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On rapid rotation in stellarators

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The conditions under which rapid plasma rotation may occur in a three-dimentional magnetic field, such as that of a stellarator, are investigated. Rotation velocities comparable to the ion thermal speed are found to be attainable only in magnetic fields which are approximately isometric. In an isometric magnetic field the dependence of the magnetic field strength B on the arc length l along the field is the same for all field lines on each flux surface ψ. Only in fields where the departure from exact isometry, $B = B(\psi, l)$, is of the order of the ion gyroradius divided by the macroscopic length scale are rotation speeds comparable to the ion thermal speed possible. Moreover, it is shown that the rotation must be in the direction of the vector $\nabla\psi \times \nabla B$.

Keywords: plasma rotation, kinetic MHD, drift kinetic equation

1 Introduction

It is well known in tokamak research that the plasma tends to rotate faster in the toroidal direction than in the poloidal direction, particularly when there is strong neutral-beam injection. It is also well known that magnetic field ripples damp toroidal rotation. Theoretically, it is expected that an axisymmetric plasma should be free to rotate in the toroidal direction, but that the poloidal rotation should be damped. To be more precise, if the gyroradius ρ is small compared with the macroscopic scale length L, so that $\delta = \rho/L \ll 1$, then the toroidal rotation can be comparable to the ion thermal speed $v_T = (2T_i/m_i)^{1/2}$, but the poloidal rotation velocity is of order δv_T. This result follows from the drift-kinetic equation, and will be rederived below. The purpose of the present paper is to clarify under what conditions rapid rotation ($V \sim v_T$) is possible if the magnetic field is not axisymmetric.

We find that rapid rotation can only occur in a certain class of magnetic fields, namely, in fields that are approximately "isometric". The definition of an isometric magnetic field is that the field strength depends on the arc length l along $\mathbf{B}$ in the same way for all field lines on the same flux surface [1]. Thus in an isometric field $B = B(\psi, l)$, where ψ is a flux-surface label. Interestingly, such fields have attracted attention because of their favourable confinement properties. They are an important subclass of "omnigenous" magnetic fields, which are fields where the time-averaged cross-field drift vanishes for all particle orbits [2, 3]. Quasi-axisymmetric [4] and quasi-helically symmetric [5, 6] fields are examples of isometric fields, but isometry is a weaker condition than quasisymmetry. We also find that the the rotation velocity vector must point in the direction $\nabla\psi \times \nabla B$, so that the streamlines coincide with lines of constant magnetic field strength. These contraints follow from the drift kinetic equation in the limit of zero gyroradius (so-called kinetic MHD), and are therefore independent of the cross-field transport collisionality. A somewhat different version of this calculation is being published in Physics of Plasmas (October 2007).

2 Expansion of the kinetic equation

First of all, it is important to choose the correct plasma model from which to proceed. Plasma equilibrium is usually described by ideal MHD, which is however not sufficient for our present purposes. Ideal MHD neglects transport altogether, both within and across flux surfaces, and therefore permits arbitrary toroidal and poloidal rotation as well as temperature variation within flux surfaces. In reality, parallel transport is many orders of magnitude larger than perpendicular transport, which implies that flux surfaces must be approximately isothermal and also constrains plasma rotation. These features are incorporated in so-called *kinetic MHD*, which follows from the terogyroradius limit of the ion kinetic equation

$$\frac{\partial f}{\partial t} + (\mathbf{V} + \mathbf{v}) \cdot \nabla f + \frac{e}{m} \left(\mathbf{E}' + \mathbf{v} \times \mathbf{B} - \frac{\partial \mathbf{V}}{\partial t} \right.$$

$$\left. - (\mathbf{V} + \mathbf{v}) \cdot \nabla \mathbf{V} \right) \cdot \frac{\partial f}{\partial \mathbf{v}} = C(f) + S, \qquad (0)$$

where $\mathbf{v} = \mathbf{u} - \mathbf{V}(\mathbf{r}, t)$ is the velocity vector measured relative to velocity field $\mathbf{V}$, $\mathbf{E}' = \mathbf{E} + \mathbf{V} \times \mathbf{B}$ the electric field in the moving frame, e and m the ion charge and mass, respectively, C the collision operator and

