$\phi = \frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty}\Phi(\eta)\exp\left[-\frac{(x-\eta)^2}{4\nu t}\right]d\eta. \qquad (4\text{-}2\text{-}5)$$

Therefore, we finally have

$$u(x,t) = \frac{\int_{-\infty}^{\infty}\frac{x-\eta}{t}e^{-\frac{G}{2\nu}}d\eta}{\int_{-\infty}^{\infty}e^{-\frac{G}{2\nu}}d\eta}, \quad G(\eta,x,t) = \int_0^{\eta}F(\eta')d\eta' + \frac{(x-\eta)^2}{2t}. \qquad (4\text{-}2\text{-}6)$$

<u>Behavior as $\nu -> 0$</u>

Using the **steepest descent method** (will be explained later), we carry out the integration in the following way,

$$\frac{\partial G}{\partial \eta} = F(\eta) - \frac{(x-\eta)}{t} = 0, \quad \eta \equiv \xi(x,t) \text{ at stationary point, } F(\xi) - \frac{(x-\xi)}{t} = 0$$

is the stationary point of function G. Asymptotic solution is given as

$$\int_{-\infty}^{\infty} g(\eta)e^{-\frac{G(\eta)}{2\nu}}d\eta \approx g(\xi)\sqrt{\frac{4\pi\nu}{|G''(\xi)|}}e^{-\frac{G(\xi)}{2\nu}} \quad u(x,t) \approx \frac{x-\xi}{t}, \quad ---> u = F(\xi),\ x = \xi + F(\xi)t$$

where $\xi(x,t)$ becomes the characteristic variable.

4.3 Steepest descent method

We here introduce briefly the steepest descent method based on the lecture note by **Alfred Banos, Jr.** (UCLA) (see, Ref. (11)) " Selected Topics on Asymptotic Methods" , (1983/12/15).

Prof. Banos was the MIT group member together with Morse & Fethbach, who were the authors of the famous textbook, " Method of Theoretical Physics, Vol. 1 and 2" . To explain the steepest descent method, we apply an integration,

$$I = \int_C F(w)e^{\phi(w)}dw, \qquad (4\text{-}3\text{-}1)$$

where w is a complex variable of integration and w=0 is a saddle point, namely, $\phi'(w) = 0$ when w=0 , i. e. w=0 is the SP. C is the path of steepest descent in w-plane.

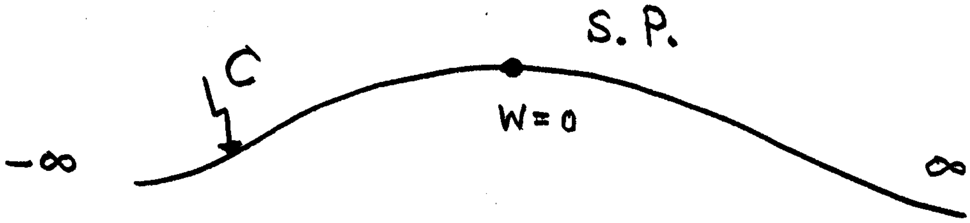


Fig. 22 Path of integration and saddle point, where $\mathrm{Im}\{\phi(w)\} = \mathrm{Im}\{\phi(0)\}$ on C.

Since the integration (4-3-1) can be written as

$$I = \int_C F(w)e^{\phi(w)}dw = e^{\phi(0)}\int_C F(w)e^{\phi(w)-\phi(0)}dw, x^2 = \phi(0) - \phi(w), \qquad (4\text{-}3\text{-}2)$$

where x is a new variable, we can expand the kernel of integration as

$$I \cong e^{\phi(0)} \int_C \{f(0) + ---\} e^{\frac{1}{2}\phi''(0)w^2} dw. \qquad (4\text{-}3\text{-}3)$$

To lowest order, we put $-x^2 = \phi''(0)w^2/2$ or $x = \sqrt{-\phi''(0)/2}w$ and carry out the integration Approximately, then we obtain the following formula,

$$I = \frac{F(0)e^{\phi(0)}}{\sqrt{-\phi''(0)/2}} \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{2\pi} \frac{F(0)e^{\phi(0)}}{\sqrt{-\phi''(0)}}. \qquad (4\text{-}3\text{-}4)$$

If the saddle point occurs at $w = w_s$ instead of w=0, then we have a general formula instead of (4-3-4)

$$I \cong \sqrt{2\pi} \frac{F(w_s)e^{\phi(w_s)}}{\sqrt{-\phi''(w_s)}}, \quad \phi'(w = w_s) = 0 ---> w = w_s \text{ is a SP.} \qquad (4\text{-}3\text{-}5)$$

4.4 Application of Cole- Hopf transformation

Topics of plasma rotations were one of the highlights during 1960's and 1970's associated with the topic of plasma confinement improvement. It was pointed out that for $M_p \approx 1$, shock -like structure in density and potential profiles occurs. Here, M_p is the poloidal Mach number.

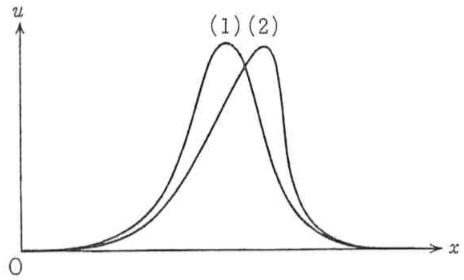
It should be noted that the slow shock problem during 1960-1970 and the strong shock formation from 1990-to the present are studied actively both theoretically and experimentally.

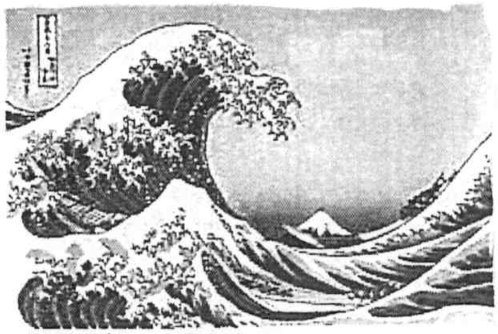
Experiments:

B-3 (1966) in Princeton, JIPP-1 (1974) in IPP at Nagoya where both were the l-3 stellarator devices, and CCT (1990) in UCLA, HYBTOK-II in Nagoya University, H-E (1993) in Kyoto, and CHS (1991) in NIFS associated with the biasing experiments. There are lots of other experimental observations.

Theory:

Hazeltine (1971), Green (1972), Stix (1973), Asano & Taniuti (1972) discussed the **week shock** theory, and Shaing, Hazeltine and Sanuki (1992), Taniuti, Wakatani et al. (1992) discussed the **strong shock** theory.





Wave steepening　　　　　　Wave breaking

Fig. 21 Wave steepening and wave breaking(http://www.sanjo.co.jp/hum/hokusai/kanagawa.jpg)

Example 5. **Burgers equation**

$$u_t + uu_x = \nu u_{xx}. \tag{4-1-7}$$

nonlinear term　　viscosity term

This equation is the 1D Navier- Stokes equation and one of well-known equations in fluid dynamics.

4.2 Cole-Hopf transformation

Cole (1951) and Hopf (1950) noted the following remarkable results: ″ Nonlinear equation may be reduced to linear equation by **nonlinear transformation**″ in the form

$$u = -2\nu \frac{d}{dx} \log \phi. \tag{4-2-1}$$

The transformation in this case has the following two steps

$u = \psi_x$, $\psi_t + \frac{1}{2}\psi_x^{\ 2} = \nu \psi_{xx}$ (first step),

$\psi = -2\nu \log \phi$ (second step) ,

then the equation (4-1-7) reduces to

If we evaluate the poloidal viscosity based on the solution of **linear theory,** we note that the viscosity is in proportion to the derivative with respect to the poloidal angle of density and/or potential within the linear theory, and then the viscosity may become too large at the shock region and we consequently have to discuss the nonlinear analysis to evaluate the poloidal viscosity near $M_p \approx 1$.

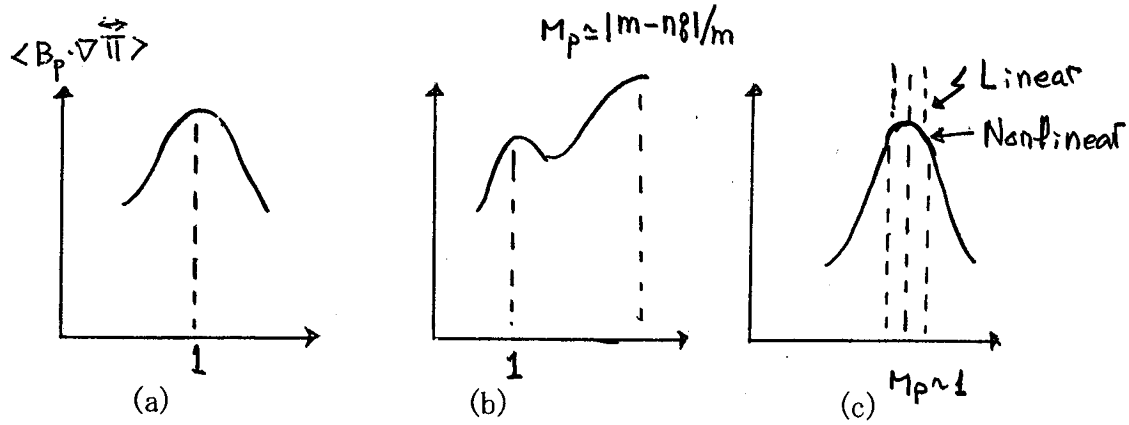


Fig. 23 Poloidal viscosity for tokamaks (a), helical systems (b) and nonlinear theory (c) is illustrated briefly.

Following the paper by **Shaing Hazeltine and Sanuki** (Ref. (23)), we briefly explain how to solve the strong shock problem and to derive the formula applicable even near $M_p \approx 1$ by using the Hope-Cole Transformation method. If we introduce the function χ in the form

$$\chi = \ln\left(N/\overline{N}\right) \cong e\phi/T \ , \qquad (4\text{-}4\text{-}1)$$

we can derive an equation describing the function χ as

$$\frac{2}{3}D\frac{d\chi}{d\eta} + (1 - M_p^{\ 2})\chi + 2A'(\chi^2 - \langle\chi^2\rangle) = 2\varepsilon G\cos\eta . \qquad (4\text{-}4\text{-}2)$$

If we apply the well-known Hopf-Cole Transformation in the following,

$$\chi = \left(D/3A'\right)\left(Z'/Z\right), \qquad (4\text{-}4\text{-}3)$$

the nonlinear equation (4-4-2) reduces to the following homogeneous equation

$$Z'' + \frac{1-M_p^{\,2}}{2A'F} Z' - \left[\left\langle \left(\frac{Z'}{Z}\right)^2 \right\rangle + \frac{b}{F^2}\cos\eta \right] Z = 0 . \qquad (4\text{-}4\text{-}4)$$

Noting that $\left\langle (Z'/Z)^2 \right\rangle \cong \langle Z''/Z \rangle$, at $M_p = 1$, we finally have the following **Mathieu equation**

$$Z'' - \left[\left\langle \left(\frac{Z'}{Z}\right)^2 \right\rangle + \frac{b}{F^2}\cos\eta \right] Z = 0, \qquad (4\text{-}4\text{-}5)$$

which is the **periodic Mathieu function of order zero,** ce_0 that satisfies the conditions that χ must be finite and periodic. We consequently obtain at $M_p = 1$, the following solution

$$\chi = \frac{D}{3A'}\left(\frac{dce_0}{d\eta}\right)\frac{1}{ce_0} . \qquad (4\text{-}4\text{-}6)$$

Note that the **nonlinear equation (4-4-2) may be reduced to linear equation (4-4-4)** or **(4-4-5) by** Hopf-Cole **transformation (4-4-3)** and we finally obtain the exact solution (4-4-6).

