Theory of Stellarators and Tokamaks in Three Dimensions

Paul R. Garabedian

Courant Institute of Mathematical Sciences, New York University
New York, New York 10012 USA

The NSTAB computer code applies the MHD variational principle to calculate equilibrium and stability of toroidal plasmas in three dimensions. Differential equations are solved in a conservation form that describes force balance correctly across islands that are treated as discontinuities. The method has been applied to stellarators, including LHD, and tokamak configurations such as DIII-D and ITER. Sometimes the solution of the equations turns out not to be unique, and there may exist bifurcated equilibria that are nonlinearly stable when theory predicts linear instability. With appropriate pressure profiles, the LHD plasma is found to be stable at $\beta = 0.045$, which correlates with recent experiments. Hence reactor values of pressure can be confined stably in a stellarator configuration with robust flux surfaces. A similar analysis shows that tokamak configurations like ITER do not remain axially symmetric at finite β, for they develop helical islands, which may manifest themselves experimentally as neoclassical tearing modes (NTMs) and edge localized modes (ELMs). These results motivate a continuing search for improved stellarator configurations which (a) employ relatively simple coils to generate good flux surfaces even at low aspect ratios typical of a tokamak; (b) retain the favorable high β equilibrium and stability characteristics demonstrated in LHD; and (c) promise reduced transport by virtue of quasiaxial symmetry.

1. Computational Science of Magnetic Fusion

For a long time it has been recognized that fusion of hydrogen to form helium is what powers the sun and the stars. Now a community of scientists believes that a similar process releasing energetic neutrons can be exploited to construct nuclear reactors that might eliminate global warming by fossil fuels. A very hot plasma of deuterium and tritium is to be confined in toroidal geometry by a strong magnetic field that separates the ionized gas from material walls. The most successful fusion experiments have been axially symmetric tokamaks with net toroidal current induced in the plasma itself to produce a poloidal field required for equilibrium. The stellarator is a more stable configuration with the external field generated by a system of complicated coils in three dimensions. Modern methods of computational science enable physicists to perform parameter studies necessary to design practical configurations for experiments to establish the feasibility of this concept of magnetic fusion.

Three-dimensional computer codes have been written to study equilibrium, stability and transport of plasmas in toroidal geometry for magnetic fusion. Advanced numerical methods have been employed to solve systems of differential equations that model these configurations. Despite the simplicity of some of the models, good correlation has been obtained with observations from experiments. We shall discuss comparisons with the Large Helical Device built and operated in Japan and the International Thermonuclear Experimental Reactor planned for construction in France by a consortium of countries. Afterwards we shall propose an alternate device, the Modular Helias-like Heliac 2, a stellarator that seems to perform better according to the predictions of a mathematical theory which we have developed.

2. Equations

The Maxwell stress tensor enables one to put the differential equations describing force balance in the conservation form

$$\nabla \cdot \Big[\mathbf{BB} - (B^2/2 + p)\mathbf{I}\Big] = 0, \nabla \cdot \mathbf{B} = 0.$$

Comparable finite difference equations, summed over a test volume, telescope down to a correct statement of force balance over the boundary. This way we capture discontinuities in solutions

$$\mathbf{B} = \nabla s \times \nabla \theta = \nabla \phi + \zeta \nabla s$$

of the MHD variational principle

$$\delta \int\int\int (B^2/2 - p(s))\, dV = \ 0$$

for a plasma with separatrix defined by the Fourier series

$$r + iz \ = \ e^{iu} \sum \Delta_{mn}\, e^{-imu+inv} \ .$$

Islands occur where the spectrum B_{mn} defined by

$$\frac{1}{B^2} = \sum B_{mn}(s) \cos\left[m\theta - (n - \iota m)\phi\right]$$

activates small denominators of the parallel current

$$\frac{\mathbf{J}\cdot\mathbf{B}}{p' B^2} = \sum \frac{m B_{mn}(s)}{n - \iota m} \cos\left[m\theta - (n - \iota m)\phi\right].$$

