Zonal flows in 3D toroidal systems

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A range of techniques for mitigating stellarator neoclassical (nc) transport has been developed, and attention is now turning to also reducing the turbulent transport fluxes. As for tokamaks, zonal flows (ZFs) will be an important tool in achieving this, and understanding the effect of machine geometry on these is important. This paper discusses a theory of the shielding and time evolution of zonal flows in stellarators and tokamaks, which attains greater generality and conciseness by use of the action-angle formalism. The theory supports the earlier perspective that neoclassically-optimized devices should have less ZF damping, but it is pointed out that the further view, that this implies that optimized devices should therefore also have less turbulent transport, is overly simplistic, neglecting the additional configuration dependence of the nonlinear source which drives the ZFs.

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Since the 1980s, a range of techniques for mitigating stellarator neoclassical (nc) transport has been developed, and attention is now turning to also reducing the turbulent transport fluxes. As for tokamaks, an important tool for achieving this will be producing strong zonal flows (ZFs), which can act to suppress the turbulence producing the transport. It is thus of interest to understand how machine geometry will affect the strength of these flows, and their effectiveness in suppressing turbulent transport. This paper describes an analytic study of the shielding and time evolution of ZFs in stellarators and tokamaks,[1] discusses some of its implications, and its context in recent work on ZFs.

ZFs are primarily poloidal $E \times B$ flows due to a radially-varying electrostatic potential $\phi_z(r,t)$ driven by the nonlinearity in the kinetic equation. For present purposes, this is the Vlasov equation,

$$(\partial_t + \hat{H}_0)\delta f(\mathbf{z},t) = -\hat{h} f_0 - \hat{h}\delta f, \qquad (1)$$

with nonlinearity $-\hat{h}\delta f \equiv -\{\delta f, h\}$. Here, $\hat{b}a \equiv \{a,b\}$ is the Poisson bracket of $a(\mathbf{z})$ with $b(\mathbf{z})$ in the 6-dimensional phase space $\mathbf{z}$, and the Hamiltonian $H(\mathbf{z},t) = H_0 + h$ and distribution function $f(\mathbf{z},t) = f_0 + \delta f$ are divided into their unperturbed (subscript 0) and perturbed portions, with f_0 satisfying $\hat{H}_0 f_0 = 0$. Here we consider electrostatic perturbations only, $h(\mathbf{z},t) = e\delta\phi(\mathbf{r},t)$, where $\mathbf{r}(\mathbf{z})$ is the particle position. The dynamics of ZFs are determined by a self-consistent loop between ϕ_z and the potential fluctuations $\phi_{\mathbf{k}}$ of the turbulence. Via the nonlinearity in Eq.(1), the $\phi_{\mathbf{k}}$ produce a source $S \sim |\phi_{\mathbf{k}}|^2$ for ϕ_z, which in turn affects the growth and amplitudes of the $\phi_{\mathbf{k}}$.[2, 3, 4] The theory developed in Ref. [1] follows earlier work[5, 6, 7, 8] in primarily addressing the former of these 2 legs of the loop, taking the nonlinearity $-\hat{h}\delta f$ as a known source $S f_0$ and computing the resultant ϕ_z as a linear response problem, $k^2\phi_z = 4\pi\delta\rho^{xt}/\mathcal{D}$, where $\delta\rho^{xt}$ is the external charge-density perturbation driven by S, $\delta\rho^{xt} \sim \int dt S(t)$, and $\mathcal{D}$ is the dielectric function. Regarding the latter leg, the effect of machine geometry on S is relatively unexplored to date, but is an important issue in understanding the overall dynamics. This is discussed further later in this paper.

In Ref. [1] the action-angle (aa) formalism[9] was used to solve the kinetic equation without expansion of that equation in small parameters of radial excursions and timescale, resulting in more general expressions for the dielectric shielding, and extending results from earlier work.[5, 6, 7, 8, 10] From these expressions, it was found that for each mechanism of collisional transport, there is a corresponding shielding mechanism, of closely related form and scaling. Assuming the amplitude of the nonlinear source is unchanged, this correspondence supports the suggestion raised in earlier work[11, 6, 12, 10] that neoclassically-optimized stellarators will also have larger ZFs, and consequently lower turbulent transport. On a longer, diffusive timescale, ZF evolution was shown to be governed by a Langevin-like equation, with radial electric field $E_r(t)$ moving diffusively about roots E_a of the ambipolarity equation. The resultant probability distribution function is bounded, a balance between the turbulent fluctuations inducing diffusion and the neoclassical fluxes providing a restoring force toward $E_r = E_a$. A fuller exposition of the analytic theory is given in Ref. [1]. Here we recap the elements of that theory, and discuss some of its implications and related issues yet to be addressed.