4.5 Periodic Mathieu function of order zero

We here briefly introduce the general characteristics of the periodic Mathieu functions, which have been discussed in the session **4.4.**
To explain this function, we refer the table by A. Abramowitz and I. A. Stegun: ″ Handbook of Mathematical Function″ (Dover, New York, 1968)) pp722 (Ref. 9)

We now consider the following canonical form of the differential equation

$$\frac{d^2y}{dv^2} + (a - 2q\cos 2v)y = 0, \qquad (4\text{-}5\text{-}1)$$

which is the Mathieu equation as the same as (2-4-2).
If (4-5-1) has non-zero periodic solutions for a given constant q when a

is selected as an appropriate eigenvalue, we call these periodic solutions as the Mathieu function. Noting that $a = n^2$ is the eigenvalue in case of q=0 and the fundamental solutions are described by $\cos n\pi, sin n\pi$, the solutions of (4-5-1) approach $\cos n\pi, sin n\pi$ as q approaches zero. These Mathieu functions are represented by $ce_n(v,q), se_n(v,q)$, respectively where these are called the nth order Mathieu functions. It should be noted that it is convenient to separate the characteristic curves into two major subsets: $a = a_r$, associated with even periodic solutions and $a = b_r$, associated with odd periodic functions, respectively.

If q is real, then the Sturmian theory of second order linear differential equations yields the following key features:

1) For a fixed real q, characteristic values a_r and b_r are real and distinct, provided $q \neq 0;\ a_0 < b_1 < a_1 < b_2 < a_2 ......, q > 0$ and $a_r(q), b_r(q)$ approach r^2 as q approaches zero.
2) A solution of (4-5-1) associated with a_r or b_r has r zeros in the interval $0 \leq z < \pi$, (q: real)
3) For a given point (a, q) there can be at most one periodic solution of period π or 2π if $q \neq 0$. This no longer holds for solutions of period $s\pi$, s>3 for these all solution are periodic, provided one is.

We now introduce the first few power series for characteristic values (See, pp722 of Ref.9). These are given as

$$a_0(q) = -\frac{q^2}{2} + \frac{7}{128}q^4 + \frac{29}{2304}q^6 + ..., \tag{4-5-2}$$

$$a_1(-q) = b_1(q) = 1 - q - \frac{q^2}{8} + \frac{q^4}{64} - ...., \tag{4-5-3}$$

$$a_2(q) = 4 + \frac{5}{12}q^2 - \frac{763}{13824}q^4 + ..., \tag{4-5-4}$$

$$b_2(q) = 4 - \frac{1}{12}q^2 + \frac{5}{13824}q^4 - .... \quad . \tag{4-5-5}$$

The explicit forms for these characteristic values and higher order values are given in Ref.9. For $r > 7$, and $|q|$ not too large, a_r is approximately equal to b_r, and the following approximation may be used.

$$(a_r\ , b_r) = r^2 + \frac{q^2}{2(r^2-1)}\frac{(5r^2+7)}{32(r^2-1)(r^2-4)}q^4 + ...\quad . \qquad (4\text{-}5\text{-}6)$$

These characteristic values are illustrated in Fig. 24.

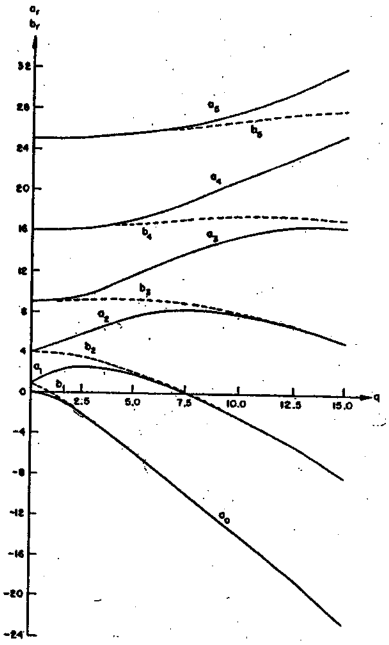


Fig. 24 Characteristic values, a_r and b_r (Ref. 9)

Also, the power series in q for the periodic functions for sufficient small $|q|$ are given as (see, page 725 in Ref. 9)

$$ce_0(z,q) = \sqrt{2}\left[1 - \frac{1}{2}q\cos 2z + q^2\left(\frac{\cos 4z}{32} - \frac{1}{16}\right) - q^3(.....)\right] + \ldots\ , \qquad (4\text{-}5\text{-}7)$$

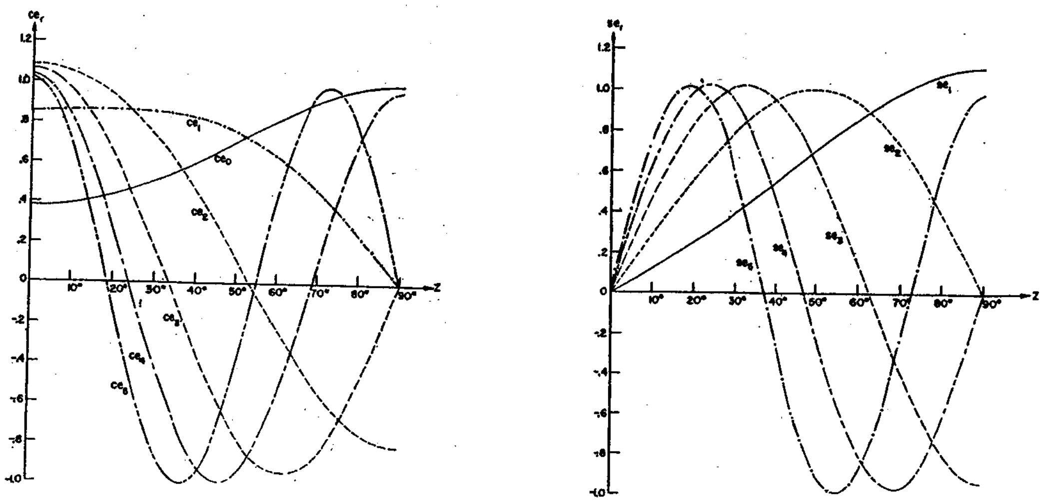
$$ce_1(z,q) = \cos z - \frac{q}{8}\cos 3z + q^2\left[\frac{\cos 5z}{192} - \frac{\cos 3z}{64} - \frac{\cos z}{128}\right] - q^3(......) + ..., \qquad (4\text{-}5\text{-}8)$$

$$sc_1(z,q) = sinz - \frac{q}{8}sin3z + q^2\left[\frac{sin5z}{192} + \frac{sin3z}{64} - \frac{sinz}{128}\right] - q^3(.....) + \ldots\ , \qquad (4\text{-}5\text{-}9)$$

$$ce_2(z,q) = \cos 2z - q\left(\frac{\cos 4z}{12} - \frac{1}{4}\right) + q^2(.......) + \ldots, \qquad (4\text{-}5\text{-}10)$$

$$se_2(z,q) = sin2z - q\frac{sin4z}{12} + q^2(......) + \ldots\ , \qquad (4\text{-}5\text{-}11)$$

where the explicit forms for these periodic functions are given in Ref. 9. Typical functions are plotted in Fig. 25 for fixed value of q=1.



(a) (b)

Fig. 25 Even periodic Mathieu functions (a) with orders 0-5 and odd periodic functions (b) with orders 1-5 are plotted for fixed q=1. (Ref. 9)

5. Nonlinear Waves --A La Carte-

5.1 Dawn of “solitons”

We here list up a couple of topics associated with nonlinear waves, particularly, “solitons” phenomena.

At the second half of 19th century, idea of wave packet, group velocity

In 1834, J. Scott Russell (1808~1882), discovery of ″soliton″.
Note that Scott is the name of mother, Russell is the name of father.

J. Boussinesq, Sir Stokes, Lord Rayleigh, Lord Kelvin, Sir Airy and others discussed whether Scott Russell's discovery is true or not. Sir G. B. Airy claimed that his discovery of ″soliton″ might be a solution of the shallow water wave. Lord Rayleigh supported the existence of this solution analytically.

Korteweg de Vries (K-dV) (1895) developed an equation for shallow water waves, which provided the basis for an analytical study of

solitary waves. After the paper by Korteweg de Vries, no much attention has not been paid on these topics.

Recent discovery, which was due to the modern computer, was an outstanding example of what Ulam referred to us ″synergetics” , which means the intelligence, selective use of computer for exploring idea (N. Zabusky)

Note: I(Sanuki) started the ″**soliton physics**″ from 1970 and leaned lots of physics and mathematical tools associated with solitons at so-called **Toda School**, which was organized by **Prof. M. Toda** in Tokyo.

Solitary Disturbances (Solitaires)

We here introduce briefly a historical event associated with “ **Discovery of Solitons**” by **J. Scott Russell** in the month of August 1834[from the book by E. Atree Jackson, Vol.2 pp349, Ref.1].

“I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped--not the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overlook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figures some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the winding of the channel.”

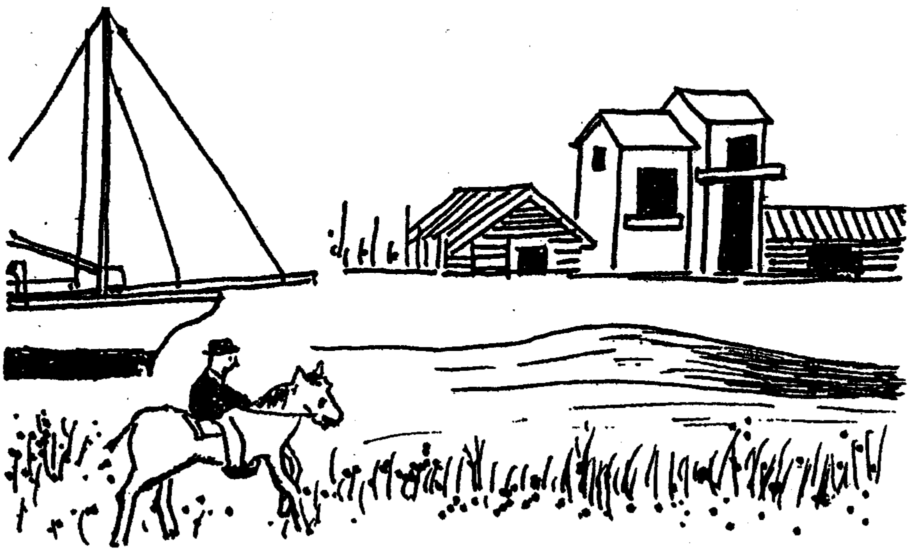


Fig. 26 M. Toda, 物理読本４、『ソリトン、カオス、フラクタル、非線形の世界』、岩波書店、"ラッセル、孤立波を発見（１９８４年)" p198 より

Much later (after 60 years from the discovery by J. Scott Russell), Korteweg and de Vries (1895) developed an equation for shallow water waves, which provided the basis for an analytic study of solitary waves (Solitons).

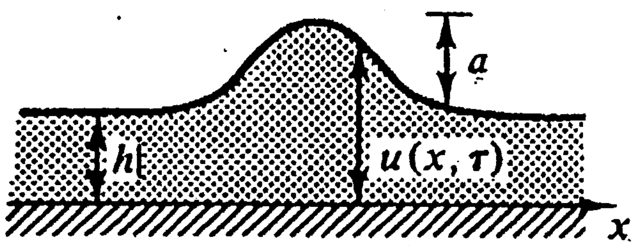


Fig. 27 Profile for Russell' s heap of water

The solution as shown in Fig. 27 is described by the following formula

$$u(x,t) = h + a\,sech^2\left(\frac{x-ct}{b}\right) \quad , \qquad (5\text{-}1\text{-}1)$$

where c is the velocity, h the ambient depth, a the amplitude, and b is the width of the heap of water, respectively.

5.2 Korteweg de Vries (K-dV) equation

We briefly introduce the experiment of solitary wave by Scott-Russell. The experiment by Scott-Russell is illustrated in Fig. 28. The three typical states (state 1, state 2 and state 3) are plotted in Fig. 28. There are two partition walls (a and b) at both sides as shown in this figure. At the state 1, the partition (a) is closed and the water is stored between the left hand end wall and the partition (a), and then the heap of water propagates keeping its profile toward the right side when one opens the partition (a), which is shown in the state (b). Finally, if one closes the partition (b) when the heap of water reaches the right end of the wall, it turns out that the water stored at the state 1 propagates to the region between the partition (b) and the right end wall. They called this nonlinear wave as the "**solitary wave**" .

