Geodesic acoustic modes in multi-ion system

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The eigenmodes of geodesic acoustic modes (GAMs) in the multi-ion sysytem are investigated. The high-frequency branch of the eigenfrequency decreases due to the increase of effective ion mass. The low-frequency branch (ion sound wave (ISW)) of the damping rate bemoces small. The ratio between the damping rate of GAM and ISW is found to become order unity at $q \sim 3$ (q is the safety factor).

Keywords: geodesic acoustic modes, ion sound wave, multi-ion system, impurity, eigenmode

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1 Introduction

Recently, the self-organizing meso-scale structure has attracted much attention. In particular, zonal flows (ZFs) is thought to reduce the anomalous transport driven by turbulence [1]. In toroidal plasma, geodesic acoustic modes (GAMs) exist [2], which are oscillatory modes. The study of GAMs is also essential for plasma research, because GAMs and ZFs share the energy from turbulence. The radial eigenmode of GAM is investigated [3, 4, 5]. In addision, the new diagnostic method, GAM spectroscopy is proposed [6]. This method enables to detect the effective ion mass, which is equal to abundance ratio of ions by detecting radial eigenmode of GAM. It is relevant to investigate whether the method of GAM spectroscopy provides information of impurities. Therefore, the GAMs eigenmodes in multi-ion system is very important.

In this study, the GAMs eigenfrequencies in the collisionless plasma with multiple ion species is investigated. The plasma is assumed to have circular cross section and high aspect ratio. The analytical expression for eigenfrequency of GAM is derived. In addition, the lower-frequency branch, which is the ion sound wave (ISW), is also analyzed. As a result, GAM eigenfrequency is found to become small due to the increase of the effective ion mass. The damping rate of GAM eigenmode becomes small around $q \sim 3$ (q is the safety factor). The analytical expression of low-frequency branch is obtained, which is the extension of [7]. The damping rate of ISW is found to become small in multi-ion system to such a degree that the ratio between the damping rate of ISW and GAM becomes order of unity. This result indicates that the ISW can be detected experimentaly, in the case that the energy is injected from the turbulence to this branch as well as GAM. The general formula of this study is explained in section2, GAM eigenmode is in section3, ISW eigenmode is in section4, and summary is given in section5.

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2 General formula

The model is explained in this section. The magnetic field is assumed to be written as

$$\boldsymbol{B} = \frac{B_0}{1+\epsilon\cos\theta}\left(\boldsymbol{e}_\zeta + \frac{\epsilon}{q}\boldsymbol{e}_\theta\right), \tag{1}$$

where $\boldsymbol{e}_\theta,\ \boldsymbol{e}_\zeta$ are poloidal and toroidal directions respectively. $\epsilon,\ q$ are the inverse aspect ratio and the safety factor. The basic equation, Gyrokinetic equation and quasi-neutral condition can be wriiten as

$$\begin{aligned}\left\{\frac{\partial}{\partial t} \right. &+ \left. v_\parallel \boldsymbol{b}\cdot\nabla + i\boldsymbol{k}_\perp\cdot\boldsymbol{v}_{dr}\right\}\delta f_{k_\perp}^{(j)} = -\left\{v_\parallel \boldsymbol{b}\cdot\nabla \right.\\ &+ \left. i\boldsymbol{k}_\perp\cdot\boldsymbol{v}_{dr}\right\}\left(F_0^{(j)} J_0(k_\perp\rho)\frac{e\phi_{k_\perp}}{T}\right), \end{aligned}\tag{2}$$

$$\begin{aligned}\sum_j Z_j \int dv^3 \delta f_{k_\perp}^{(j)} - \sum_j Z_j n_j \left(1-\Gamma_0(k^2\rho_j^2/2)\right) \\ \times \frac{e\phi_{k_\perp}}{T_i} = \frac{n_e}{T_e}\left(\phi_{k_\perp} - <\phi_{k_\perp}>\right), \end{aligned}\tag{3}$$