Action-Angle formalism

In the aa formalism one parametrizes phase points $\mathbf{z}$ with the 3 invariant actions $\mathbf{J}$ of the unperturbed motion and their 3 conjugate angles $\boldsymbol{\theta}$, instead of the more directly physical particle position $\mathbf{r}$ and momentum $\mathbf{p}$. The unper-

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turbed Hamiltonian is then independent of $\boldsymbol{\theta}$, $H_0 = H_0(\mathbf{J})$. The key feature of aa variables is that they make the description of particle motion very simple. Hamilton's equations are:

$$\dot{\boldsymbol{\theta}} = \partial_{\mathbf{J}} H = \boldsymbol{\Omega}(\mathbf{J}) + \partial_{\mathbf{J}} h \simeq \boldsymbol{\Omega}(\mathbf{J}), \qquad (2)$$

$$\dot{\mathbf{J}} = -\partial_{\boldsymbol{\theta}} h = -i \sum_{\mathbf{l}} \mathbf{l} h_{\mathbf{l}}(\mathbf{J}, t) \exp(i\mathbf{l} \cdot \boldsymbol{\theta}), \qquad (3)$$

where $\partial_{\mathbf{J}}$ $(\partial_{\boldsymbol{\theta}})$ denotes a gradient in $\mathbf{J}$ $(\boldsymbol{\theta})$-space, $\boldsymbol{\Omega}(\mathbf{J}) \equiv \partial_{\mathbf{J}} H_0$, and $\mathbf{l}$ is the 3-component vector index, specifying the harmonic of each component of $\boldsymbol{\theta}$ in the Fourier decomposition $h(\mathbf{z}) = \sum_{\mathbf{l}} h_{\mathbf{l}}(\mathbf{J}) \exp(i\mathbf{l} \cdot \boldsymbol{\theta})$.

Following Refs. [5, 6, 7] in replacing the nonlinear term $-\{\delta f, h\}$ with a specified source $S(\mathbf{z}, t) f_0$, and Laplace transforming in time and Fourier transforming in $\boldsymbol{\theta}$, one readily obtains a solution for δf, nonperturbative in the excursions made in a particle orbit,

$$\delta f_{\mathbf{l}}(\mathbf{J}, \omega) = G_0[i\mathbf{l} \cdot \partial_{\mathbf{J}} f_0 h_{\mathbf{l}}(\mathbf{J}, \omega) + \delta f_{\mathbf{l}}(\mathbf{J}, t = 0) + S_{\mathbf{l}}(\mathbf{J}, \omega) f_0], \qquad (4)$$

with propagator $G_0 \equiv (-i\omega + i\mathbf{l} \cdot \boldsymbol{\Omega} + \nu_f)^{-1}$, in which we include an effective damping rate ν_f, to later consider the effect of collisions. Inserting Eq.(4) into the expression for the charge density, (now showing species label s) $\delta\rho_s(\mathbf{x}) = \int d\mathbf{z}\rho(\mathbf{x}|\mathbf{z})\delta f_s(\mathbf{z})$, where $\rho(\mathbf{x}|\mathbf{z}) \equiv e_s\delta(\mathbf{x} - \mathbf{r}(\mathbf{z}))$ is the charge density kernel and e_s is the species charge, one obtains 3 contributions, labelled A, B, and C, corresponding to the 3 terms on the right side of (4):

$$\delta\rho_{sA}(\mathbf{x}, \omega) = \int d\mathbf{x}' K_s(\mathbf{x}, \mathbf{x}', \omega)\delta\phi(\mathbf{x}', \omega) \qquad (5)$$

$$\delta\rho_{s,B+C}(\mathbf{x}, \omega) = (2\pi)^3 \int d\mathbf{J} \sum_{\mathbf{l}} \rho^*_{\mathbf{l}}(\mathbf{x}|\mathbf{J}) G_0 \times [\delta f_{s\mathbf{l}}(\mathbf{J}, t = 0) + S_{s\mathbf{l}}(\mathbf{J}, \omega) f_{s0}], \qquad (6)$$

with response kernel[9]

$$K_s(\mathbf{x}, \mathbf{x}', \omega) = K^{ad}_s(\mathbf{x}, \mathbf{x}') + (2\pi)^3 \int d\mathbf{J} \sum_{\mathbf{l}} \times \rho^*_{\mathbf{l}}(\mathbf{x}|\mathbf{J}) \frac{\omega\partial_{H_0} f_{s0} + \mathbf{l} \cdot \partial_{\mathbf{J}})_{H_0} f_{s0}}{\mathbf{l} \cdot \boldsymbol{\Omega} - \omega - i\nu_f} \rho_{\mathbf{l}}(\mathbf{x}'|\mathbf{J}). \qquad (7)$$