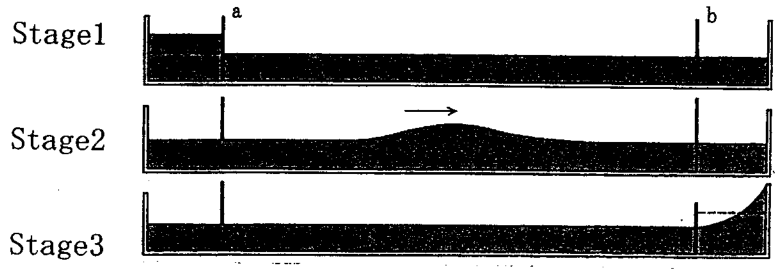


Fig. 28 Experiment of Scott-Russell

Scott-Russell Empirical Scaling:

The velocity of **shallow water wave** is given by $c_0 = \sqrt{gh}$, where h is the water depth from the bottom and g is the gravity. On the other hand, the velocity of solitary Wave is described by the formula, $c = \sqrt{g(h+\eta_0)}$, where η_0 is the height of the solitary wave from the average water surface. The configuration of the water wave is plotted in Fig. 29.

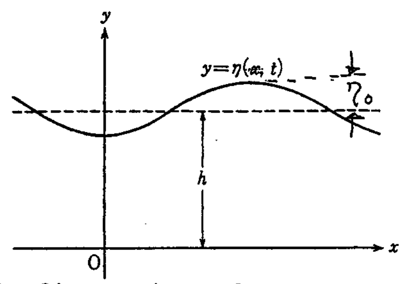


Fig. 29 Configuration of water wave

The K-dV equation is described by

$$\frac{\partial \eta}{\partial t}+\frac{3c_0}{2h}\eta\frac{\partial \eta}{\partial \xi}+\frac{c_0h^2}{6}\frac{\partial^3\eta}{\partial \xi^3}=0 \quad , \qquad (5\text{-}2\text{-}1)$$

and its solution is given as

$$\eta=\eta_0\operatorname{sec}h^2\left(\frac{1}{2}\sqrt{\frac{3\eta_0}{h^3}}(x-ct)\right) \quad , \qquad (5\text{-}2\text{-}2)$$

with the abbreviation of

$$c=c_0\left(1+\frac{\eta_0}{2h}\right)=\sqrt{gh}\left(1+\frac{\eta_0}{2h}\right) \quad . \qquad (5\text{-}2\text{-}3)$$

If we take the shallow water limit, $\eta_0 << h$, the velocity (5-3-3) is in good agreement with the Scott -Russell **empirical scaling** mentioned above.

5.3 Discovery of soliton

Nonlinear interactions among **" solitary-wave pulses"**, propagating in nonlinear dispersive media were actively investigated in various fields. These phenomena were observed in the numerical solution of the K-dV euaqtion

$$u_t+uu_x+\delta^2u_{xxx}=0. \qquad (5\text{-}3\text{-}1)$$

As was discussed in the previous session, this equation can be used to describe the one-dimensional, long-time asymptotic behavior of small but finite amplitude disturbances such as shallow-water waves, collisionless plasma MHD waves, long waves in anharmonic crystals and so on.

N. J. Zabusky and M. D. Kruskal discussed numerically the interaction of "solitons" in collisionless plasma in Ref. 25. They sought stationary solutions of (5-3-1) in a frame moving with velocity c. If we substitute

$$u=U(x-ct), \qquad (5\text{-}3\text{-}2)$$

we obtain a third order nonlinear ordinary differential equation, which has periodic solutions representing **wave trains**. However. If we study a solution, which is asymptotically constant at infinity, the solution is described by the equation

$$u=u_\infty+(u_0-u_\infty)sech^2[(x-x_0)/\Delta], \qquad (5\text{-}3\text{-}3)$$

with the abbreviations of

$$\Delta = \delta[(u_0 - u_\infty)/12]^{-1/2},\ c = u_\infty + (u_0 - u_\infty)/3\ . \qquad (5\text{-}3\text{-}4)$$

Thus, the larger the pulse amplitude and the smaller δ, the narrower is the pulse. As was shown by Zabusky and Kruskal, it should be noted that the computational phenomena observed is characterized in terms of three time intervals.

Stage (1): initially, the first two terms of (5-3-1), as already discussed in session 4-1(see, (4-1-6)), dominate and u steepness in regions where it has a negative slope.

Stage (2): after u steepened sufficiently, the third term (dispersive) becomes important and serves to prevent the discontinuity. Oscillations with small wavelength develop on the left of the front due to the dispersive effect. Finally, each oscillation achieves almost steady amplitudes and has a shape almost identical to that of an individual solitary- wave solution of (5-3-1) in a form (5-3-3).

Stage (3): finally, each such solitary-wave pulse begins to move uniformly at a rate, which is linearly proportional to its amplitude. (See, (5-3-4))

The surprising thing is that these pulses (5-3-3) with (5-3-4), which are strict solution only when completely isolated, can exist in close proximity and interact without losing their form or identity. To demonstrate the time evolution of the form, the numerical calculation was carried out by Zabusky and Kruskal, in which these phenomena were started with $\delta = 0.022$ and the periodic initial condition, $u(t=0) = \cos \pi x$. Since the third term is negligible for this parameter, its solution is given approximately by the following implicit relation,

$$u = \cos \pi (x - ut), \qquad (5\text{-}3\text{-}5)$$

and u becomes discontinuous at $x = 1/2$ and $t = T_B = 1/\pi$, the breaking time. The time evolution of the solution is plotted in Fig. 31. The curve A gives the initial condition, and curve B shows the solution at $t = T_B = 1/\pi$. Curve C at $t = 3.6T_B$ shows a train of "solitons" (numbered 1-8), which developed from the oscillations.

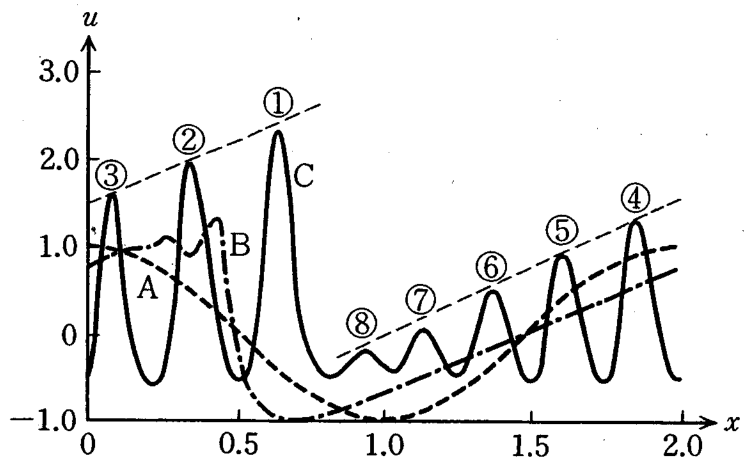


Fig. 30 Temporal development of the wave form (from Fig. 1 of Ref. 25)

$u(x,0) = \cos \pi x, \quad \delta = 0.022,$

$t = 0(A, dotted), \quad t = 1/\pi\,(B, dotpoint), \quad t = 3.6/\pi\,(C, solid)$

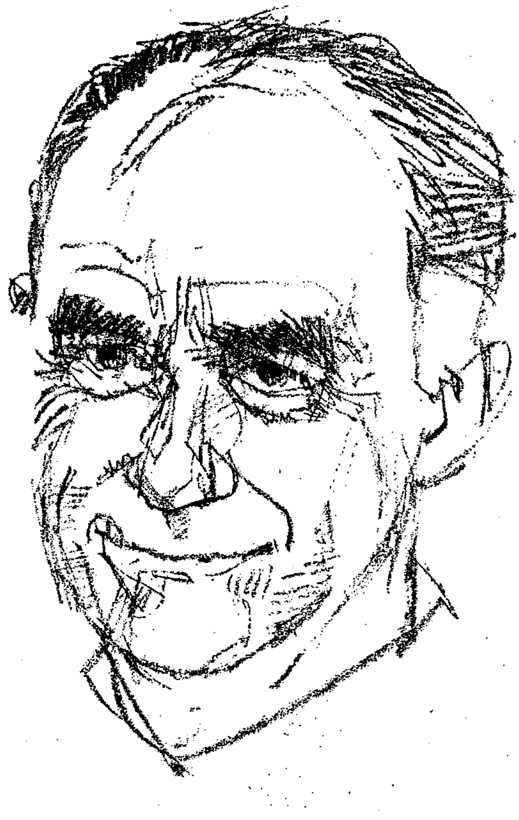


Fig. 31 Sketch of Enrico Fermi (from the pamphlet of Fermi National Accelerator Laboratory

5.4 Fermi-Pasta-Ulam recurrence phenomena

We here consider the 1D lattice system as shown below

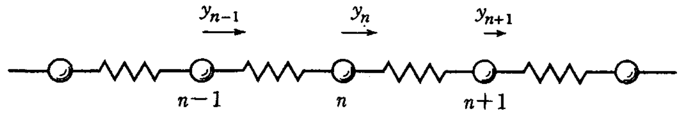


Fig. 32 1D lattice system

"Hooke Formula" describes the motion in linear and/or nonlinear systems. In the early nineteenth century, **Enrico Fermi** analyzed numerically the nonlinear 1D lattice problem instead of analytically by using " MANIAC1 Computer" associated with " Ergode Problem". If we discuss the linear spring lattice model, the oscillations in this system is composed of the normal modes and each mode is independent, and the system is not ergodic. When the nonlinear spring model is employed, however, one expect this system to exhibit thermalization, namely, complete energy sharing among the corresponding normal modes after long times (Ergotic). To study this problem, **Fermi, J. R. Pasta and S. M. Ulam** carried out the numerical analysis based on the 1D nonlinear lattice model, where the system is composed of 32 or 64 atoms, and fixed boundary condition is assumed.

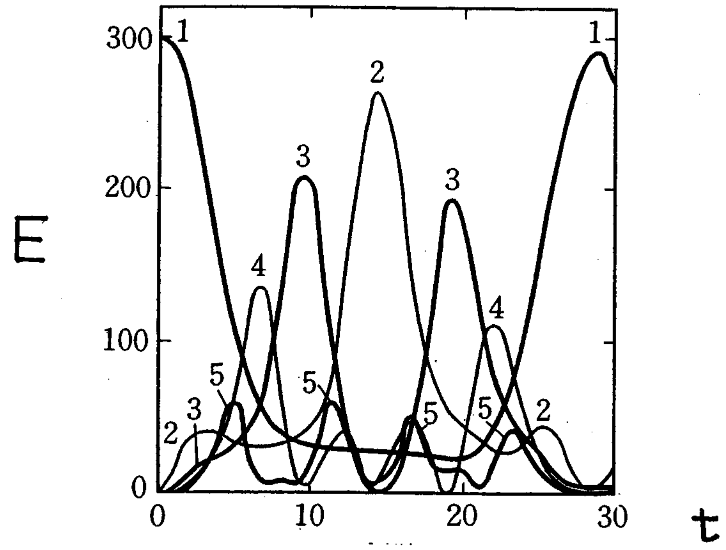


Fig. 33 Fermi-Pasta-Ulam recurrence phenomena (Ref. 25)

At t=0, **mode 1** is excited and it evolves in time. Then, a couple of other **modes (2, 3, 4, 5)** are also excited with suppression of **mode 1** but system returns to only initial state, namely, **mode 1** is only excited after long time. This result was different one from that expected. This is called as the **Fermi-Pasta-Ulam recurrence phenomena** and it demonstrated that the computer is a powerful tool to analyze nonlinear phenomena.

The discovery of solitons by Zabusky et al. and recurrence phenomena by Fermi-Pasta-Ulam were the remarkable examples to find new physics by computer science. J. Von Neumann developed a new idea, namely, "**Synergetics**" with combined **numerical experiment (計算機実験)** and **analytical tool (解析的手法)**.