where $\delta f_{k_\perp}^{(j)},\ \phi_{k_\perp}$ are the response of distribution of jth ion and ZF potential, resectively. Z_j, n_j, n_e, T_i, T_e are jth ion charge, density, ion temperature, electron density, and electron temperature. The blaket $<>$ represents the average over the magnetic surface. $F_0^{(j)}$ is Maxwell distribution, which is written as $F_0^{(j)} = n_j/\pi^{3/2}\exp\left(-\hat{v}_j^2\right)$. $\hat{v}_j$ is the velocity normalized by jth ion thermal velocity. The thermal velocity can be written as $\hat{v}_j = \sqrt{T_i/m_j}$. m_j is jth ion mass. Here, $\delta f_{k_\perp}^{(j)},\ \phi_{k_\perp}$ are expanded by Foureir series as

$$\delta f_{k_\perp}^{(j)} = \sum_{m=-\infty}^{\infty} e^{im\theta - i\omega t}\delta f_m^{(j)}(\omega) \tag{4}$$

$$\phi_{k_\perp} = \sum_{m=-\infty}^{\infty} e^{im\theta - i\omega t}\hat{\phi}_m(\omega). \tag{5}$$

Using Eqs. (2), (3), the response of ion to ZFs potential is obtained as [8]

$$i\hat{\omega}\delta f_0^{(j)} = \frac{1}{2}J_0\left(s\frac{\hat{v}_{\perp j}}{q}\right)k\hat{u}_j(\delta f_{-1}^{(j)} - \delta f_1^{(j)}$$
$$+ \quad \hat{\phi}_{-1} - \hat{\phi}_1), \tag{6}$$
$$\delta f_m^{(j)} = \sum_{l,l'} J_0(s\hat{v}_{\perp j}/q)F_0\frac{\hat{v_{\parallel j}}(m-l)}{\omega - \hat{v_{\parallel j}}(m-l)}i^{l'-l}$$
$$J_l(k\delta_1)J_{l'}(k\delta_1)\phi_{m-l-l'}., \tag{7}$$

where s_j is the finite orbit width, which is defined by $s_j = kv_jq/\Omega_j$. $\delta_j = -s_j\left(\hat{v}_{\parallel j} + \hat{v}_{\perp j}^2/2\hat{v}_{\parallel j}\right)$ represents the Doppler shift due to toroidal effect. $\hat{\omega}_j = \omega R_0 q/v_j$ is the normalized frequency. Here, the poloidal harmonics are truncated, and the poloidal modes $m = 0, \pm 1$ are kept. The ion response can be written as

$$i\hat{\omega}\delta f_0^{(j)} = n_j\left(C_{00}^{(j)}\hat{\phi}_0 + C_{01}^{(j)}\hat{\phi}_1\right) \tag{8}$$
$$\delta f_1^{(j)} = n_j\left(C_{10}^{(j)}\hat{\phi}_0 + C_{11}^{(j)}\hat{\phi}_1\right). \tag{9}$$

C_{ij} is the coefficient which is the function of velocity. Combining Eqs. (8), (9) with the quasi-neutral condition Eq. (3), the dispersion relation is given by

$$\begin{aligned}
\Delta \quad = \quad & \left\{\sum_j \frac{Z_jn_j}{n_e}i\frac{s_j^2}{2}\left[\frac{\hat{\omega}_j}{q^2} + \frac{1}{2}\right.\right. \\
\times \quad & \left.\left.\left\{\frac{3}{2}\hat{\omega}_j + \hat{\omega}_j^3 + \left(\frac{1}{2} + \hat{\omega}_j^2 + \hat{\omega}_j^4\right)Z(\hat{\omega}_j)\right\}\right]\right\} \\
\times \quad & \left\{\frac{1}{\tau_e} - \sum_j \frac{Z_jn_j}{n_e}\left(1 + \hat{\omega}_jZ(\hat{\omega}_j)\right)\right\} \\
- \quad & \sum_j \frac{Z_jn_j}{n_e}s_j\left\{\hat{\omega}_j^2 + \left(\frac{1}{2}\hat{\omega}_j + \hat{\omega}_j^3\right)Z(\hat{\omega}_j)\right\} \\
\times \quad & \sum_j \frac{Z_jn_j}{n_e}i\frac{s_j}{2}\left\{\hat{\omega}_j + \left(\hat{\omega}_j^2 + \frac{1}{2}\right)Z(\hat{\omega}_j)\right\} \\
= 0. &
\end{aligned} \tag{10}$$