$\delta\rho_{sA}$, proportional to h or $\delta\phi$, gives the self-consistent response of the plasma, with response kernel K_s. $\delta\rho_{sB}$, due to the initial conditions of δf, gives the transient ballistic response, and the third term, $\delta\rho_{sC}$, is due to the nonlinear drive. $K^{ad}_s(\mathbf{x}, \mathbf{x}') \equiv e_s\delta(\mathbf{x} - \mathbf{x}') \int d\mathbf{z}\rho(\mathbf{x}|\mathbf{z})\partial_{H_0} f_{s0}$ is the (generalized) adiabatic term, reducing to the usual adiabatic term when f_0 is specialized to the local Maxwellian form

$$f_M(\mathbf{J}) \equiv \frac{n_0}{(2\pi MT)^{3/2}} \exp[-(H_0 - e\Phi_a)/T], \qquad (8)$$

where density n_0, ambipolar radial potential Φ_a, and temperature T are functions of the drift-averaged minor radius $r_d(\mathbf{J})$, and M is the particle mass.

Specialization to Toroidal Geometry

The expressions given thus far are valid for any system where a complete set $\mathbf{J}$ of constants of the motion exists. We now specialize to toroidal geometries, including tokamaks and stellarators. We represent position in terms of flux coordinates $\mathbf{r} = (\psi, \theta, \zeta)$, where $2\pi\psi$ is the toroidal flux within a flux surface, and θ and ζ are the poloidal and toroidal azimuths. In terms of these, the magnetic field may be written $\mathbf{B} = \nabla\psi \times \nabla\theta + \nabla\zeta \times \nabla\psi_p = \nabla\psi \times \nabla\alpha_p$, with $2\pi\psi_p$ the poloidal flux, Clebsch angle $\alpha_p \equiv \theta - \iota\zeta$, constant along a field line, and $\iota \equiv q^{-1} \equiv d\psi_p/d\psi$ the rotational transform. α_p and momentum $(e/c)\psi$ form a canonically conjugate pair for motion perpendicular to the field line. It is also useful to define an average minor radius $r(\psi)$ by $\psi \equiv \bar{B}_0 r^2/2$, with $\bar{B}_0 \equiv \bar{B}(r = 0)$ the average magnetic field strength on axis. We consider toroidal systems with the nonaxisymmetric portion of magnetic field strength B dominated by a single helical phase $\eta_0 \equiv n_0\zeta - m_0\theta$,

$$B(\mathbf{x}) = \bar{B}(r)[1 - \epsilon_t(r)\cos\theta - \delta_h(\mathbf{x})\cos\eta_0], \qquad (9)$$

but with ripple strength $\delta_h(\mathbf{x})$ allowed to vary slowly over a flux surface, with flux-surface average $\epsilon_h(r) \equiv \langle\delta_h\rangle$.

In such configurations, collisionless particle motion occurs on 3 disparate timescales, the gyro, transit/bounce, and perpendicular drift scales, denoted by subscripts g, b, and d. The characteristic gyro, bounce, and drift frequencies satisfy $\Omega_g \ll \Omega_b \ll \Omega_d$, and the corresponding radial excursions particles make on those scales are the gyroradius $\rho_g = v_\perp/\Omega_g$, the radial bounce excursion/banana width $\rho_b \simeq v_{Bt}/\Omega_b$, and the "superbanana width" excursion $\rho_d \simeq \sigma_h v_{Bt}/\Omega_d$. Here, $v_{Bt} = \epsilon_t\mu\bar{B}/(M\Omega_g r)$ is the grad-B drift amplitude, μ is the magnetic moment, and σ_h equals 1 for a ripple-trapped particle (in trapping state $\tau = h$), and 0 otherwise. Thus, passing or toroidally-trapped particles ($\tau = p$ and t, resp.) have $\rho_d = 0$.