5.5 **Gardner- Morikawa transformation**

We discuss the 1D anharmonic lattice. Taking account of the" **discreetness** "of the lattice, the motion in this system is given as the following nonlinear discrete equation

$$m\ddot{y}_n = \kappa[y_{n+1} - y_n + \alpha(y_{n+1} - y_n)^2] - \kappa[y_n - y_{n-1} + \alpha(y_n - y_{n-1})^2], \quad (5\text{-}5\text{-}1)$$

Defining a new coordinate through $x = na$ (a is the lattice distance) and taking the **continuum limit**, we have the nonlinear differential equation

$$y_{tt} = c_0^2\left(y_{xx} + \frac{a^2}{12}y_{xxxx} + \frac{a^4}{360}y_{xxxxxx} + \ldots + 2\alpha a y_x y_{xx} + \frac{1}{3}\alpha a^3 y_{xx}y_{xxx} + \ldots\right), \quad (5\text{-}5\text{-}2)$$

which is reduced to the following approximate equation

$$y_{tt} = c_0^2\left(y_{xx} + \frac{a^2}{12}y_{xxxx} + 2\alpha a y_x y_{xx}\right). \quad (5\text{-}5\text{-}3)$$

This equation is called the ″**Boussinesq Equation**″. It should be noted that the original Boussinesq Equation (1972), which is well known in the fluid dynamics, is given by

$$y_{tt} = c_0^{\,2}\left(y_{xx} + \frac{a^2}{12}y_{xxtt} + 2\alpha a y_x y_{xx}\right), \qquad (5\text{-}5\text{-}4)$$

and (5-5-4) reduces to (5-5-3) under ″Boussinesq Approximation″, which is often used in the shallow water wave analysis.

Introducing the new slow variables (ξ,τ) instead of fast ones (x, t),

$$\xi = \varepsilon^p\left(x - c_0 t\right)/\,a,\ \ \tau = \varepsilon^q c_0 t\,/\,a,\ \ y(x,t) = \varepsilon^r y^{(1)}(\xi,\tau), \qquad (5\text{-}5\text{-}5)$$

where these coordinates is called "Stretching Coordinates". Then, (5-5-3) can be written in the form

$$\begin{aligned}&\frac{\varepsilon^{2p}c_0^{\,2}}{a^2}\frac{\partial^2 y}{\partial\xi^2} - 2\frac{\varepsilon^{p+q}c_0^{\,2}}{a^2}\frac{\partial^2 y}{\partial\xi\partial\tau} + \frac{\varepsilon^{2q}c_0^{\,2}}{a^2}\frac{\partial^2 y}{\partial\tau^2}\\ &= c_0^{\,2}\left[\frac{\varepsilon^{2p}}{a^2}\frac{\partial^2 y}{\partial\xi^2} + \frac{\varepsilon^{4p}}{12a^2}\frac{\partial^4 y}{\partial\xi^4} + 2\alpha\frac{\varepsilon^{3p+r}}{a^2}\frac{\partial y}{\partial\xi}\frac{\partial^2 y}{\partial\xi^2} + O(\varepsilon^{6p}) + \ldots\right]\end{aligned} \qquad (5\text{-}5\text{-}6)$$

From each order of (5-5-6), we have the following relations,

1) From 0 (ε^{2p}), it is automatically satisfied,
2) If p>q, we assume the ordering, p+q=4p=3p+r,
3) If we choose r=1/2, then we obtain p=1/2, q=3/2 and we finally have the following equation,

$$u + uu_x + \frac{1}{24}u_{xxx} = 0,\ \ u = \alpha y_\xi, \qquad (5\text{-}5\text{-}7)$$

which is the Korteweg deVries (K-dV) equation. It should be noted how to choose what kind of **Gardner-Morikawa Transformation** is determined by the fact that we consider what kind of dispersion relation. To demonstrate this situation, we consider the linear dispersion relation, which is given as

$$\omega^2 = 4\frac{\kappa}{m}sin^2\left(\frac{ka}{2}\right). \qquad (5\text{-}5\text{-}8)$$

If k is small, the dispersion relation can be approximately given by

$$\omega(k) \approx c_0 k - \frac{1}{24} c_0 k (ka)^2 + ... , \qquad (5\text{-}5\text{-}9)$$

provided we keep the terms up to the second order with respect to k. If we put $k = \varepsilon^p \tilde{k}$, we have the phase relation

$$kx - \omega t = \tilde{k}\varepsilon^p (x - c_0 t) + \frac{1}{24} \tilde{k}^3 c_0 a^2 \varepsilon^{3p} t + ... \qquad (5\text{-}5\text{-}10)$$

In order to discuss the slow variation of wave propagation, we have to select the slow variables as

$$\xi = \varepsilon^p (x - c_0 t)/a, \quad \tau = \varepsilon^{3p} c_0 t / a, \qquad (5\text{-}5\text{-}11)$$

which is the same as the Gardner-Morikawa Transformation (5-5-5). If we select p=1/2, q=3/2 and r=0, we have well known **modified K-dV** equation,

$$u_t + u^2 u_\xi + \frac{1}{24} u_{\xi\xi\xi} = 0. \qquad (5\text{-}5\text{-}12)$$

So, the **Gardner-Morikawa Transformation** is the ordering associated with **mutual relation among the amplitude, space and time.**

5.6 Sine-Gordon equation (S-G eq.)

One of the well-known equations in the field theory is the 1 dimensional **Klein-Gordon equation**

$$\phi_{tt} - \phi_{xx} + m^2 \phi = 0, \qquad (5\text{-}6\text{-}1)$$

and if we replace the term $m^2\phi$ by $m^2 sin\phi$, we have the following equation

$$\phi_{tt} - \phi_{xx} + m^2 sin\phi = 0. \qquad (5\text{-}6\text{-}2)$$

This is the Sine-Gordon equation as known as a nonlinear model equation describing the soliton phenomena. It should be noted that the S-G equation satisfies the invariance for the following **"Lorentz Transformation"**,

$$x' = \gamma(x - vt), \quad t' = \gamma(t - vx), \quad \gamma = (1 - v^2)^{-1/2}.$$

We next discuss the stationary solution of the S-G equation, which is described by

$$\phi_{xx} - m^2 sin\phi = 0, \qquad (5\text{-}6\text{-}3)$$

under the boundary condition, namely, $\phi(x) -> 0 \ (\mathrm{mod}\ 2\pi), (|x| -> \infty)$. If we introduce $\phi/2 = \Phi$, $\xi/a = \eta$, (5-6-3) reduces to

$$\frac{d}{d\eta}\log\left(tan\frac{\Phi}{2}\right) = -1, \qquad (5\text{-}6\text{-}4)$$

which gives the solution

$$\phi(x) = 4\arctan\left[e^{\pm m(x-x_0)}\right]. \qquad (5\text{-}6\text{-}5)$$

By taking account of the Lorentz Invariant, we finally obtain

$$\phi(x,t) = 4\arctan\left[e^{\pm m\gamma(x-vt-x_0)}\right], \qquad (5\text{-}6\text{-}6)$$

which is composed of the so-called "**Kink**" and "**Anti-Kink**" solutions.

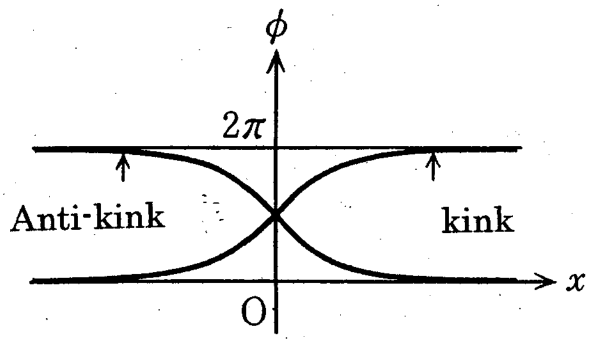


Fig.34 Kink and anti-kink solutions

We hare introduce a couple of examples, which are described by the S-G equation, namely

Nonlinear pendulum,

Slip dislocation model (Frenkel-Kontorova equation (1939)),

Josephson junction phenomena, etc..

5.7 Bäcklund transformation

A. B. Bäcklund discussed the differential geometry theory associated with So called " **Bäcklund transformation**" , 1875. Let us briefly explain the scheme of Bäcklund Transformation. We now consider two coupled differential equations

$$u_{xt} = f(u, u_x, u_{xx}, \dots\dots\dots), \qquad (5\text{-}7\text{-}1)$$

$$v_{xt} = g(v, v_x, v_{xx}, \dots\dots\dots). \qquad (5\text{-}7\text{-}2)$$

If we can obtain the following coupled equations from (5-7-1) and (5-7-2)

$$F(u_x, v_x, ...) = 0, \qquad (5\text{-}7\text{-}3)$$

$$G(u_t, v_t, ...) = 0, \qquad (5\text{-}7\text{-}4)$$

where only first order derivative with respective to "t" is included in (5-7-3) and (5-7-4). Then, we call the relations (5-7-3) and (5-7-4) the " Bäcklund Transformation" . In this case, the solution "u" of (5-7-1) is transformed into the solution "v" of (5-7-2) through the relations (5-7-3) and (5-7-4).

5.7.1 Bäcklund transformation for S-G equation

We here consider the S-G equation

$$\phi_{TT} - \phi_{XX} + sin\phi = 0. \qquad (5\text{-}7\text{-}5)$$

Introducing new variables, $t = (T + X)/2$ and $x = (T - X)/2$, (5-7-5) reduces to

$$\phi_{xt} + sin\phi = 0. \qquad (5\text{-}7\text{-}6)$$

Since the detailed derivation of the Bäcklund Transformation for the S-G equation is reported in Ref. 12 and Ref. 27, we briefly introduce the Bäcklund Transformation in the form

$$\begin{aligned} (\phi' - \phi)_t &= 2asin[\frac{1}{2}(\phi' + \phi)] \\ (\phi' + \phi)_x &= -\frac{2}{a}sin[\frac{1}{2}(\phi' - \phi)] \end{aligned}, \qquad (5\text{-}7\text{-}7)$$

If $\phi(\phi')$ is a solution, $\phi'(\phi)$ is also a solution. Here, ϕ_0 and ϕ_1 are assumed to be the solutions of (5-7-7) but we choose $\phi_0 = 0$ (vacuum solution) of the S=G equation. Then, (5-7-7) reduces to

$$\frac{1}{2}\phi_{1t} = asin\left(\frac{1}{2}\phi_1\right), \ \frac{1}{2}\phi_{1x} = -\frac{1}{a}sin\left(\frac{1}{2}\phi_1\right), \qquad (5\text{-}7\text{-}8)$$

which yields the solution

$$\phi_1(\xi) = 4\arctan[e^{-(\xi - \xi_0)/a}], \qquad (5\text{-}7\text{-}9)$$

where $\xi = x - a^2 t$, a is the constant and the solution with a<0 is the Kink solution and one with a>0 corresponds to the anti-Kink solution, as was shown in Fig. 34.

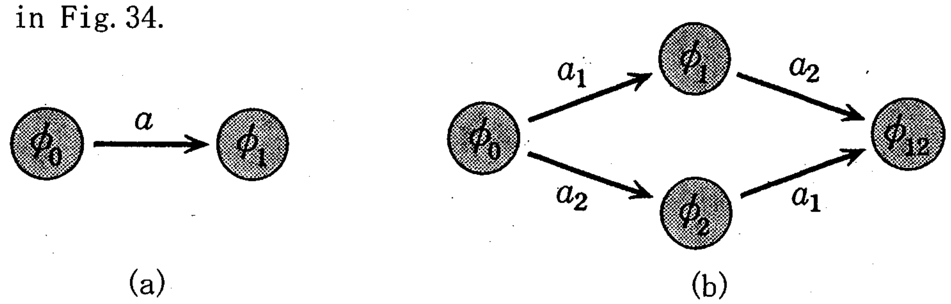


Fig. 35 Diagram of Bäcklund Transformation for two (a) and four (b) solutions.