3 GAM eigenmode

The high-frequency branch, standard GAM is the solution derived under the assumption $\hat{\omega} >> 1$. In this case, the plasma dispresion function $Z(\hat{\omega})$ can be expanded as

$$Z(\hat{\omega}) \approx -\left(\frac{1}{\hat{\omega}} + \frac{1}{2\hat{\omega}^3} + \frac{3}{4\hat{\omega}^5}\right) + i\sqrt{\pi}e^{-\hat{\omega}^2}. \tag{11}$$

The dispersion relation Eq.(10) can be expanded explicitly as

$$\begin{aligned}
\Delta \quad = \quad & A\hat{\omega} - (B + \delta B)\frac{1}{\hat{\omega}} - \frac{C}{\hat{\omega}^3} \\
+ \quad & i\left(De^{-\hat{\omega}^2} + Ee^{-\hat{\omega}^2/4}\right),
\end{aligned} \tag{12}$$

where $A,\ B,\ \delta B,\ C,\ D,\ E$ can be written as

$$A = \frac{1}{2q^2\tau_e}\sum_j \frac{Z_jn_j}{n_e}\left(\frac{Z_{main}}{Z_j\zeta_j}\right)^2\frac{1}{\zeta_j} \tag{13}$$

$$\begin{aligned}
B \quad & = \frac{7}{8\tau_e}\sum_j \frac{Z_jn_j}{n_e}\left(\frac{Z_{main}}{Z_j\zeta_j}\right)^2\zeta_j \\
+ \quad & \frac{1}{2}\sum_{j,j'} \frac{Z_jn_j}{n_e}\frac{Z_{main}}{Z_j\zeta_j}\frac{Z_j'n_j'}{n_e}\frac{Z_{main}}{Z_j'\zeta_j'}\zeta_j'
\end{aligned} \tag{14}$$

$$\begin{aligned}
\delta B \quad & = \frac{1}{4q^2}\sum_{j,j'} \frac{Z_jn_j}{n_e}\left(\frac{Z_{main}}{Z_j\zeta_j}\right)^2\frac{1}{\zeta_j} \\
\times \quad & \frac{Z_j'n_j'}{n_e}\frac{Z_{main}}{Z_j'\zeta_j'}\zeta_j'^2
\end{aligned} \tag{15}$$

$$\begin{aligned}
C \quad & = \frac{23}{16\tau_e}\sum_j \frac{Z_jn_j}{n_e}\left(\frac{Z_{main}}{Z_j\zeta_j}\right)^2\zeta_j^3 \\
- \quad & \frac{7}{16}\sum_{j,j'} \frac{Z_jn_j}{n_e}\left(\frac{Z_{main}}{Z_j\zeta_j}\right)^2\zeta\frac{Z_j'n_j'}{n_e}\zeta_j'^2 \\
+ \quad & \frac{1}{2}\sum_{j,j'} \frac{Z_jn_j}{n_e}\frac{Z_{main}}{Z_j\zeta_j}\frac{Z_j'n_j'}{n_e}\frac{Z_{main}}{Z_j'\zeta_j'} \\
\times \quad & \left(\zeta_j'^3 + \zeta_j^2\zeta_j'\right)
\end{aligned} \tag{16}$$

$$\begin{aligned}
D \quad & = \sqrt{\pi}\frac{Z_{main}n_{main}}{n_e}\frac{1}{2\tau_e}\hat{\omega}^4 + \left\{\frac{1}{2} - \frac{\tau_e}{4}\right. \\
& \sum_j \frac{Z_jn_j}{n_e}\zeta_j^2 + \frac{\tau_e}{2}\frac{Z_jn_j}{n_e}\frac{Z_{main}}{Z_j\zeta_j} \\
& \left. + \frac{\tau_e}{2}\frac{Z_jn_j}{n_e}\frac{Z_{main}}{Z_j}\right\}\hat{\omega_G}^2
\end{aligned} \tag{17}$$