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S represents any sources present in the plasma. Although $\mathbf{V}$ is in principle arbitrary, we shall choose it to be equal to the lowest-order mean ion velocity. As in MHD, the electic field is ordered to be so large that the $E \times B$ velocity is comparable to the thermal speed, $E \sim v_T B$, while the collsion frequency is taken to be comparable to the transit frequency, v_T/L, in order to allow for all conventional collisionality regimes. The dependent variables $f = f_0 + f_1 + \ldots$, $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1 + \ldots$ and, unconventionally, also the magnetic field $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1 + \ldots$ are expanded in the smallness of $\delta = v_T/\Omega L \ll 1$, where $\Omega = eB/m$. In order to study equilibrium (rather than the approach to it), the time derivatives of zeroth-order quantities are assumed to be small, $\partial f_0/\partial t \ll (v_T/L) f_0$, whilst higher-order quantities may vary more rapidly (to allow for turbulence, for instance). The electric field is thus electrostatic in lowest order, $\mathbf{E}_0 = -\nabla\Phi_0$.

The largest terms in Eq. (2) are of order Ωf, and the others of order $\delta\Omega f = v_T f/L$ or smaller. In lowest order, then, our kinetic equation becomes simply

$$\frac{e}{m}\left(\mathbf{E}'_0 + \mathbf{v}\times\mathbf{B}_0\right)\cdot\frac{\partial f_0}{\partial \mathbf{v}} = 0,$$

and can only hold for all $\mathbf{v}$ if

$$(\mathbf{v}\times\mathbf{B_0})\cdot\frac{\partial f_0}{\partial \mathbf{v}} = 0, \tag{1}$$

and $\mathbf{E}'_0 = 0$, so that

$$\mathbf{V}_\perp = \mathbf{V} - \mathbf{V}\cdot\mathbf{bb} = \frac{\mathbf{B_0}\times\nabla\Phi_0}{B_0^2} \tag{2}$$

and $\mathbf{b}\cdot\nabla\Phi_0 = 0$, where $\mathbf{b} = \mathbf{B}_0/B_0$. We shall assume that the magnetic field at least approximately (i.e., in lowest order) possesses flux surfaces, so that $\Phi_0 = \Phi_0(\psi, t)$.

A drift kinetic equation can now be derived in the conventional way [7, 8, 9, 10] by writing

$$\mathbf{v} = v_\parallel \mathbf{b} + v_\perp(\mathbf{e}_1\cos\vartheta + \mathbf{e}_2\sin\vartheta),$$

where $\mathbf{e}_1$, $\mathbf{e}_2$ are unit vectors with $\mathbf{e}_1\times\mathbf{e}_2 = \mathbf{b}$, and averaging over the gyro-angle ϑ. Equation (1) first implies that f_0 is independent of gyro-angle, $\partial f_0/\partial\vartheta = 0$, and Eq. (2) becomes

$$\Omega\frac{\partial f}{\partial\vartheta} = \frac{\partial f}{\partial t} + \Lambda(f) + \frac{e}{m}(\mathbf{v}\times\mathbf{B}_1)\cdot\frac{\partial f}{\partial\mathbf{v}} = C(f) + S(3)$$

where

$$\Lambda(f) = (\mathbf{v}+\mathbf{V})\cdot\left[\nabla f + \nabla\mathbf{b}\cdot\mathbf{v}\left(\frac{\partial f}{\partial v_\parallel} - \frac{v_\parallel}{v_\perp}\frac{\partial f}{\partial v_\perp}\right)\right.$$

$$\left. + (\cos\vartheta\nabla\mathbf{e}_2 - \sin\vartheta\nabla\mathbf{e}_1)\cdot\frac{\mathbf{v}}{v_\perp}\frac{\partial f}{\partial\vartheta}\right]$$

$$+\left[\frac{e\mathbf{E}'_1}{m} - \frac{\partial\mathbf{V}}{\partial t} - (\mathbf{V}+\mathbf{v})\cdot\nabla\mathbf{V}\right]$$

$$\cdot\left(\mathbf{b}\frac{\partial f}{\partial v_\parallel} + \frac{\mathbf{v}_\perp}{v_\perp}\frac{\partial f}{\partial v_\perp} + \frac{\mathbf{b}\times\mathbf{v}}{v_\perp^2}\frac{\partial f}{\partial\vartheta}\right). \tag{4}$$