The aa variables can be chosen so that motion on each timescale can be described by one of the 3 pairs (θ_i, J_i). A suitable choice is $\boldsymbol{\theta} = (\theta_g, \theta_b, \bar{\alpha}_p)$, $\mathbf{J} = (J_g, J_b, (e/c)\bar{\psi})$, with $J_g \equiv (Mc/e)\mu$ the gyroaction, θ_g the gyrophase, describing the fastest time scale of the motion, J_b the bounce action, θ_b its conjugate bounce phase, $\bar{\psi}$ the drift-orbit averaged value of ψ, and its conjugate phase $\bar{\alpha}_p$, the orbit-averaged Clebsch coordinate α_p, describing the slow, drift timescale. To make the periodicity of the drift angle 2π as for the other 2 phases, instead of $(\bar{\alpha}_p, (e/c)\bar{\psi})$ we use the closely related canonical pair $(\theta_d, J_d = (e/c)\bar{\psi}_d)$, with $\theta_d \equiv \bar{\alpha}_p/(1 - \iota q_{mn0})$, $\bar{\psi}_d \equiv \bar{\psi} - \bar{\psi}_p q_{mn0}$, where $q_{mn0} \equiv m_0/n_0$. For typical parameters, $\iota q_{mn0} \ll 1$, so that $(\theta_d \simeq \bar{\alpha}_p, \bar{\psi}_d \simeq \bar{\psi})$. Correspondingly one has the characteristic frequencies of motion $\boldsymbol{\Omega} \equiv (\Omega_g, \Omega_b, \Omega_d)$, and vector index $\mathbf{l} \equiv (l_g, l_b, l_d)$.

We adopt an eikonal form for the structure of any mode a,

$$\phi_a(\mathbf{x}) = \bar{\phi}_a(r) \exp i\eta_a(\mathbf{x}), \qquad (10)$$

with wave phase $\eta_a(\mathbf{x}) \equiv [\int^r dr' k_r(r') + m\theta + n\zeta]$, and slowly-varying envelope $\bar{\phi}_a(r)$. Then using Eqs. (5,6,7), and (10) in the Poisson equation, one obtains the radially-local response equation[1]:

$$k^2\mathcal{D}(\mathbf{k},\omega)\frac{e_i\overline{\phi}_a(r)}{T_i} = \sum_s \lambda_{si}^{-2}\sum_{\mathbf{l}}\langle G^*_{\mathbf{l}a}(\mathbf{J})\times \frac{i[\delta f_{s\mathbf{l}}(t=0)/f_{s0} + S_{s\mathbf{l}}(\omega)]}{(\omega - \mathbf{l}\cdot\boldsymbol{\Omega} + i\nu_{fs})}\rangle \qquad (11)$$

Here, $\lambda_{si}^2 \equiv T_i/(4\pi n_{s0}e_s e_i)$, $k^2 \equiv |\mathbf{k}|^2$, $\langle\ldots\rangle \equiv (2\pi)^{-2}\oint d\theta d\zeta \int d\mathbf{p}(f_0/n_0)\ldots$ is the flux surface and momentum-space average over the unperturbed distribution function f_0, $G_{\mathbf{l}a}(\mathbf{J}) \equiv (2\pi)^{-3}\oint d\boldsymbol{\theta}\exp(-i\mathbf{l}\cdot\boldsymbol{\theta})\exp i\delta\eta_a(\mathbf{z})$ the "orbit-averaging factor", a concise expression for the interaction of mode a with particles with actions $\mathbf{J}$, and $\delta\eta_a$ the portion of η_a oscillatory in $\boldsymbol{\theta}$ (so having zero $\boldsymbol{\theta}$–average). Dielectric function $\mathcal{D}$ is given by $\mathcal{D}(\mathbf{k},\omega) \equiv 1 + \sum_s \chi_s(\mathbf{k},\omega)$, with susceptibility $\chi_s(\mathbf{k},\omega) = (k\lambda_s)^{-2}g_s(\mathbf{k},\omega))$, and shielding function

$$g_s(\mathbf{k},\omega) = 1 - \sum_{\mathbf{l}}\langle |G_{\mathbf{l}a}(\mathbf{J})|^2\frac{\omega - \omega^f_{*s}}{\omega - \mathbf{l}\cdot\boldsymbol{\Omega} + i\nu_{fs}}\rangle. \qquad (12)$$

Here, $\omega_*^f \equiv \omega_*[1 + \eta(u^2 - 3)/2]$, with $\omega_* \equiv -k_\alpha cT/(eBL_n)$ the diamagnetic drift frequency, $\eta \equiv d\ln T/d\ln n$, $u \equiv v/v_s$ the particle velocity, normalized to the thermal speed v_s, $k_\alpha \equiv l_d/r$, and $L_n^{-1} \equiv -\partial \ln n_0/\partial r$. The 1 in g_s comes from the adiabatic term K_s^{ad} in Eq.(7). The 2 terms on the right side of Eq.(11) arise from $\delta\rho_{s,B+C}$. This response equation is of the same form as that obtained in Refs. [5, 6], or of any linear response calculation. The differences lie in the form of the dielectric $\mathcal{D}$, and in the use of the aa form, which facilitates dealing with the range of timescales and of orbit-averaging effects in complex geometries in a general manner.