The diagram of Bäcklund Transformation for two and four solutions are illustrated in Fig. 35. Figure 35(a) corresponds to the Bäcklund transformation from ϕ_0 to ϕ_1 with constant a. Figure35 (b) is the diagram for Bäcklund transformation for four solutions, ϕ_0 to ϕ_1 with a_1, ϕ_0 to ϕ_2 with a_2, ϕ_1 to ϕ_{12} with a_2 and ϕ_2 to ϕ_{12} with a_1. These diagrams are represented by the following relations

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{1}{2}(\phi_1+\phi_0) &= -\frac{1}{a_1}sin\frac{1}{2}(\phi_1-\phi_0),\\
\frac{\partial}{\partial x}\frac{1}{2}(\phi_2+\phi_0) &= -\frac{1}{a_2}sin\frac{1}{2}(\phi_2-\phi_0),\\
\frac{\partial}{\partial x}\frac{1}{2}(\phi_{12}+\phi_1) &= -\frac{1}{a_2}sin\frac{1}{2}(\phi_{12}-\phi_1),\\
\frac{\partial}{\partial x}\frac{1}{2}(\phi_{12}+\phi_2) &= -\frac{1}{a_1}sin\frac{1}{2}(\phi_{12}-\phi_2),
\end{aligned} \qquad (5\text{-}7\text{-}10)$$

By eliminating the x-derivative terms in (5-7-10), we have the following solution for ϕ_{12}

$$\phi_{12} = \phi_0 + 4\arctan\left(\frac{a_1+a_2}{a_1-a_2}tan[\frac{1}{4}(\phi_1-\phi_2)]\right). \qquad (5\text{-}7\text{-}11)$$

It should be noted that the solution ϕ_{12} can be constructed from known solutions, ϕ_0, ϕ_1 and ϕ_2 through the relation (5-7-11). For an example, if one assume the vacuum solution ϕ_0=0 and one soliton solution for ϕ_1 and ϕ_2, which are given as

$$\phi_i = 4\arctan[\exp(-\xi_i/a_i)],\ \xi_i = x - a_i^2 t - \xi_{i0},\ \ (i=1,2), \qquad (5\text{-}7\text{-}12)$$

we obtain the following solution from (5-7-11)

$$\phi_{12} = 4\arctan\left[\frac{a_1+a_2}{a_1-a_2}\frac{-sinh\frac{1}{2}(\xi_1/a_1-\xi_2/a_2)}{\cosh\frac{1}{2}(\xi_1/a_1+\xi_2/a_2)}\right], \qquad (5\text{-}7\text{-}12)$$

which is the two-soliton solution. Thus, we can construct the N-soliton solution by repeating the same way. This solution is generally described in the form, $\phi = 4\arctan(f/g)$.

5.7.2 Bäcklund transformation for K-dV equation and conservation law

We now write the K-dV equation in the form

$$u_t - 6uu_x + u_{xxx} = 0. \qquad (5\text{-}7\text{-}13)$$

If we introduce the functions,

$$u(x,t) = w_x(x,t),\ u'(x,t) = w_x{}'(x,t)\ , \qquad (5\text{-}7\text{-}14)$$

we have the following Bäcklund transformation for K-dV equation

$$w_x + w_x{}' = -2\eta^2 + \frac{1}{2}(w-w')^2, \qquad (5\text{-}7\text{-}15)$$

$$w_t + w_t{}' = 2[w_x{}^2 + w_x w_x{}' + (w_x{}')^2] - (w-w')(w_{xx} - w_{xx}{}'), \qquad (5\text{-}7\text{-}16)$$

where η is an arbitrary constant.

In order to derive the formula for the conservation law, we rewrite (5-7-15) and (5-7-16) in the forms

$$\begin{aligned} &w_x + w_x{}' = -2\eta^2 + \frac{1}{2}(w-w')^2 \\ &w_t - w_t{}' = -[2w_{xx} - 2w_x(w-w') + 4\eta^2(w-w')]_x \end{aligned}\ . \qquad (5\text{-}7\text{-}17)$$

If we expand the term $w - w'$ as

$$w - w' = 2\eta + \sum_{n=1}^{\infty} f_n \eta^{-n}, \qquad (5\text{-}7\text{-}18)$$

substitution of (5-7-18) into (5-7-17) yields

$$\begin{aligned} &f_{n+1} = u\delta_{n,0} - \frac{1}{2}f_{n,x} - \frac{1}{4}\sum_{m=1}^{n-1} f_m f_{n-m} \\ &\frac{\partial f_n}{\partial t} + \frac{\partial}{\partial x}(-2uf_n + 4f_{n+2}) = 0 \end{aligned}\ , \qquad (5\text{-}7\text{-}19)$$

where f_n is the conserved density. We here show the explicit forms of some conserved quantities,

$$f_1 = u,\ f_2 = -\frac{1}{2}u_x,\ f_3 = -\left(\frac{1}{2}\right)^2 (u^2 - u_{xx}),\ f_4 = \left(\frac{1}{2}\right)^3 (2u^2 - u_{xx})_x. \quad (5\text{-}7\text{-}20)$$

5.7.3 Konno- Sanuki transformation

We consider a nonlinear one-dimensional lattice system under a weak dislocation potential associated with the slip dislocation model, in which the equation of motion of the n-th atom is of the form (Ref. 12 and Ref. 29)

$$\frac{\partial^2 u_n}{\partial T^2} = u_{n+1} - 2u_n + u_{n-1} + \frac{1}{24}\{(u_{n+1} - u_n)^3 - (u_n - u_{n-1})^3\} - 2\alpha\varepsilon^4 sin(u_n), \quad (5\text{-}7\text{-}21)$$

where u_n is a dimensionless displacement of the n-th atom and a measure of the size of the displacement. In case of disturbance of long wavelength compared to the spacing of the atoms in the anharmonic lattice, a continuum approximation may be adopted. If we introduce a stretching

$$\xi_n = \varepsilon(nh - \bar{\lambda}T),\ \ \tau = \varepsilon^3 T, \quad (5\text{-}7\text{-}22)$$

where h is a lattice spacing and $\bar{\lambda}$ is the group velocity, the displacement may be expanded as a Taylor series in the small parameter εh. Keeping the terms up to $O(\varepsilon^4)$ and introducing the new variables, $x = (24)^{1/4}\xi / h,\ t = h\tau / [(24)^{1/4}\bar{\lambda}]$, we finally obtain

$$u_{xt} + \frac{2}{3}{u_x}^2 u_{xx} + u_{xxxx} - \alpha sin(u) = 0 \ . \quad (5\text{-}7\text{-}23)$$

In (5-7-23), the equation without the last term, namely, dislocation potential term, reduces to the modified K-dV equation. Also, if we neglect the second and third terms, the equation tends to the S-G equation. It is well known that both modified K-dV equation and S-G equation can be solved and have the soliton solutions. How to solve the complicated nonlinear equation including nonlinear effects of both anharmomic lattice and dislocation potential such as (5-7-23) had not solved yet.

Although (5-7-23) was solved by Konno, Kameyama and Sanuki (Ref. 29) by the inverse scattering method, the Backlund transformation for (5-7-23) was later derived by Konno and Sanuki [Ref. 30], which is represented by the following Backlund transformation

$$u_x + u'_x = -4\lambda sin\frac{1}{2}(u-u')$$
$$u_t - u'_t = 2[C-B] - 2[C+B]\cos\frac{1}{2}(u-u') + 4\lambda sin\frac{1}{2}(u-u') \quad , \qquad (5\text{-}7\text{-}24)$$

where the coefficient A, B, C are given as

$$A = -4\lambda^3 - \frac{1}{2}\lambda(u_x)^2 + \left(\frac{\alpha}{4\lambda}\right)\cos(u)$$
$$B = \frac{1}{2}u_{xxx} + \lambda u_{xx} + 2\lambda^2 u_x + \frac{1}{4}u_x^{\ 3} + \left(\frac{\alpha}{4\lambda}\right)sin(u) \ . \qquad (5\text{-}7\text{-}15)$$
$$C = -\frac{1}{2}u_{xxx} + \lambda u_{xx} - 2\lambda^2 u_x - \frac{1}{4}u_x^{\ 3} + \left(\frac{\alpha}{4\lambda}\right)sin(u)$$

The permutability theorem based on the spatial part (5-7-24) of this Bäcklund transformation may now be generated in the manner earlier indicated for the S-G equation. The second- generated solution is given by (5-7-11) with some modifications of coefficients. Note that (5-7-23) has a kink solution and its derivative is given by a soliton solution. The head-on collision of the two-component soliton solution in the spatial derivative of the two-kink solution is illustrated in Fig. 36 (a). The collision process of two soliton solutions having amplitude of opposite sign in the spatial derivative of the kink- antikink solutions is also shown in Fig. 36 (b).

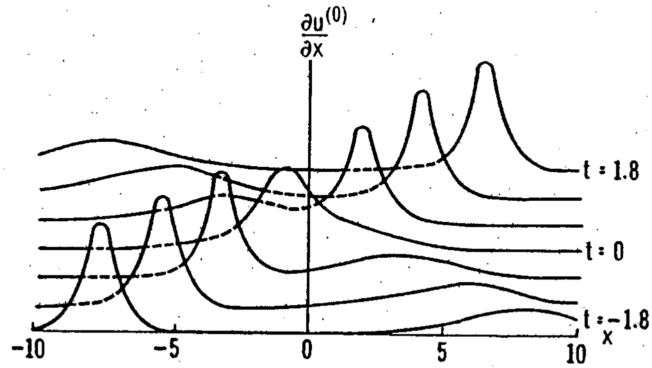


Fig. 36 (a) Collision of two solitons having amplitudes of same sign (repulsive collision) (Ref. 29)

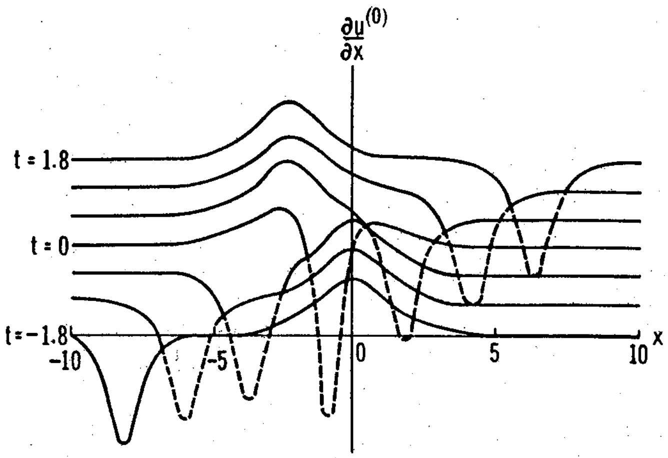


Fig. 36 (b) Collision of two solitons having amplitudes of opposite sign (attractive collision) (Ref. 29)

6. Reductive Perturbation Method (逓減摂動論)

The reductive perturbation method is a perturbation theory based on **multi-time and -space expansion technique** and a general perturbation formula which is extended from "**Gardner-Morikawa Transformation**". (See, references, 2, 3, 4 and 31)

We here introduce two examples of the application of reductive perturbation theory associated with the shallow water wave and convective cell formation in plasmas.

6.1 "Propagation of nonlinear gravitational wave" (Ref. 32)

The gravitational wave propagating in the water with a uniform depth h is described by the following Lap lace equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0, \qquad (6\text{-}1\text{-}1)$$

which can be derived from $\nabla\cdot\vec{v}=0$ (imcompressive) and $\nabla\times\vec{v}=0$ (vortex free), which satisfies $\vec{v}=\nabla\phi$. The configuration under consideration is illustrated in Fig. 37.

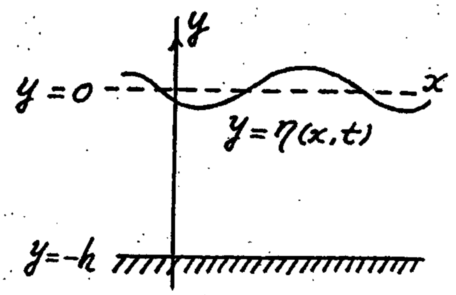


Fig. 37 Configuration for gravitational wave.