As an example, consider the RFP model problem

$$(\Psi_x^2)_x \ = \ \eta \Psi_{xxx} \quad , \quad \Psi \ = \ 1 - |x| \ .$$

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Figures

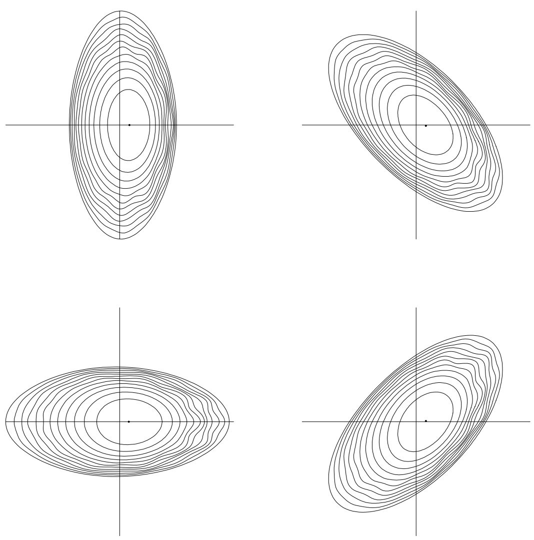


Figure 1: Poincaré sections of the flux surfaces of a bifurcated, nonlinearly stable LHD equilibrium at $\beta = 0.048$. The ripple in the flux surfaces on the right suggests that ballooning modes appear in the solution, but a reliable prediction of the β limit cannot be made without more information about the experiment. The existence of several solutions of the ideal MHD equilibrium equations is considered to be evidence of linear instability.

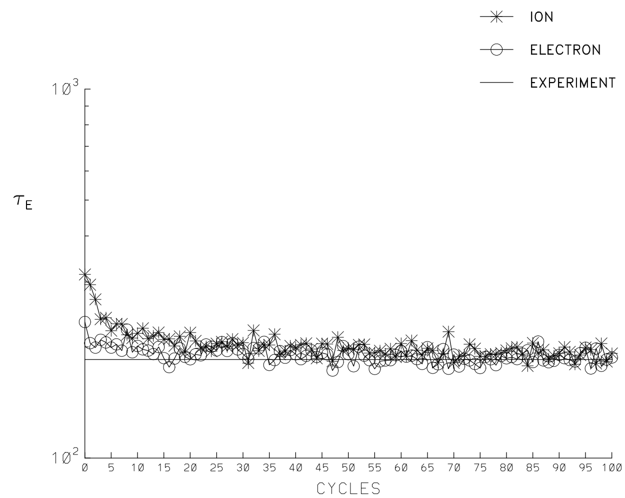


Figure 2: Cycles of a calculation of the energy confinement time τ_E in milliseconds for an NBI shot of the LHD experiment using a quasineutrality algorithm to adjust the electric potential Φ. Oscillations of Φ along the magnetic lines model anomalous transport, so there is good agreement with the observed value. Results from the LHD experiment have served to validate the numerical simulations of equilibrium, stability and transport that are provided by runs of the NSTAB and TRAN computer codes. The TRAN code models thermal transport by performing a random walk among complicated orbits of the ions or electrons, which are found from Runge-Kutta solutions of a system of ordinary differential equations for guiding centers. The results depend primarily on the magnetic spectrum of the plasma, which is obtained from runs of the NSTAB code.