Taking the ϑ-average gives the desired drift kinetic equation

$$\bar{\Lambda}(f_0) = C(f_0) + S, \tag{5}$$

where

$$\bar{\Lambda}(f_0) = (\mathbf{V} + v_\parallel\mathbf{b})\cdot\nabla f_0 + \frac{e\tilde{E}_\parallel}{m}\frac{\partial f_0}{\partial v_\parallel}$$

$$+\frac{v_\perp^2}{2}(\nabla\cdot\mathbf{b})\left(\frac{\partial f_0}{\partial v_\parallel} - \frac{v_\parallel}{v_\perp}\frac{\partial f_0}{\partial v_\perp}\right)$$

$$-(\mathbf{V} + v_\parallel\mathbf{b})\cdot\nabla\mathbf{V}\cdot\mathbf{b}\frac{\partial f_0}{\partial v_\parallel}$$

$$+\frac{v_\perp}{2}(\mathbf{b}\cdot\nabla\mathbf{V}\cdot\mathbf{b} - \nabla\cdot\mathbf{V})\frac{\partial f_0}{\partial v_\perp},$$

with $\tilde{E}_\parallel = \mathbf{b}\cdot\mathbf{E}'_1 = \mathbf{b}\cdot(\mathbf{E}_1 + \mathbf{V}\times\mathbf{B}_1)$. This equation, and its derivation, agrees with that derived in Ref. [9] except for the addition of the term containing $\mathbf{B}_1$. It looks simpler if one of the independent variables is chosen to be the magnetic moment measured in the moving frame, and then agrees with Refs. [7, 10].

3 Constraints on the flow velocity

In equilibrium, the source term balances transport losses and is therefore also relatively small, usually of order δ or δ^2. The solutions to the resulting equilibrium equation (5),

$$\bar{\Lambda}(f_0) = C(f_0) \tag{6}$$

are found from a familiar H-theorem argument. Multiplying the equation by $\ln f_0$, integrating over velocity space, and taking a flux surfaces average gives

$$\left\langle \int \ln f_0 C(f_0)\, 2\pi v_\perp dv_\perp dv_\parallel \right\rangle = 0, \tag{7}$$

and it follows that f_0 must be a Maxwellian, whose density n and temperature T may vary over each flux surface. Substituting this Maxwellian into Eq. (6) gives the equation $\bar{\Lambda}(f_0) = 0$, or

$$(\mathbf{V} + v_\parallel\mathbf{b})\cdot\left[\nabla\ln n + \left(\frac{mv^2}{2T} - \frac{3}{2}\right)\nabla\ln T\right]$$

$$-\frac{e\tilde{E}_\parallel v_\parallel}{T} + \frac{mv_\parallel}{T}(\mathbf{V} + v_\parallel\mathbf{b})\cdot\nabla\mathbf{V}\cdot\mathbf{b}$$

$$-\frac{mv_\perp^2}{2T}(\mathbf{b}\cdot\nabla\mathbf{V}\cdot\mathbf{b} - \nabla\cdot\mathbf{V}) = 0,$$

which can only be satisfied if the following relations are satisfied [9, 10]:

$$\mathbf{b}\cdot\nabla\ln n - \frac{e\tilde{E}_\parallel}{T} + \frac{m}{T}\mathbf{V}\cdot\nabla\mathbf{V}\cdot\mathbf{b} = 0,$$

$$\mathbf{b}\cdot\nabla T = 0,$$

$$\mathbf{V}\cdot\nabla\left(\ln n - \frac{3}{2}\ln T\right) = 0,$$

$$\nabla\cdot(n\mathbf{V}) = 0,$$

$$\mathbf{b}\cdot\nabla\mathbf{V}\cdot\mathbf{b} - \frac{1}{3}\nabla\cdot\mathbf{V} = 0.$$

These equations imply

$$\mathbf{V}\cdot\nabla\ln n = \nabla\cdot\mathbf{V} = \mathbf{b}\cdot\nabla\mathbf{V}\cdot\mathbf{b} = 0. \tag{8}$$