The following boundary conditions are applied,

$$\frac{\partial \phi}{\partial y}=0 \quad \text{at } y=-h, \tag{6-1-2}$$

$$\frac{\partial \phi}{\partial y}=\frac{\partial \eta}{\partial t}+\frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} \quad \text{at } y=\eta(x,t), \tag{6-1-3}$$

$$\frac{\partial \phi}{\partial t}+\frac{1}{2}\left\{\left(\frac{\partial \phi}{\partial x}\right)^2+\left(\frac{\partial \phi}{\partial y}\right)^2\right\}+G\eta=0 \quad \text{At} \quad y=\eta(x,t), \tag{6-1-4}$$

$$\phi(x,y,t)=\phi\Big|_{y=0}+\frac{\partial \phi}{\partial y}\Big|_{y=0}\eta+\frac{1}{2}\frac{\partial^2 \phi}{\partial y^2}\Big|_{y=0}\eta^2+. \tag{6-1-5}$$

Introducing stretched new variables (ξ,τ) in addition to fast scale variables (x, y, t), $\xi=\varepsilon(x-\lambda t)$, $\tau=\varepsilon^2 t$, and we expand the velocity potential ϕ and the displacement η around free surface as

$$U=\sum \varepsilon^n U^{(n)},\ U^{(n)}=\sum U_l^{(n)}(\xi,\tau;y)\exp[il(kx-\omega t)]\ . \tag{6-1-7}$$

The procedure to get the solution by the reductive perturbation method is roughly explained as

1) O (ε): the dispersion relation, $\omega=\omega(k)$ is obtained,

2) O(ε^2): the moving velocity λ in the stretched variable, $\xi=\varepsilon(x-\lambda t)$ is determined as the group velocity,

3) O(ε^3): the equation for slow variation with (ξ,τ) is obtained.

From the first order with respect to ε:

$$\phi_l^{(1)} = A_l(\xi,\tau)\frac{\cosh\{lk(y+h\}}{\cosh(lky)}, \quad A_{l=\pm1} \neq 0,\ A_{l\neq\pm1} = 0, \qquad (6\text{-}1\text{-}8)$$

and the dispersion relation for the fundamental mode with $l=\pm1$ is determined as

$$\omega(k)^2 = Gk\cdot tanh(kh), \qquad (6\text{-}1\text{-}9)$$

which is the linear **gravitational wave dispersion relation** and k is the wavenumber and h is the depth of the water from the bottom.

From the second order:

Substitution of (6-1-7) into (6-1-2)-(6-1-5) together with stretched variables give the following relations

$$\eta_0^{(2)} = -\frac{k^2}{G}sech^2(kh)|A_1|^2, \qquad (6\text{-}1\text{-}10)$$

$$\phi_{\pm1}^{(2)} = \mp(y+h)\frac{sinh\{k(y+h)\}}{\cosh(kh)}i\frac{\partial A_{\pm1}}{\partial\xi}, \qquad (6\text{-}1\text{-}11)$$

$$\phi_2^{(2)} = \frac{\cosh\{2k(y+h)\}}{\cosh(2kh)}A_2^{(2)}, \qquad (6\text{-}1\text{-}12)$$

$$A_2^{(2)} = i\frac{3}{4}\frac{k^2}{\omega}\frac{1+tanh^2(kh)}{sinh^2(kh)}A_1A_1, \qquad (6\text{-}1\text{-}13)$$

$$\eta_2^{(2)} = -\frac{k^2}{2G}\frac{3+2sinh^2(kh)}{sinh^2(kh)}A_1A_1. \qquad (6\text{-}1\text{-}14)$$

Since we also have the relation for the equation for $l=1$ in the form

$$i\{2\omega\lambda - G(tanh(kh)+khtanh(kh))\}\frac{\partial A_{+1}}{\partial\xi} = 0, \qquad (6\text{-}1\text{-}15)$$

In order for this equation to have a non-trivial solution, λ should be $\lambda = \partial\omega/\partial k$, which is the **group velocity** of the fundamental mode.

From the third order:

We obtain the nonlinear equation, which describes the fundamental mode (here, we put $A = A_{\pm1}$ for simplicity)

$$i\frac{\partial A}{\partial \tau}+p\frac{\partial^2 A}{\partial \xi^2}+q|A|^2 A=0, \qquad (6\text{-}1\text{-}16)$$

$$p=\frac{1}{2}\frac{\partial^2 \omega}{\partial k^2},$$
$$q=-\frac{k^4}{2\omega_0}\left[\frac{9-10tanh^2(kh)+9tanh^4(kh)}{2tanh^2(kh)}-sech^4(kh)\right]. \qquad (6\text{-}1\text{-}17)$$

Equation (6-1-16) is so called " Nonlinear Schrödinger equation. Whether the solution is stable or not against amplitude modulation, is determined by the relation between coefficients p and q (6-1-17), namely the case with $pq<0$ is modulationally stable but the case with $pq>0$ follows the modulational instability. Since $pq>0$ for (6-1-16)with (6-1-17), the plane wave solution for the nonlinear gravitational wave may become **modulational unstable.**

6.2 Elecroromagnetic drift wave turbulence and convective cell formation

Nonlinear properties of electrostatic drift waves and convective cell formation have attracted much interest associated with the turbulence and related anomalous transport. Hasegawa and Mima discussed the electrostatic convective cell formation based on a nonlinear drift wave model, so-called **Hasegawa-Mima eq.** (See, Ref.32), which is given as

$$\frac{\partial}{\partial t}(\Delta_\perp \hat{\phi}-\hat{\phi})=[(\nabla_\perp \hat{\phi}\times \hat{z})\cdot \nabla_\perp][\Delta_\perp \hat{\phi}-\ln\left(\frac{n_0}{\omega_{ci}}\right)]=0. \qquad (6\text{-}2\text{-}1)$$

Sanuki and Weiland extended the model for electrostatic convective cells into the electromagnetic convective cells(see. **Ref.33**) based on so-called " **Sanuki-Weiland Model**" , which describes the drift Alfven wave driven electromagnetic convective cell formation.

In Cartesian co-ordinates, the model equation is given by

$$\left(\frac{\partial^2}{\partial t^2}-v_A{}^2\frac{\partial^2}{\partial z^2}-V_{Di}\nabla_\perp\frac{\partial}{\partial t}-\mu\Delta_\perp\frac{\partial}{\partial t}\right)\Delta_\perp \phi$$
$$=\frac{c}{B_0}\left(1-\frac{k_{//}v_A{}^2}{\omega^2}\right)\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial \phi}{\partial x}\frac{\partial}{\partial y}\right)\Delta_\perp \phi. \qquad (6\text{-}2\text{-}2)$$

Applying the reductive perturbation method to (6-2-2), we analyze the electromagnetic convective cell formation and its spatial structure.

We here briefly explain the convective cell formation process by introducing the following stretching coordinates and ordering

$$\phi = \phi^{(0)}(x) + \sum_{\alpha} \varepsilon^{\alpha} \phi^{(\alpha)},$$

$$\phi^{(\alpha)} = \sum_{l} \phi_l^{(\alpha)}(x,\xi,\tau)\exp(il(k_{//}z + k_y y - \omega t)), \qquad (6\text{-}2\text{-}3)$$

$$x = x,\ \xi = \varepsilon(y - \lambda t),\ \tau = \varepsilon^2 t,\ \mu = O(\varepsilon^2)$$

where the ordering for viscosity is assumed as the second order. In the x-direction, we apply periodic boundary conditions, i.e.

$$\phi^{(\alpha)}(0,\xi,\tau) = \phi^{(\alpha)}(L,\xi,\tau) = 0,\ (\alpha \neq 0), \qquad (6\text{-}2\text{-}4)$$

where L is the dimension in the x-direction and the reality condition is assumed for the potential.

Derivation of solution of convective cells

From the first Order, we have the linear dispersion relation for **drift Alfven wave** as

$$\omega^2 - k_{//}^2 v_A^2 + k_y(v_{Di} - c_0)\omega = 0. \qquad (6\text{-}2\text{-}5)$$

Assuming a sinusoidal variation of the solution in x-direction, we have the first order solution as

$$\phi_l^{(1)} = \hat{\phi}_l^{(1)}(\xi,\tau)\left(\frac{2}{L}\right)^{1/2} sin(k_m x),\ k_m = m(2\pi/L), \qquad (6\text{-}2\text{-}6)$$

where m is the radial mode number.

From the second Order, λ is determined by the compatibility condition ,which yields

$$\lambda = \partial\omega/\partial k\ . \qquad (6\text{-}2\text{-}7)$$

After lengthy and tedious calculations, from **the third order**, the amplitude for **shear Alfven wave** and **convective cell mode** are given

$$\phi_0^{(2)} = \frac{k_y^2}{k_m L(\lambda + v_{Di} - c_0)} \frac{c}{B_0} (1 - \frac{k_{//}^2 v_A^2}{\omega^2}) |\hat{\phi}_1^{(1)}(\xi,\tau)|^2 sin(2k_m x), \quad (6\text{-}2\text{-}8)$$

$$is\frac{\partial \hat{\phi}_1^{(1)}}{\partial \tau} + U\frac{\partial^2 \hat{\phi}_1^{(1)}}{\partial \xi^2} + Q|\hat{\phi}_1^{(1)}|^2 \hat{\phi}_1^{(1)} = -iv\hat{\phi}_1^{(1)}, \quad (6\text{-}2\text{-}9)$$

with the abbreviations

$$\begin{aligned}
s &= (2\omega + k_y v_{Di} - k_y c_0)(k_m^2 + k_y^2), \\
U &= -\lambda(\lambda + v_{Di} - c_0)(k_m^2 + k_y^2), \\
Q &= -(1 - \frac{k_{//}^2 v_A^2}{\omega^2}) \frac{\omega k_y^2 (3k_m^2 - k_y^2)}{L(\lambda + v_{Di} - c_0)} \left(\frac{c}{B_0}\right)^2, \\
v &= -\omega\mu(k_m^2 + k_y^2)
\end{aligned} \quad (6\text{-}2\text{-}10)$$

Using an analytical method by **Sanuki et al. (1972, Ref. 35)**, we can solve (6-9-2) and have the solution

$$\hat{\phi}_1^{(1)} = a(t) sech[({}^{g}\!/_{2p})^{1/2} a(t)(\xi - 2vt) \exp(i\frac{v}{p}(\xi - vt) + i\frac{g}{2}\int_0^t a^2(t)dt), \quad (6\text{-}2\text{-}11)$$

where $p = \frac{U}{s}$, $g = \frac{Q}{s}$, ${}^{\partial a}\!/_{\partial t} = -2va$. It should be noted that the **structure of created vortex** is determined by the **modulational instability condition**, which is given by

$$UQ \propto (3k_m^2 - k_y^2) \geq 0. \quad (6\text{-}2\text{-}12)$$

We always have modulational instability for $k_y^2 < 3k_m^2$ and stability for $k_y^2 > 3k_m^2$. The characteristic shape of the created vortices is shown in Fig. 38.

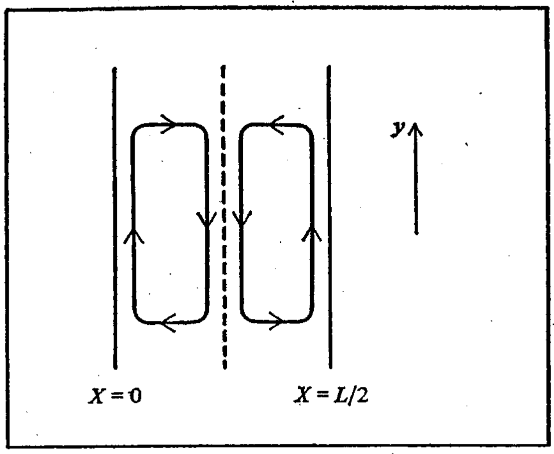


Fig. 38 Created vortex structure of electromagnetic convective cells (Ref. 34)

We note that the typical size in y-direction of created vortex is determined by the nonlinear effect through the modulational instability although the size in x-direction is characterized by the boundary condition, giving dimensional scaling between x and y with $k_y \approx \sqrt{3}k_m$. This characteristic structure has some similarity to the elongated structure of zonal flow and/or streamer, which is actively discussed, associated with turbulence in plasmas (Ref. 36).