$$E = \sqrt{\pi}\frac{Z_{main}n_{main}}{n_e}\frac{1}{2\tau_e}\frac{\hat{\omega}^6}{1024} \tag{18}$$

$$\zeta_j = \sqrt{\frac{m_{main}}{m_j}\tau_j}, \tag{19}$$

where ζ_j is the thermal velocity ratio, τ_j is the j ion temperature normalized by main ion temperature $\tau_j = T_j/T_{main}$. $m_{main}, Z_{main}, n_{main}$ are the mass, the charge and the density of main ion. The solution of this dispersion relation can be obtained as

$$\omega_G = \frac{v_T}{R_0q}\frac{B}{A}\left\{1 + \left(\frac{B\delta B}{A^2} + \frac{C}{A}\right)\frac{B}{A}\right\} \tag{20}$$

$$\begin{aligned}
\gamma_G \quad \approx \quad & -\sqrt{\pi}\frac{Z_{main}n_{main}}{n_e}q\frac{v_T}{R_0}\left\{\sum_j \frac{Z_jn_j}{n_e}\left(\frac{Z_{main}}{Z_j\zeta_j}\right)^2\right. \\
\times \quad & \left.\frac{1}{\zeta_j}\right\}^{-1}\left[\left\{\frac{\hat{\omega_G}^4}{2} + \left\{\frac{1}{2} - \frac{\tau_e}{4}\sum_j \frac{Z_jn_j}{n_e}\zeta_j^2\right.\right.\right. \\
+ \quad & \left.\frac{\tau_e}{2}\frac{Z_jn_j}{n_e}\frac{Z_{main}}{Z_j\zeta_j} + \frac{\tau_e}{2}\frac{Z_jn_j}{n_e}\frac{Z_{main}}{Z_j}\right\} \\
\times \quad & \left.\left.\hat{\omega_G}^2\right\}e^{-\hat{\omega_G}^2} + s^2\frac{\hat{\omega_G}^6}{1024}e^{-\hat{\omega_G}^2/4}\right].
\end{aligned} \tag{21}$$

The real eigenfrequency ω_G agrees with the result of [8, 9] in the limit of single-ion $\sum_j Z_j n_j \rightarrow n_{main}$. In the limit of $\sum_j Z_j n_j \rightarrow Z_1 n_1$, which is impurity limit, ω_G becomes

$$\omega_G \rightarrow \zeta\omega_{G0}\left(1 + \frac{\zeta}{Z_1}\frac{\alpha}{q^2}\right), \tag{22}$$

where $\omega_{G0} = v_T/R_0\sqrt{7/4+\tau_e}, \alpha = (23 + 16\tau_e + 4\tau_e^2)/(7+4\tau_e)^2$. Approximately, the frequency of leading order becomes ζ times small as compared with that for hydrogen plasma. Higher order term, which is $1/q^2$ order term becomes ζ/Z_1 times small. In short, the effect of mass on the real eigenfrequency is more significant than the the effect of charge. The result of the real eigenfrequency in two-ion system is shown in Fig. 1. The behavior of the damping rate is shown in Fig. 2 in the case of $Z_{eff} = 4$. The damping rate in multi-ion system is found to be larger. This is because the effect of the Landau damping becomes large due to the decrease of $\hat{\omega}_G$. Especially, the dampng rate becomes smaller around $q \sim 3,\ 4$, because the $m = \pm 2$ coupling becomes effective.

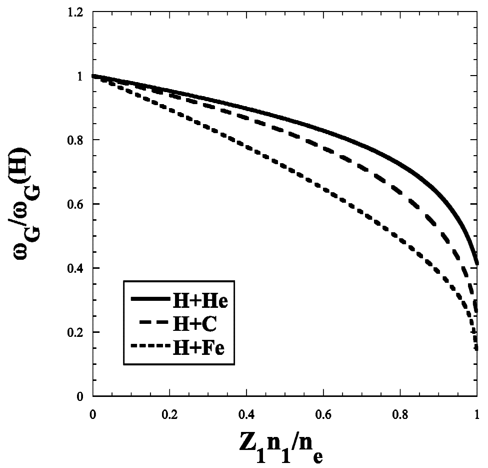

Fig. 1 The multi-ion effect on the GAM real frequency.