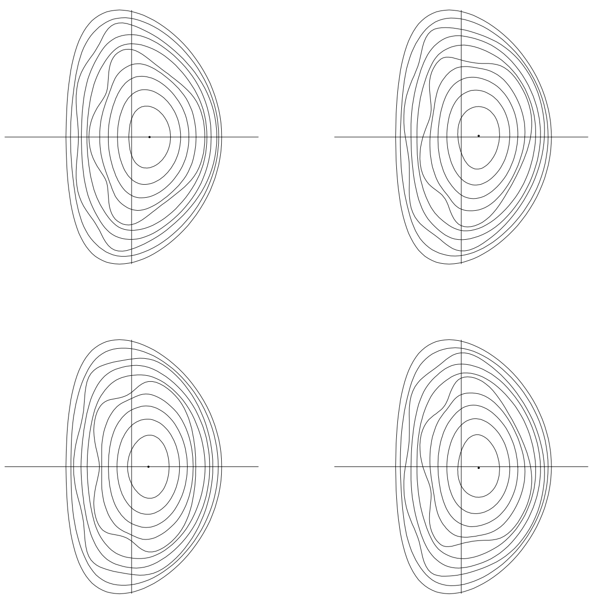


Figure 3: Four Poincaré sections of the magnetic surfaces over one out of three artificial field periods of a bifurcated, nonlinearly stable ITER equilibrium at $\beta = 0.03$ with $0.9 > \iota > 0.4$. The ripple in the flux surfaces suggests that there may be NTMs and ELMs in this three-dimensional solution of an axially symmetric problem. Extensive studies of numerical examples produce ample evidence that bifurcated equilibria can be expected to appear in most tokamak problems, so provision should be made for that. Convergent runs of the NSTAB code can capture islands whose widths are smaller than the radial mesh size and compute bifurcated equilibria in examples where there is marginal linear instability, but global nonlinear stability, of the ideal MHD model.

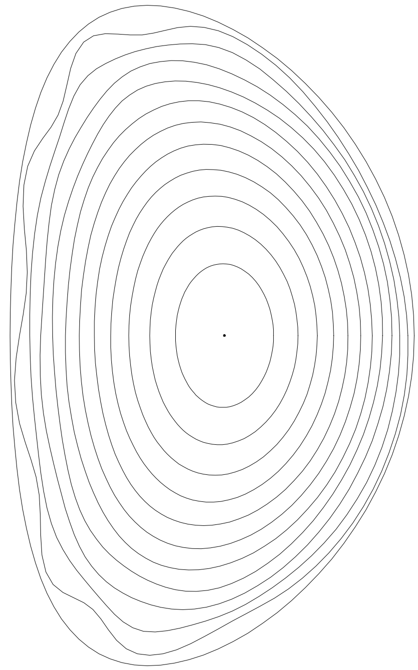
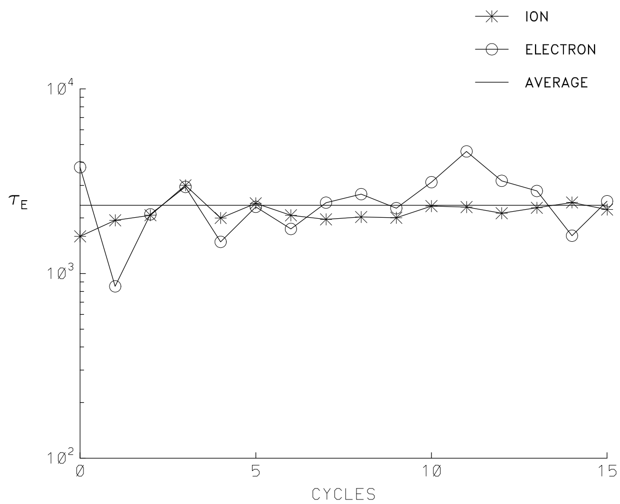


Figure 4: Poincaré section of the flux surfaces of a bifurcated ITER equilibrium at $\beta = 0.027$ with net current bringing the rotational transform into the interval $0.93 > \iota > 0.37$. There are helical islands in this three-dimensional solution of an axially symmetric MHD problem, which may model ELMs. A forcing term was used early in the run to trigger a mode that appears as a discontinuity in the three-dimensional solution. This double precision calculation converges to the level of round off error, showing that the discrete problem has been solved. Islands are captured numerically by finite difference equations in a conservation form that works despite the nested surface hypothesis present in our mathematical formulation of the problem. The results are plausible on the long time scale of a magnetic fusion reactor.