6.3 Simple derivation of nonlinear Schrödinger equation

In nonlinear dispersive media, the nonlinear dispersion relation is generally given as

$$\varepsilon(\omega,k;|A|^2)=0,\quad \omega=\omega(k;|A|^2)\ . \qquad (6\text{-}3\text{-}1)$$

Here, the derivation of (6-3-1) was discussed in Ref. 37 and Ref. 38.

Assuming the **monochromatic approximation** for nonlinear waves, we can expand the dispersion relation around (k_0,ω_0) of $\exp[i(k_0 r-\omega_0 t)]$ as

$$\omega-\omega_0 \cong \left(\frac{\partial\omega}{\partial k}\right)_0 (k-k_0)+\frac{1}{2}\left(\frac{\partial^2\omega}{\partial k^2}\right)_0 (k-k_0)^2+\left(\frac{\partial\omega}{\partial|A|^2}\right)_0 |A|^2+\ldots \qquad (6\text{-}3\text{-}2)$$

If we put $\omega-\omega_0 \cong i\partial/\partial t$, $k-k_0 \cong -i\partial/\partial x$, we have the following **nonlinear Schrödinger equation (NLS equation)**,

$$i\left\{\frac{\partial A}{\partial t}+\left(\frac{\partial \omega}{\partial k}\right)_0 \frac{\partial A}{\partial x}\right\}+\frac{1}{2}\left(\frac{\partial^2 \omega}{\partial k^2}\right)_0 \frac{\partial^2 A}{\partial x^2}+\left(\frac{\partial \omega}{\partial |A|^2}\right)_0 |A|^2 A=0. \qquad (6\text{-}3\text{-}3)$$

In the wave frame, (6-3-3) reduces to the general form of NLS equation.

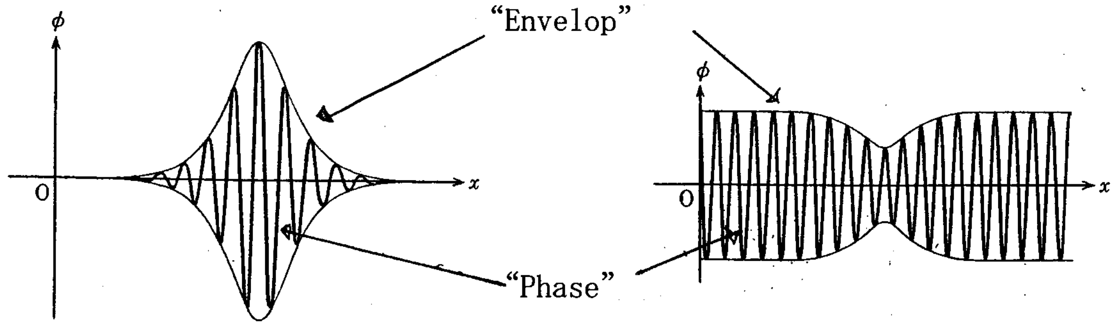
A Solution of NLS equation

The solution of nonlinear Schrödinger equation is described by

$$\varphi(x,t)=\sqrt{\Omega}sech[\sqrt{\Omega}(x-Vt-x_0)]\exp[i\frac{V}{2}x-i(\frac{V^2}{4}-\Omega)t]. \qquad (6\text{-}3\text{-}4)$$

It should be noted that the solution (6-3-4) has two characteristic parameters, namely, one is V, which is the velocity parameter and he other is Ω, which is related to the wave profile. The existence of these two parameters is different from the case of K-dV equation (one parameter) and it is associated with the fact that ϕ is complex and the NLS equation has one more freedom than K-dV equation.

Since the solution (6-3-4) is characterized by a solitary envelope shape, (6-3-4) is called the **envelop soliton solution**. Depending on two parameters (Ω, V), the envelop soliton solutions has two typical shapes, Namely, the bright soliton and dark soliton solutions. These solutions are illustrated in Fig. 39.



Bright Soliton **Dark Soliton**

Fig. 39 Bright and dark solitons

6.4 2D-nonlinear wave equations

We here introduce a couple of typical examples of 2 dimensional nonlinear equations and its properties of solutions.

(1) Kadomtsev-Petviashivili (K-P) equation

Obliquely propagating nonlinear waves are discussed in shallow water and in plasmas. The K-P equation is given as

$$(u_t + 6uu_x + u_{xxx})_x + \alpha u_{yy} = 0, \ (\alpha = \pm 1). \qquad (6\text{-}4\text{-}1)$$

The equation (6-4-1) reduces to the K-dV equation in case of $\alpha = 0$.
For example, we consider the two-soliton solution with $\alpha = 1$, which propagates obliquely in x-y plane with the angle $tan^{-1}p$ and the solution is described by

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\log f(x,y,t), \qquad (6\text{-}4\text{-}2)$$

with abbreviations of

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2},$$
$$\eta_i = k_i(x + p_i y - \omega_i t), \ \omega_i = k_i^2 + \alpha p_i^2, \ . \qquad (6\text{-}4\text{-}3)$$
$$A_{12} = \frac{3(k_1 - k_2)^2 - \alpha(p_1 - p_2)^2}{3(k_1 + k_2)^2 - \alpha(p_1 - p_2)^2}$$

The two-soliton solution of K-P equation is plotted in Fig. 40. We note that the first two terms in f gives one soliton solution and we obtain the two soliton solution by keeping e^{η_1}, e^{η_2} terms. However, each soliton is far away and the last term (A_{12}) describes the two-soliton solution at the overlapping phase.

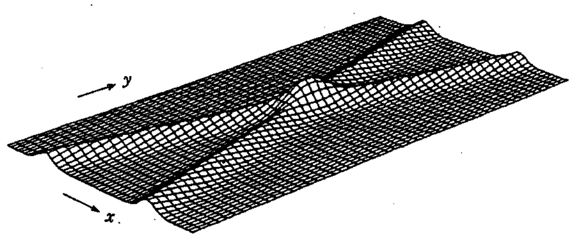


Fig. 40 2- soliton solution of K-P equation(Ref. 6)

(2) Flierl- Petviashvili equation

It is well known that the one-dimensional equation has the solution such as K-dV soliton as discussed in Sec. 5. The two dimensional equation has vortical monopolar solutions and Rossby wave in rotating atmosphere is one of another examples. Starting from the one-dimensional solution (K-dV soliton solution), possible extension to an analytical closed form of a two dimensional solution has been discussed by **F. Spineanu et al.** based on so-called the Flierl-Petviashivili equation (Ref. 40) associated with the generation of radially localized layers of sheared flow (zonal flow). In plasma physics, the Flierl-Petviashivili equation has the form

$$\Delta\phi = \alpha\phi - \beta\phi^2, \qquad (6\text{-}4\text{-}4)$$

where α, β are physical parameters, which are function of x and y. When $\partial^2/\partial x^2 << \partial^2/\partial y^2$, (6-4-4) is simplified and the solution can be obtained in terms of Jacobian elliptic functions. In this case, we have the solution

$$\phi(x,y) = \left(\frac{3\alpha}{2\beta}\right) sech^2[(\sqrt{3}/2)y], \qquad (6\text{-}4\text{-}5)$$

where the dependence on x is only parametric, through coefficient α, β. The properties of the elliptic functions will be discussed in **Appendix**. This solution (6-4-5) has the same form as the K-dV equation. Using the method of positions of the singularities, a class of new exact solutions of (5-4-4) has been discussed in Ref. 40. The solutions are periodic and have the geometry of zonal flow, namely, it has a one-dimensional geometry consisting of layers of periodic flow with an orientation in the (y, x) plane. Amplitude of perturbed potential is plotted in Fig. 41.

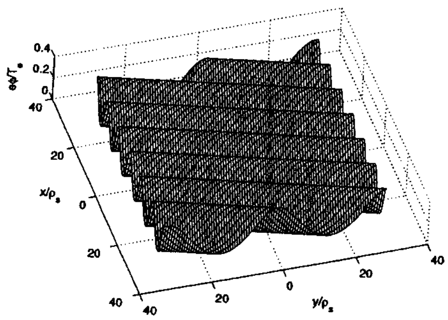


Fig. 41 Amplitude of normalized perturbed potential, $e\phi/T$ in (x, y) plane (from Ref. 40)

(3) Nonlinear drift wave in a collisionless nonuniform plasma

A nonlinear theory of the collisionless drift wave developed in a planner geometry with an arbitrary one-dimensional density profile. Since we consider the low-pressure collisionless plasma with $T_e/T_i > 1$, the electrons follow a linearized Boltzmann distribution. For the ions, we solve the drift kinetic equation. The following set of equations are employed(Ref. 41)

$$n_e = n_0 \exp(e\phi/T_e), \tag{6-4-6}$$

$$\frac{\partial f}{\partial t} + v_{//}\frac{\partial f}{\partial x} + \frac{1}{B}\{\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial \phi}{\partial y}\frac{\partial f}{\partial x}\} - \frac{e}{M}\frac{\partial \phi}{\partial z}\frac{\partial f}{\partial v_{//}} = 0, \tag{6-4-7}$$

$$\nabla_{//}{}^2\phi + \nabla_\perp \varepsilon_\perp \nabla_\perp \phi = 4\pi e\{n_e - \int f d\vec{v}\}, \tag{6-4-8}$$

where

$$\varepsilon_\perp = 1 + 4\pi Mn/B^2. \tag{6-4-9}$$

It should be noted that we have to solve the Poisson equation (6-4-8) by using the guiding center coordinates provided we solve the drift kinetic equation (6-4-7), which is described by the guiding center coordinates in a same way for Poisson equation. Although the distribution function can be described in the guiding center coordinates, the real coordinates essentially write the potential. Since we solve the drift kinetic equation described by the guiding center coordinates, we have to write the Poisson equation in the same guiding center coordinates like (6-4-8). The second term in the perpendicular component of dielectric tensor (6-4-9)is associated with this situation.

If we introduce the following stretched coordinates,

$$x = x,\ \ \xi = \varepsilon^{1/2}(y - st),\ \ \varsigma = \varepsilon z,\ \ \tau = \varepsilon^{3/2}, \tag{6-4-10}$$

and assume the first order of the potential in the form

$$\phi^{(1)}(x,\xi,\varsigma,\tau) = \Psi(\xi,\varsigma,\tau)g(x), \tag{6-4-11}$$

we have the following equations

$$\frac{d}{dx}\{N(\frac{\Omega_i}{\omega_{pi}})^2 + n_0\}\frac{d}{dx}g(x) - \frac{M\Omega_i^2}{T_e}n_0(x)\left(1 - \frac{v_*(x)}{s}\right)g(x) = 0, \tag{6-4-12}$$

$$\alpha\frac{\partial^2\Psi}{\partial\xi\partial\tau} + \beta\frac{\partial^2\Psi^2}{\partial\xi^2} + \gamma\frac{\partial^2\Psi}{\partial\varsigma^2} + \delta\frac{\partial^4\Psi}{\partial\xi^4} = 0. \tag{6-4-13}$$

Here, $v_*(x) = -(T_e/eB)(1/n_0)dn_0/dx$ is the drift velocity, g (x) should be solved as eigenvalue problem for given density profile and we get the eigenfunction g (x) and eigenvalue s by solving (6-4-12). This corresponds to the case of localized modes in plasmas with strong inhomogeneity. The time evolution with slow variation with (ξ,ς,τ) is determined by (6-4-13). Although the explicit forms for $\alpha,\beta,\gamma,\delta$ are given in Ref.41, these coefficients are determined by integration form of the eigenfunction g (x) and eigenvalue s. Without the third term due to ion motion along x-direction, (6-4-13) reduces to the K-dV equation as was discussed in Session 5.2. Assuming we consider the solution in wave frame as $\psi = \psi(\eta = \xi - \lambda\tau)$ and rewrite (6-4-13) in a dimensionless form, we obtain

$$\frac{\partial^2\psi}{\partial\varsigma^2} - \frac{\partial^2\psi}{\partial\eta^2} + 6\frac{\partial^2\psi^2}{\partial\eta^2} + \frac{\partial^4\psi}{\partial\eta^4} = 0. \qquad (6\text{-}4\text{-}14)$$

The equation (6-4-14) is a **Boussinesq equation**, which is well known in fluid dynamics (see, (5-5-4)). We can solve (6-4-14) and finally obtain the 2D soliton solution

$$\psi = \frac{\kappa^2}{4} sech^2[\frac{1}{2}(\kappa\eta - \beta\varsigma + \sigma]. \qquad (6\text{-}4\text{-}15)$$

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Note: These topics have been presented in the Sokendai lecture (September1-3, 2004). Some topic of these lectures was also presented at the seminars in ASIPP and SWIP, China, 2005/May/23 and 27

Appendix

"Elliptic Functions"

We here explain briefly the elliptic functions, which has been discussed in Sec. 2. 2, (see, (2-1-6)) and associated with the solution of K-dV equation in Sec. 5. 1 and 5. 2 and Flierl-Petviashivili equation in Sec. 6. 4.