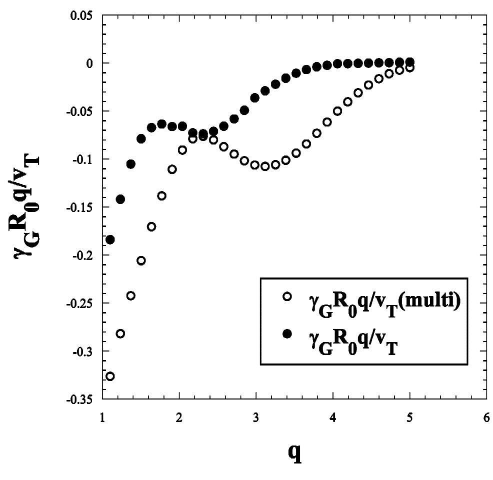

Fig. 2 The multi-ion effect on the GAM damping rate. Ions consist of hydrogen and carbon. $Z_eff = 4$.

4 ISW eigenmode

The low-frequency branch, ISW, has the real frequency which has a resonant frequency with heavier ions. The solution is derived under the assumption $\hat{\omega} << 1$. In the case of single-ion system, the plasma dispersion function $Z(\hat{\omega})$ can be expanded as

$$Z(\hat{\omega}) \approx -2\hat{\omega} + \frac{4}{3}\hat{\omega}^3 - \frac{8}{15}\hat{\omega}^5 + i\sqrt{\pi}e^{-\hat{\omega}^2}. \tag{23}$$

However, in multi-ion system, the argument of Z-function is $\hat{\omega}/\zeta_j$. The coefficient ζ_j is less than unity. Therefore, it is difficult to estimate the analytical representation of eigenfrequency in multi-ion system. Here, the eigenmode in single-ion system is investigated. The dispersion relation is written as

$$\Delta = a\hat{\omega} + b\hat{\omega}^3 + i\left(c + d\hat{\omega}^2\right) \tag{24}$$

Coefficients a to d can be written as

$$a = \left(\frac{1}{2} + \frac{1}{q^2}\right)\left(\frac{1}{\tau_e} + 1\right) - \frac{\pi}{4} \tag{25}$$

$$b = \frac{-2}{q^2} - \frac{1}{3\tau_e} + \frac{\pi}{2} \tag{26}$$

$$c = \frac{\sqrt{\pi}}{2\tau_e} \tag{27}$$

$$d = \sqrt{\pi}\left(\frac{1}{q^2} + \frac{1}{\tau_e} - \frac{1}{2}\right). \tag{28}$$

The solution of $\Delta = 0$ is obtained by following procedure. In this case, due to $\omega_{isw} \sim \gamma_{isw}$, ω is replaced by $\omega = \omega_r + i\gamma$. Eq.(24) can be expressed as

$$\begin{aligned}\Delta &= \hat{\omega}_r\left\{a + b\left(\hat{\omega}_r^2 - 3\hat{\omega}_r\hat{\gamma}^2\right) - 2d\hat{\gamma}\right\} + i\left\{a\hat{\gamma}\right.\\ &+ \left. b\left(3\hat{\omega}_r^2\hat{\gamma} - \hat{\gamma}^3\right) + c + d(\hat{\omega}_r^2 - \hat{\gamma}^2)\right\}\end{aligned} \tag{29}$$

The obvious solution for $\omega_r = 0$ is

$$\omega \sim 0 \tag{30}$$

$$\gamma = \frac{1}{2}\frac{v_T}{R_0 q}\left(\frac{a}{d} - \sqrt{\frac{a^2}{d^2} + \frac{4c}{d}}\right). \tag{31}$$