To evaluate the $G_{\mathbf{l}}$, we describe the radial motion by $r \simeq r_d + \delta r^{(d)}(\theta_d) + \delta r^{(b)}(\theta_b) + \delta r^{(g)}(\theta_g)$, where for each $i = g, b, d$, we make a harmonic approximation of the motion in that phase, $\delta r^{(i)}(\theta_i) \simeq \rho_i\cos\theta_i$. This is a very good approximation for gyromotion, and a good approximation for bounce motion not too near a trapping-state boundary. For simplicity, we assume that superbanana ($\tau = h$) particles do not detrap, but precess poloidally dominated by $E\times B$ poloidal drift, $\Omega_d \simeq \Omega_{dE}$, which is roughly constant on a given orbit, while drifting radially as $v_{Bt}\sin\theta$, as usual. This yields radial motion of the given harmonic form, with superbanana width $\rho_d = \sigma_h v_{Bt}/\Omega_{dE}$, as noted above. For completeness, one may also include in this description the finite banana widths ρ_{bh} of $\tau = h$ particles, which give rise to the helically–symmetric nc transport branch in straight stellarators[13], and a second type of superbanana width, the finite radial excursions ρ_{dt} made by $\tau = t$ particles on the drift timescale, which give rise to the "banana-drift" transport branch.[14, 15, 16] One then finds

$$G_{\mathbf{l}a}(\mathbf{J}) = J_{l_g}(z_g)J_{l_b}(z_b)J_{l_d}(z_d)e^{-i\xi_a}, \qquad (13)$$

with $z_{g,b,d} = k_r\rho_{g,b,d}$, and ξ_a a phase factor, whose value is irrelevant, since $G_{\mathbf{l}}$ enters only as $|G_{\mathbf{l}}|^2$.

For drift turbulence, which is driving the ZFs, one typically has $k_\perp^d\rho_{gi} \sim 0.3$, and frequencies $\omega_{\mathbf{k}} \sim \omega_*(k_\perp^d)$. For ZFs, one has much smaller k_r and frequencies ω_Z, down by an order of magnitude, perhaps by the "mesoscale" ratio, $k_r^Z/k_\perp^d \sim \sqrt{\rho_{gi}/a}$. Thus, for both species, one has the ordering $\omega_Z, \Omega_d \ll \Omega_b \ll \Omega_g$, and $z_g < z_b < 1$. For the moment we leave the relative sizes of ω_Z and Ω_d unspecified. Also, one may have $z_d \gtrsim 1$ for trapped particles, for ions and also, notably, for electrons, as noted by [6]. Thus, as opposed to tokamaks, in stellarators electrons can participate in orbit averaging, because their radial excursions on the drift timescale can be comparable with those of ions.

Because $\omega_Z \ll \Omega_{b,g}$, the sum over $\mathbf{l}$ in Eq.(12) is dominated by the terms with $l_{g,b} = 0$, an approximation strengthened for $z_{g,b} \ll 1$, for which the factors $J^2_{l_{g,b}}$ in $|G_{\mathbf{l}}|^2$ in Eq.(12) are negligible unless $l_{g,b} = 0$. These reduce the triple sum there to a single sum $\sum_{l_d}$. In that sum, if one has $\omega \gg \Omega_d$, then over the l_d–range $\Delta l_d \sim z_d$ over which $J^2_{l_d}$ in Eq.(12) is appreciable the integrand does not change greatly, so that one can perform the summation, using the identity $\sum_l J_l^2 = 1$, which eliminates the $J^2_{l_d}$ factor, leaving only the factor $J^2_{l_g}J^2_{l_b}$. In the other limit $\omega \ll \Omega_d$, the sum is dominated by the $l_d = 0$ term, and the effect of $J^2_{l_d}$ survives. Thus, for $\omega \ll \Omega_d$, all of gyro-, bounce- and drift-averaging contribute. Neglecting ν_{fs}, Eq.(12) becomes

$$\begin{aligned} g_s(\mathbf{k},\omega) &\simeq 1 - \Lambda_{0b}(b_g,b_b),\ (\Omega_d \ll \omega), \qquad (14)\\ g_s(\mathbf{k},\omega) &\simeq 1 - \Lambda_{0d}(b_g,b_b,b_d),\ (\omega \ll \Omega_d), \end{aligned}$$

where $\Lambda_{0d}(b_g,b_b,b_d) \equiv \langle J_g^2J_b^2J_d^2\rangle$, $\Lambda_{0b}(b_g,b_b) \equiv \Lambda_{0d}(b_g,b_b,b_d=0) \equiv \langle J_g^2J_b^2\rangle$, $J^2_{g,b,d} \equiv J_0^2(z_{g,b,d})$, $b_g \equiv k_r^2\rho^2_{gT}$, $b_b = b_g q^2/(F_t\epsilon_t^{1/2})$, and $b_d \equiv k_r^2\rho^2_{dT}$, with $\rho_{gT} \equiv v_T/\Omega_g$, v_T the species thermal velocity, $\rho_{dT} \equiv \rho_d(v = v_T) \propto v_T^2$, and F_t the fraction of toroidally-trapped particles.