We now recall Eq. (2) and note that

$$0 = \nabla\times(\mathbf{V}\times\mathbf{B}_0) = \mathbf{B}_0\cdot\nabla\mathbf{V} - \mathbf{V}\cdot\nabla\mathbf{B}_0,$$

which combined with Eq. (8) leads to

$$\mathbf{V}\cdot\nabla\mathbf{B}_0\cdot\mathbf{B}_0 = 0.$$

Since $(\nabla\mathbf{B}_0)\cdot\mathbf{B}_0 = B_0\nabla B_0$ we thus conclude that

$$\mathbf{V}\cdot\nabla B_0 = 0.$$

In other words, the streamlines of the flow are given by the intersection between flux surfaces and surfaces of constant B_0. This means that the velocity field can be written as

$$\mathbf{V}(\mathbf{r}) = g(\mathbf{r})\nabla\psi\times\nabla B_0$$

for some function $g(\mathbf{r})$ of the spatial coordinates $\mathbf{r}$. The parallel component of the flow is thus

$$V_\parallel\mathbf{b} = g(\mathbf{r})\nabla\psi\times\nabla B_0 - \frac{d\Phi_0}{d\psi}\frac{\mathbf{b}\times\nabla\psi}{B_0}.$$

Taking the scalar product of this equation with $\mathbf{b}\times\nabla\psi$ gives an expression for g,

$$g\mathbf{b}\cdot\nabla B_0 + \frac{1}{B_0}\frac{d\Phi_0}{d\psi} = 0,$$

and thus enables us to write down an explicit expression for the lowest-order flow velocity,

$$\mathbf{V} = -\frac{d\Phi_0}{d\psi}\frac{\nabla\psi\times\nabla B_0}{\mathbf{B}_0\cdot\nabla B_0}. \tag{9}$$

If $\mathbf{B}_0$ is written in Clebsch coordinates, $\mathbf{B}_0 = \nabla\psi\times\nabla\alpha$, then $\mathbf{V}$ becomes

$$\mathbf{V} = \frac{\nabla\Phi_0\times\nabla B_0}{(\nabla\psi\times\nabla B_0)\cdot\nabla\alpha}.$$

The requirement (8) that this flow field should be incompressible now implies a constraint on the spatial variation of the magnetic field strength,

$$(\nabla\psi\times\nabla B_0)\cdot\nabla(\mathbf{B}_0\cdot\nabla B_0) = 0. \tag{10}$$

If B_0 is expressed in coordinates (ψ,α,l), where l is the arc length along $\mathbf{B}_0$ then it follows from Eq. (10) that

$$(\nabla\psi\times\nabla B_0)\cdot\nabla\dot{B}_0 = 0,$$

where $\dot{B}_0 = \partial B_0/\partial l$. Hence

$$\frac{\partial B_0}{\partial\alpha}\frac{\partial\dot{B}_0}{\partial l} - \frac{\partial B_0}{\partial l}\frac{\partial\dot{B}_0}{\partial\alpha} = 0, \tag{11}$$

and it follows that $\dot{B}_0$ must be expressible as a function of ψ and B_0, i.e., $\dot{B}_0 = \dot{B}_0(\psi, B_0)$, at least locally. This implies, in turn, that B_0 is isometric. To see this formally, we note that Eq. (11) can be written as

$$\frac{\partial\ln\dot{B}_0}{\partial l} = \frac{\partial}{\partial l}\ln\left(\frac{\partial B_0}{\partial\alpha}\right),$$

and integrated once, to yield

$$\frac{\partial B_0}{\partial l} = F(\psi,\alpha)\frac{\partial B_0}{\partial\alpha},$$

with F an arbitrary function. The general solution is

$$B_0 = B_0(\psi, l'),$$

where $l' = l - l_0(\psi,\alpha)$ is an arc length coordinate whose origin l_0 is related to F by $F\partial l_0/\partial\alpha = -1$. We conclude that rotation at a speed comparable to the thermal speed is only possible if the magnetic field is isometric in lowest order. The converse is also true: the flow field (9) satisfies the conditions (8) if B_0 is isometric, and our theorem can thus be stated in the following way. *The lowest-order drift kinetic equation admits solutions where the mean flow velocity is comparable to the thermal speed if, and only if, the magnetic field is approximately isometric.*