This solution is the extension of [7] for finite τ_e. The solution for the dispersion relation Eq.(24) whose frequency is finite (ISW) is difficult to obtain in an analytic expansion to this order. To get the analytical expression, it needs to be expanded in higher order terms. The solution of ISW is obtained by Eq.(10) numerically. The compasion between the eigenfrequency of multi-ion system and single-ion system is shown in Fig.3. The damping rate in multi-ion system becomes much smaller than that in single-ion system. This attributes the fact that the Landau damping becomes smaller due to the decrease of real frequency. Discontinuity in the calculated damping rate arises the fact that we chose the minimum damping rate solution among many solutions. The ratio of damping rates of ISW and GAM is found to become the order of unity. This result indicates that ISW can be observed experimentally.

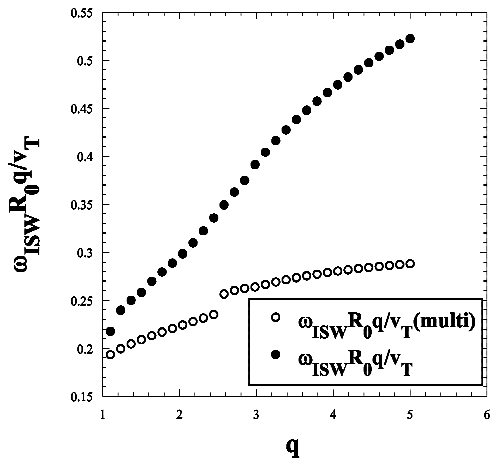

Fig. 3 The comparison between the real frequency of ISW in multi-ion (hydrogen and carbon. $Z_eff = 4$) and single-ion.

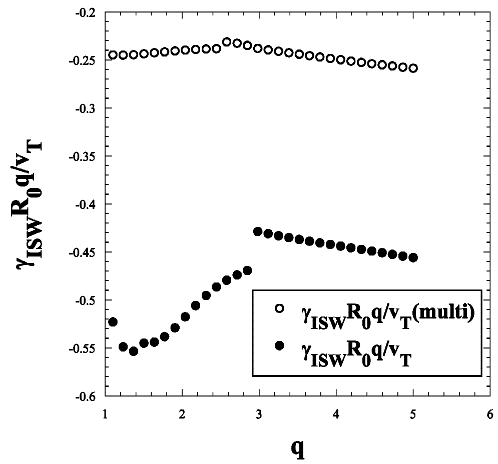

Fig. 4 The comparison between the damping rate of ISW in multi-ion and single-ion.

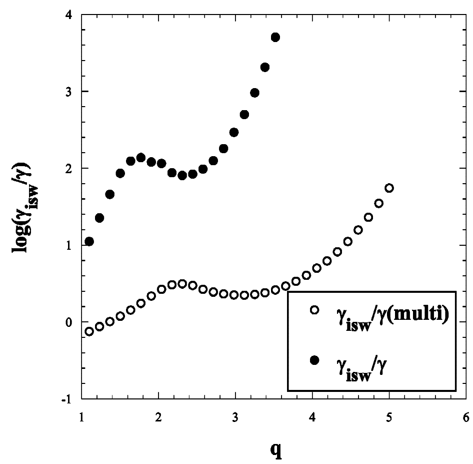

Fig. 5 The ratio between the damping rate of ISW and GAM.

5 Summary

The GAMs eigenmode in multi-ion system is investigated in the collisionless limit. The plasma is assumed to have circular cross section, and high aspect ratio. The high-frequency branch, standard GAM, is analyzed. The analytic expression for eigenfrequency is obtained. With the increase in effective ion mass, GAM frequency decreases. The mass dependence is more significant than the charge dependence. These results are essential for GAM spectroscopy. The damping rate of standard GAM becomes smaller than single-ion system. This influences the energy partition between ZFs and GAMs. In addition, the low-frequency branch, ISW, is analyzed. The damping rate in multi-ion system becomes small, and their damping rates become comparable. Thus, if the turbulent energy is injected to this brach, this mode can be observed in experiment.

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