The physics represented by Eqs.(14) is that if the the ZF drive in a stellarator has a time variation slow compared with Ω_d [*cf.* Eq.(14b)], $\tau = h$ particles have time to partially shield out ϕ_z by drifting along their collisionless superbanana orbits, an averaging mechanism not available to tokamaks. If the ZF drive varies rapidly compared with Ω_d [Eq.(14a)], this new mechanism for radial averaging is lost.

The functions Λ_{0b} and Λ_{0d} succinctly describe the additional contributions from finite b_b, corresponding to shielding due to the "bounce" contribution g^b to the shielding function computed in Refs. [5] and [6], and from finite b_d, corresponding to a "drift" contribution g^d to g, extending the result in Ref. [6]. Table 1 synopsizes some of the limiting cases covered by earlier work, extended in Ref. [1], of which Eqs.(14) here are Eqs.(15) in [1] noted in the table. One notes that most of the entries are for collisionless theory, $\nu_f = 0$. In the tokamak limit ($\epsilon_h, b_d \to 0$), $\Lambda_{0d}(b_g,b_b,b_d) \to \Lambda_{0b}(b_g,b_b)$, so that Eq.(14a) again holds. In the further cylindrical limit ($\epsilon_t \to 0$), b_b vanishes, and the Λ's in Eqs.(14) reduce to the more familiar $\Lambda_0(b_g) \equiv \Lambda_{0b}(b_g, b_b = 0) \equiv \langle J_g^2\rangle = I_0(b_g)e^{-b_g}$, with $I_0(b)$ the modi-

Parameter range	Ref.
$\nu_f = 0$:	
tokamak limit ($\epsilon_h, b_d = 0$), $\omega < \Omega_b$:	
$b_b < 1$	[5]
b_b arbitrary	[17]
stellarators ($\epsilon_h, b_d > 0$) :	
$b_d \to \infty$, $\omega < \Omega_d$	[6, 7]
$b_d < 1$, $\omega < \Omega_d$	[1](16b)
$b_d < 1$, $\Omega_d < \omega$	[1](16a)
b_d arbitrary, $\omega < \Omega_d$	[1](15b)
b_d arbitrary, $\Omega_d < \omega$	[1](15a)
$\nu_f > 0$:	
$\nu_h/\Omega_d > 1$	[8]
ν_h/Ω_d arbitrary	[1]

Table 1 Cases covered by the present theory

fied Bessel function of the first kind. For $b_g < 1$, one has $\Lambda_0(b_g) \simeq 1 - b_g$, and thus $g \simeq b_g$, the gyro- contribution g^g to g, corresponding to the classical (gyro) polarization current $J^{p,g}$.

Using the small-argument expansion $J_0(z) \simeq 1-(z/2)^2$ for z_d as well as $z_{g,b}$ in Λ_{0d}, Eqs.(14) reduce to [1]

$$g_s(\mathbf{k},\omega) \simeq b_g + F_t c_b b_b,\ (\Omega_d \ll \omega), \qquad (15)$$
$$g_s(\mathbf{k},\omega) \simeq b_g + F_t c_b b_b + F_h c_d b_d,\ (\omega \ll \Omega_d),$$

where $F_h = (2/\pi)\sqrt{2\epsilon_h}$ is the fraction of particles with $\tau = h$, $c_d \simeq (15/2)$, and for a tokamak, one finds $c_b \simeq 10\sqrt{2}/(3\pi) \simeq 1.5$, in approximate agreement with the value in Ref. [5].

One notes that the drift contribution $g^d = F_h c_d b_d \simeq F_h(k_r\rho_d)^2$ in Eq.(15b) has a form analogous to the bounce and gyro contributions, differing from the scaling $g^d \simeq F_h$ found in Refs. [6, 7]. This is because in Refs. [6, 7], the term $\Omega_d\partial_{\theta_d}\delta f$ was neglected in their kinetic equation, thereby implicitly taking the limit $b_d \to \infty$ (see Table 1), also of physical interest. Taking that limit in Eq.(14b) also recovers that scaling.