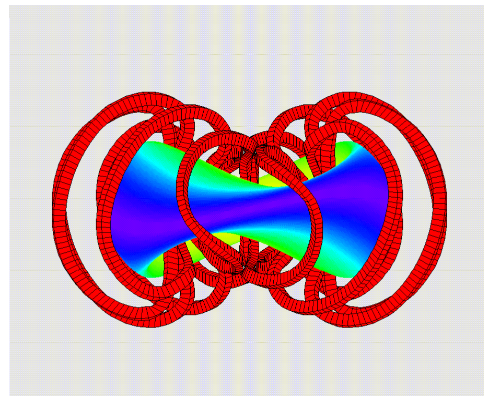
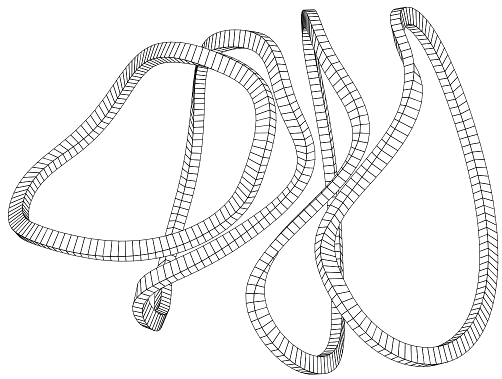


Figure 5: Cycles of a calculation of the energy confinement time τ_E in milliseconds for the ITER tokamak using a quasineutrality algorithm to adjust the electric potential Φ. The three-dimensional effect of ripple associated with a system of just twelve toroidal coils has been introduced to drive the radial electric field and cause the plasma to spin. It is not clear that two-dimensional models describe transport in tokamaks adequately. The energy confinement time is calculated from an empirical relationship with the particle confinement time. Without three-dimensional terms we have not been able to reconcile discrepancies between the ion and the electron confinement times that are computed by the Monte Carlo method in tokamaks.

Figure 6: In magnetic fusion, hot deuterium and tritium ions are combined to form helium and release neutrons intended to provide a commercial source of energy. The color map of the hydrogen plasma shown in the figure displays a symmetry property that enhances confinement. Twelve only moderately twisted coils generate a magnetic field designed to keep the plasma in stable equilibrium separated from material walls. The side view of this compact stellarator shows that there is ample space between the coils for NBI heating. Our hope is that a judiciously optimized QAS stellarator may overcome the troubles with poor transport and low ion temperature in conventional stellarators, and with poor stability and ELMs crashes in tokamaks.



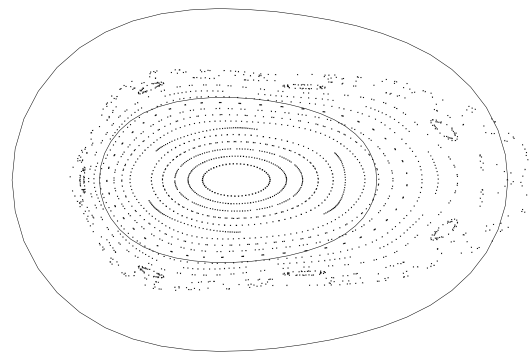


Figure 7: Four out of twelve modular coils of the MHH2 stellarator in a vacuum magnetic field given by the Biot-Savart law. The coils at the sides of the diagram are located at corners over the full torus, so the distances between all the coils can be estimated from the figure. Judicious filtering of the Fourier series used to calculate filaments specifying the geometry of the configuration defines shapes that are not excessively twisted. Parameters have been adjusted to provide ample space around each coil, and the aspect ratio of the plasma is 2.5. The spacing is adequate for superconducting coils in a reactor with major radius 7 m or 8 m, but there is difficulty fitting the coils together inside a small experiment at high magnetic field. Through rigid motions, four copies of the quadrant of coils shown in the plot can be combined to give an accurate picture over the full torus.

Figure 8: Cross section of a line tracing calculation for a QAS stellarator displaying curves that define the control surface for the coils and the shape of the plasma, together with magnetic lines computed at $\beta = 0$. The flux surfaces and islands outside the separatrix show that the magnetic field is well organized for a divertor. This works because smooth coils were found to provide the external field confining the plasma. It is essential to choose the coils both so that the stellarator can be constructed without too much difficulty and so that the flux surfaces remain robust when realistic changes are made in physical parameters. The formulas we apply to represent the plasma surface and the shape of the coils are motivated by a knowledge of conformal mapping and Runge's theorem in the theory of analytic functions of a complex variable.