Another way of stating this result is that a sufficiently large radial electric field is only possible if the magnetic field is isometric. "Sufficiently large" in this context refers to fields that are strong enough to produce flow velocities comparable to v_T (sonic rotation), and it is worth noting that this may occur for fields that are in fact much smaller than $E\sim v_T B$ (though formally of this order, in the sense of the gyroradius ordering assumed). The result (9) can be written as

$$\frac{\mathbf{V}}{E/B_0} = \frac{\mathbf{n}\times\nabla B_0}{\mathbf{b}\cdot\nabla B_0}, \tag{12}$$

where $\mathbf{n} = \nabla\psi/|\nabla\psi|$ is the unit vector normal to the flux surfaces and $E = -\mathbf{n}\cdot\nabla\Phi_0$ is the electric field. The point is that the right-hand side of (12) can be relatively large (but not infinite in an isometric field), in which case the parallel component of the velocity (9) is significantly larger than the perpendicular one. In tokamaks, for instance, $|\mathbf{n}\times\nabla B_0|/(\mathbf{b}\cdot\nabla B_0)\sim q/\epsilon\gg 1$, where q is the safety factor and ϵ the inverse aspect ratio. As is well known, sonic rotation thus occurs already for radial electric fields of order $E\sim\epsilon v_T B/q$. In a stellarator, a similar estimate tends to hold approximately, but the details depend of course on the specific magnetic configuration. Importantly, sonic rotation can occur at roughly the same electric field as when the poloidal $E\times B$ drift cancels the poloidal component of $v_\parallel$ for a thermal ion. This "resonance" condition is thought to strongly affect the neoclassical transport [11].

It is interesting to note that Eq. (9) implies a simple expression for the neoclassical polarisation current. When combined with the momentum equation

$$\rho \frac{d\mathbf{V}}{dt} = \mathbf{j} \times \mathbf{B} - \nabla \cdot \mathbf{P},$$

where $\mathbf{P}$ is the pressure tensor, it yields the perpendicular current as

$$\mathbf{j}_\perp - \frac{\mathbf{B} \times (\nabla \cdot \mathbf{P})}{B^2} = \frac{\rho \mathbf{B}}{B^2} \times \frac{d\mathbf{V}}{dt} \simeq -\frac{\rho}{B^2} \frac{\partial \nabla \Phi}{\partial t},$$

where the last, approximate, equality refers to low speeds, $\partial \mathbf{V}/\partial t \gg \mathbf{V} \cdot \nabla \mathbf{V}$.

The result that the eletric field cannot be large unless the magnetic field is isometric suggests a paradox in the low-density limit, since any electric field strength is possible in vacuum. The resolution lies in our ordering of the collision frequency, $\nu_i \sim v_T/L$. This is the standard neoclassical ordering, and is usually followed by a subsidiary ordering where the collsion frequency is taken to be smaller or larger than the transit frequency, but usually not as small as $\nu \sim \delta v_T/L$. At extremely low densities, this latter case must be allowed, in which case the lowest-order drift kinetic equation becomes $\bar{\Lambda}(f_0) = 0$ and does not constrain f_0 to be Maxwellian or B_0 to be isometric.

4 Conclusions

It is well established that plasma rotation tends to have a beneficial influence on plasma confinement. Therefore it is of interest to establish under which conditions rotation is allowed to occur. We have considered this question for general three-dimensional magnetic confinement systems, and found that sonic rotation is only possible in isometric magnetic fields, and can only occur in the direction of constant magnetic field strength. In the special case of a tokamak, plasma rotation must therefore be purely toroidal in lowest order, as is well known both theoretically and experimentally. (Although the poloidal rotation in experiments has been reported to exceed its neoclassical prediction, it is still far smaller than the toroidal rotation [12, 13].)

In stellarators, the radial electric field and rotation velocity are set by the condition of ambipolar cross-field transport, and is usually fairly slow in experiments, $V \ll v_T$. It is often the case that neoclassical transport dominates, and the magnitude and direction of the rotation then depend on the collisionality and heating channel. What we have shown in this paper is that rapid rotation can only occur in isometric magnetic fields and only in the direction $\nabla\psi \times \nabla B$. These constraints are approximate, in the sense that they only need to be satisfied to lowest order in gyroradius, but are independent of the cross-field transport. They therefore hold in all (conventional) collisionality regimes, and also in the presence of gyro-kinetic turbulence.

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