As discussed in Ref. [1] and illustrated by Eqs.(15), there is a correspondence between the contributions g^j to the shielding function and the radial collisional (classical+nc) transport coefficients D^j: the gyromotion producing the classical polarization term g^g also gives rise to classical transport, the bounce motion producing g^b gives rise to axisymmetric nc transport, and the drift motion yielding g^d also produces the "superbanana" branch of transport, dominant in conventional stellarators. For each mechanism j, one may use the heuristic form $D^j \simeq F_j\nu_{fj}(\Delta r_j)^2$, with F_j the fraction of particles participating in that mechanism, Δr_j the radial step in the random walk process, and ν_{fj} the effective stepping frequency in that random walk. For example, for the $1/\nu$ superbanana regime, one has $F_j \to F_h \simeq \epsilon_h^{1/2}$, $\Delta r_j \to v_{Bt}/\nu_h$, and $\nu_{fj} \to \nu_h \simeq \nu/\epsilon_h$. And for the shielding contributions g^j, one has the approximate form $g^j \simeq F_j(k_r\Delta r_j)^2$, exemplified by Eqs.(15). Hence, $g^{j'}/g^j \simeq (D^{j'}/D^j)(\nu_{fj}/\nu_{fj'})$. Thus, for $j \to g$ and $j' \to b$, one expects the gyro- contribution g^g in Eqs.(15) to be smaller than the bounce contribution g^b, because classical diffusion D^g is subdominant to banana diffusion D^b. Similarly, for $j \to b$, $j' \to d$, one expects the bounce contribution g^b to dominate g^d in Eq.(15b) as long as superbanana transport D^d is subdominant to D^b. For NCSX, for example, evaluations have shown[18] that, at a self-consistent radial ambipolar field E_a, NCSX should have D^d down from D^b by about an order of magnitude. Correspondingly, one expects that the ripple contribution g^d to ZF damping should be small compared with the tokamak contribution g^b.

It has been argued[11, 6, 12, 10] that neoclassically-optimized stellarators should also have lower turbulent transport, due to less damping of ZFs. The basic idea of most nc optimization techniques has been to reduce ripple transport (D^d) by reducing either F_h, or by reducing superbanana width $\rho_d \simeq v_{Bt}/\Omega_d$.[19] One notes from the resemblance of D^d to g^d, characterized by the argument $\frac{1}{2}\langle z_d^2\rangle \simeq F_h k_r^2\rho_{dT}^2$, that the present theory supports this argument.

However, as noted in the introduction, the shielding of ZFs from a given source, which most analytic ZF studies to date (including the present one) address, is only 1 of the 2 legs of the feedback loop in ZF dynamics. That work demonstrates that neoclassically-optimized stellarators will also tend to have lower damping of ZFs. However, in general, different configurations will have differing levels of drive for instabilities, and thus differing strengths of the ZF source S. Thus, recent gyrokinetic simulations[10] comparing configurations modeling LHD in its (A)standard and (B)inward-shifted operation, reported that the ZFs increased (by about 50%) in going from A to B, as might be expected from the better nc-optimization of case B. However, because case B also has much stronger (about 60%) ITG growth rates, the turbulent flux in that case is *increased* (by about 16%) over that in case A, in contrast with the experimentally–observed reduction[11] in turbulent transport. Very recent further simulations[20] of this same comparison, but with more realistic equilibrium profiles, have brought the numerical transport trends more into accord with experimental observations. Thus, it appears that the now commonly-cited correlation between nc and turbulent optimization is too simple, and a more refined understanding of this relationship must come from also accounting for the effects of machine geometry on the source strength.

Longer-time ZF evolution

As discussed in Ref. [1], the effects on ZFs from the g^j (describing dielectric shielding) and the D^j (describing radial collisional fluxes) come together in the time evolution equation for the flux-surface averaged radial electric field $E_r \equiv \langle\nabla r\cdot\mathbf{E}\rangle$, obtained from the surface average of

Ampere's law, plus an expression for the surface-averaged radial current J_r,

$$\begin{aligned} \partial_t E_r &= -4\pi J_r, \\ J_r &= (4\pi)^{-1}\chi\partial_t E_r + \sigma(E_r - E_a) + F_S/B. \end{aligned} \tag{16}$$