One of the classical problems of nonlinear analysis is the determination of motion of both linear and nonlinear pendulum. In general, the analysis of nonlinear pendulum requires the introduction of transcendental functions like **elliptic functions**. As was discussed in Sec. 2. 2, the solution of nonlinear oscillator with the cubic polynomial nonlinear force (2-1-4) is given by the elliptic function (see, (2-1-6)).

Here, we introduce two typical elliptic functions, namely, **complete elliptic integrals** and **Jacobian elliptic functions**.

As an example of the complete elliptic integrals, we consider the undamped pendulum, which is described by

$$\ddot{\theta} + sin\theta = 0 \ , \qquad \text{(A-1)}$$

where θ is the angle of pendulum. The integration of (A-1) is given as

$$\frac{1}{2}\dot{\theta}^2 - \cos\theta = E = -\cos(2\alpha), \qquad \text{(A-2)}$$

where the integration constant E is assumed to be E<1 and α is a constant given by $2\alpha = \max\theta(\dot{\theta} = 0)$. If we introduce the variable ϕ through the relation,

$$sin(\theta/2) = sin(\alpha)sin\phi, \ (\max\phi = \pi/2), \qquad \text{(A-3)}$$

we finally obtain the following relation

$$\dot{\phi} = \cos(\theta/2) = [1 - sin^2(\theta/2)]^{1/2} = [1 - k^2 sin^2\phi]^{1/2} \ , \qquad \text{(A-4)}$$

where $k = sin\theta$; $k^2 \leq 1$. Integrating (A-4) yields

$$t - t_0 = \int_0^\phi [1 - k^2 sin^2\theta]^{-1/2} d\theta$$

$$\overline{F}(\phi,k) \equiv \int_0^\phi [1 - k^2 sin^2\theta]^{-1/2} d\theta \qquad \text{(A-5)}$$

Here, $\overline{F}(\phi,k)$ is called the elliptic integration of the first kind.

Next we discuss the Jacobian elliptic functions. To do this, we consider the K-dV equation, $u_t + 6uu_x + u_{xxx} = 0$. If we study a moving frame solution of K-dV equation in the form $u(x,t) = u(y = x - \lambda t)$, K-dV equation reduces to

$$-\lambda u + 6uu' + u''' = 0 \ , \qquad \text{(A-6)}$$

and integration of (A-6) yields

$$\frac{1}{2}u'^2 + u^3 - \frac{1}{2}\lambda u^2 + Au + B = 0, \qquad \text{(A-7)}$$

where A and B are the integration constants and are determined by the boundary condition. Introducing a potential V (u) as

$$V(u) = u^3 - \frac{1}{2}\lambda u^2 + Au + B = 8u - a)(u - b)(u - c), \ [a > b > c]. \qquad \text{(A-8)}$$

The potential V (u) versus u is roughly plotted in Fig. 42.

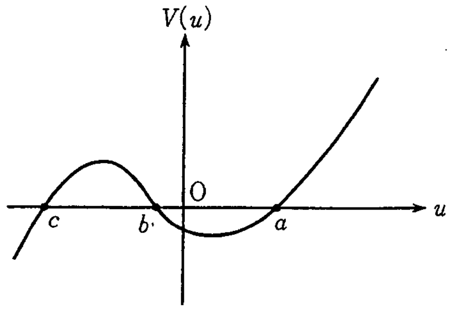

Fig. 42 Potential V (u) versus u

It should be noted that there is an oscillatory solution for a>u>b. From (A-7), we obtain

$$\int_a^u \frac{du}{\sqrt{(a-u)(u-b)(u-c)}} = \pm \int_{y_0}^y \sqrt{2} dy, \qquad \text{(A-9)}$$

which yields

$$\frac{2}{\sqrt{a-b}} sn^{-1}\left(\sqrt{\frac{a-u}{a-b}}, k\right) = \pm\sqrt{2}(y-y_0),\ k = \sqrt{\frac{a-b}{a-c}}, \quad \text{(A-10)}$$

where **sn (z, k)** is called the **Jacobian elliptic function**. We finally have the solution of K-dV equation

$$u(x,t) = b + (a-b)cn^2[\sqrt{\frac{a-c}{2}}(x-\lambda t - x_0), k]. \quad \text{(A-11)}$$

We call this type of propagating wave the **cnoidal wave**. If we search a solution, which satisfies the boundary condition, $u=0, u'=0$ as $y->\infty$, A=B=0 and b=c=0 and $a=\lambda/2$, then (A-11) reduces to

$$u(x,t) = \frac{1}{2}\lambda sech^2\left(\frac{1}{2}\sqrt{\lambda}(x-\lambda t - x_0)\right), \quad \text{(A-12)}$$

which is the soliton solution discussed in (5-3-3), (6-4-4) and (6-4-15). Noting that in this case, k tends to unity, we have the asymptotic form of the elliptic function cn(u, k=1)=sech(u). Using these properties of elliptic functions, we easily obtain (A-12) from (A-11).

Calculations involving the elliptic functions are complicated because of he lack of suitable tables of formulas. Here, we summarize some formulas in connection with the topics discussed in this lecture.

<u>Complete elliptic integrals</u>

a) **First kind**:

$$K = K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2 sin^2\theta}} = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad \text{(A-13)}$$

b) **Second kind**:

$$E = E(k) = \int_0^{\pi/2} \sqrt{1-k^2 sin^2\theta}\, d\theta = \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}}\, dx. \quad \text{(A-14)}$$

c) **Third kind**:

$$\Pi(\gamma,k) = \int_0^{\pi/2} \frac{d\theta}{(1+\gamma sin^2\theta)(1-k^2 sin^2\theta)^{1/2}} = \int_0^{\pi/2} \frac{dx}{(1+\gamma x^2)(1-x^2)\sqrt{1-k^2x^2}}. \quad \text{(A-15)}$$

It should be noted that the adiabatic invariant form in tokamak and helical configurations are expressed by these complete elliptic integrals. Particularly, the adiabatic invariant form in the helical configuration with strong helical ripples is described by the combination form of first kind and third kind elliptic integrals (see, Ref. 42).

d) **Expansions in series:**

$$K(k) = \frac{\pi}{2}\left(1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + ...\right), \ (k << 1)$$
$$E(k) = \frac{\pi}{2}\left(1 - \frac{1}{4}k^2 - \frac{9}{64}\frac{k^4}{3} - ....\right), \ (k << 1) \quad \text{(A-16)}$$

e) **Limiting forms:**

$$K(k) -> \log\frac{4}{\sqrt{1-k^2}}, \ E(k) -> 1, \ (k -> 1). \quad \text{(A-17)}$$

f) **Associated complete elliptic integrals:**

We call k the modulus and the k' is referred to the complementary modulus through the relation, $k' = \sqrt{1-k^2}$,

$$K'(k) \equiv K(k'), \ E'(k) \equiv E(k'). \quad \text{(A-18)}$$

g) **Derivatives with respect to the modulus:**

$$\frac{dK}{dk} = \frac{E - k'^2 K}{kk'^2}, \ \frac{dE}{dk} = \frac{E-K}{k},$$
$$\frac{d}{dk}(E - k'^2 K) = kE, \ \frac{d}{dk}(E-K) = -\frac{kE}{k'^2} \quad \text{(A-19)}$$

Jacobian elliptic functions

a) $sn(u)$, $cn(u)$, $dn(u)$ **functions:**

$$u = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2 sin^2\theta}} = \int_0^{sin\phi} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad \text{(A-20)}$$

$$sn(u) = sin\phi = sn(u,k), \ cn(u) = \cos\phi = cn(u,k),$$
$$dn(u) = \sqrt{1-k^2 sn^2 u} = dn(u,k) \quad \text{(A-21)}$$

b) **Limiting forms:**

$$sn(u,0) = sin(u),\ \ cn(u,0) = \cos(u),$$
$$sn(u,1) = tanh(u),\ \ cn(u,1) = dn(u,1) = sech(u) \qquad \text{(A-22)}$$

c) **Expansions in series:**

$$k^2 = m,\ \ 1-k^2 = m_1,\ \ k'^2 = m_1,$$
$$sn(u) = u-(1+m)\frac{u^3}{3!}+(1+14m+m^2)\frac{u^5}{5!}-......$$
$$cn(u) = 1-\frac{u^2}{2!}+(1+4m)\frac{u^4}{4!}-.......,\qquad \text{(A-23)}$$
$$dn(u) = 1-m\frac{u^2}{2!}+m(m+4)\frac{u^4}{4!}-.........$$

d) **Differentiations**

$$\frac{\partial}{\partial u}(sn(u)) = cn(u)dn(u),\ \ \frac{\partial}{\partial u}(cn(u)) = -sn(u)dn(u),$$
$$\frac{\partial}{\partial u}(dn(u) = -k^2 sn(u)cn(u),$$
$$\frac{\partial}{\partial k}(sn(u)) = \frac{dn(u)cn(u)}{kk'^2}[-E(u)+k'^2u+k^2sn(u)cd(u)],\ \ cd(u) = \frac{cn(u)}{dn(u)},\qquad \text{(A-24)}$$
$$\frac{\partial}{\partial k}(cn(u) = \frac{sn(u)dn(u)}{kk'^2}[-k'^2u+E(u)-k^2sn(u)cd(u)],$$
$$\frac{\partial}{\partial k}(dn(u) = \frac{ksn(u)cn(u)}{k'^2}[E(u)-k'^2u-dn(u)tn(u)],\ tn(u) = \frac{sn(u)}{cn(u)}$$

e) **Relations between the squares of Jacobian elliptic Functions**

$$sn^2(u)+cn^2(u) = 1,\ \ dn^2(u)+k^2sn^2(u) = 1,\ \ dn^2(u)-k^2cn^2(u) = 1-k^2,\qquad \text{(A-25)}$$

f) **Double and half arguments**

$$sn(2u) = \frac{2sn(u)cn(u)dn(u)}{1-k^2sn^4(u)},\ \ cn(2u) = \frac{cn^2(u)-sn^2(u)dn^2(u)}{1-k^2sn^4(u)},$$
$$dn(2u) = \frac{dn^2(u)-k^2sn^2(u)cn^2(u)}{1-k^2sn^4(u)},$$
$$sn^2\frac{1}{2}(u) = \frac{1-cn(u)}{1+dn(u)},\ \ cn^2\frac{1}{2}(u) = \frac{dn(u)+cn(u)}{1+dn(u)},\qquad \text{(A-26)}$$
$$dn^2\frac{1}{2}(u) = \frac{1-k^2+dn(u)+k^2cn(u)}{1+dn(u)}$$

h) Additional theorems

$$sn(u+v)=\frac{sn(u)cn(v)dn(v)+sn(v)cn(u)dn(u)}{1-k^2sn^2(u)sn^2(v)},$$
$$cn(u+v)=\frac{cn(u)cn(v)-sn(u)dn(u)sn(v)dn(v)}{1-k^2sn^2(u)sn^2(v)}, \quad . \qquad \text{(A-27)}$$
$$dn(u+v)=\frac{dn(u)dn(v)-k^2sn(u)cn(u)sn(v)cn(v)}{1-k^2sn^2(u)sn^2(v)}$$

The detailed formula discussed in the Appendix have been discussed in Refs. 9, 10 , 43 , other books and mathematical tables.

References (Lecture 1)

References for papers, books, mathematical tables, Functions etc. used in this lecture are summarized here.

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