The first term in J_r, proportional to the time derivative of E_r, represents the polarization current J^p, with χ containing the dielectric shielding contributions. The second term represents the nonambipolar radial current due to nc transport, from a first-order expansion in $E \equiv (E_r - E_a) = -\langle\nabla r \cdot \nabla\phi_z\rangle$ of the nc radial current $\sum e_s\Gamma_s(E_r)$, where $E_a = -\langle\nabla r \cdot \nabla\Phi_a\rangle$ is the ambipolar value at which the ion and electron particle fluxes are equal. F_S is the force, coming from the source S in Eq.(1), exerted by the turbulence within a magnetic surface normal to the magnetic field, which acts as a source driving E_r. Using Eq.(16b) in (16a) yields a Langevin-like equation, which in the ω domain may be written

$$-i\omega E(\omega) + \gamma_E E(\omega) = c_S(\omega), \tag{17}$$

where $\gamma_E(\omega) \equiv 4\pi\sigma/\mathcal{D}(\omega)$, $c_S(\omega) \equiv -4\pi F_S/B\mathcal{D}(\omega)$, and $\mathcal{D}(\omega) \equiv 1 + \chi(\omega)$ as before. γ_E, absent in a tokamak, provides the restoring force toward the point $E_r = E_a$ of ambipolarity. If $\mathcal{D}(\omega)$ is ω-independent, then γ_E is as well, and in the time domain Eq.(17) reduces to a standard Langevin equation for E,

$$\partial_t E(t) + \gamma_E E(t) = c_S(t). \tag{18}$$

The source c_S that drives the ZFs is approximated as random. Thus, ensemble averaging (18), one has

$$\partial_t\langle E\rangle_p = -\gamma_E\langle E\rangle_p, \tag{19}$$

where $\langle ...\rangle_p \equiv \int dE ... p(E,t)$ is the ensemble average with probability distribution function (pdf) $p(E,t)$. It satisfies

$$\partial_t p = \partial_E(D_E\partial_E p + \gamma_E E p), \tag{20}$$

with $D_E \equiv \int_0^\infty d\tau\langle c_S(t)c_S(t-\tau)\rangle_p$ the diffusion coefficient in E-space. From this follows Eq.(19), and

$$\partial_t \frac{1}{2}\langle E^2\rangle_p = D_E - \gamma_E\langle E^2\rangle_p. \tag{21}$$

One notes from this the balance between diffusion and the restoration toward $E_r = E_a$. In steady-state, Eqs.(20) and (21) yield $p(E) = p_0\exp(-\gamma_E E^2/2D_E)$, and $\langle E^2\rangle_p = D_E/\gamma_E$. Since $\gamma_E \sim \mathcal{D}^{-1}$ and $D_E \sim \mathcal{D}^{-2}$, one has $\langle E^2\rangle_p \sim \mathcal{D}^{-1}$. Thus, assuming the turbulent forces F_S driving the ZFs are unaffected, the larger $\mathcal{D}$ implied at low-ω by the drift-polarization shielding would reduce γ_E, but reduce the diffusion D_E even more, resulting in a smaller ZF amplitude $\langle E^2\rangle_p^{1/2}$.

Discussion

In the tokamak limit $\sigma, \gamma_E \rightarrow 0$, Eq.(18) or (21) predicts an unbounded diffusion. Then other restoring mechanisms, such as those given in the model Eq.(19) of Ref. [5], become important, and would provide an analogous bounded statistical evolution of E_r, though for that model equation the time-average value of E_r would shift from the stellarator value E_a to 0. The more robust ambipolar field E_a in a stellarator provides a turbulence-suppression mechanism additional to, enhancing or diminishing that of, the ZFs themselves. For example, internal transport barriers induced by jumps in $E_a(r)$ from the ion to the electron root[21, 22, 23] (which enhances the shear in E_r and resultant flow-shear) have been observed on W7AS[24], LHD[25], and on CHS[26]. Better understanding how the ambipolar and ZF-induced flow shear act together presents an additional issue, and potential opportunity, for stellarators.

The theory from Ref. [1], as for earlier work[5, 6, 7, 8] which it extends, treats one leg of the self-consistent ZF loop, the time evolution of zonal flows, given a specified turbulent source S. The relationship between the transport coefficients D^j and shielding function contributions g^j established by that theory indicates that neoclassically-optimized stellarators should have less ZF damping, and thus supports the earlier view[11, 6, 12, 10] that neoclassically-optimized stellarators should also have lower turbulent transport. However, as noted here, the assumption that S remains fixed from one configuration to another is unwarranted, and further study of the dependence of S on configuration is clearly indicated. In fact, the recent results[10, 20] from LHD simulations indicate that this widely-held view is too simplistic, and that a fuller perspective will include additional variables beyond only the degree of a configuration's neoclassical optimization.

Understanding the configuration-dependence of S is of course complicated, possibly the reason most analytic work has focussed mainly on the first leg of the ZF loop. Theoretical progress here can be aided greatly by analysis of further numerical simulations designed to elucidate this relationship.

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