Algebraic analysis approach for multibody problems

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Here we propose an algebraic analysis approach for multibody Coulomb interaction. The momentum transfer cross section using the algebraic approximation is close to the exact one. The CPU time for the algebraic approximation is only around 20 minutes on a PC, while the exact analysis needs 15 hours to integrate the whole set of multibody equations of motion, in which all the field particles are at rest.

Keywords: anomalous diffusion, multibody problems, algebraic analysis

Since it is difficult to rigorously deal with multibody Coulomb collisions, the current classical theory considers them as a series of temporally-isolated binary Coulomb collisions. Let us first breafly review a binary collision between ions. In the center of mass coordinate (r, θ) in the collision plane, the test particle with a reduced mass of μ moves along a hyperbola as

$$\boldsymbol{r}(\theta) = \frac{b \sin\theta_0}{\cos\theta - \cos\theta_0} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \tag{1}$$

with a velocity of

$$\boldsymbol{g}(\theta) = \frac{g_0}{\sin\theta_0} \begin{bmatrix} \cos\theta_0 \sin\theta \\ 1 - \cos\theta_0 \cos\theta \end{bmatrix}, \tag{2}$$

with which the velocity change is given by $\Delta\boldsymbol{g} = 2g_0 \cos\theta_0 \boldsymbol{e}_x$. As shown in Fig. 1 its scattering angle, $\chi \equiv \pi - 2\theta_0$, is given by $b = b_0 \tan\theta_0$, where b is the impact parameter, $b_0 \equiv e^2/4\pi\varepsilon_0\mu g_0^2$ corresponds to $\chi = \pi/2$ scattering, and g_0 the initial relative speed at $r = \infty$ and $\theta = -\theta_0$.

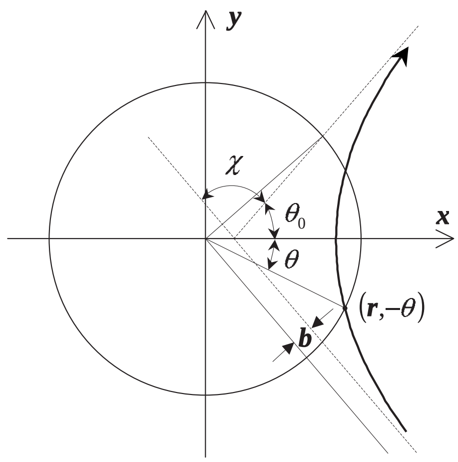

Fig. 1 Unperturbed trajectory $r = r(\theta)$ in an orbital plane. The scattering center is at the origin. An impact parameter is $b = b_0 \tan\theta_0$. Interaction region is inside the circle with a radius $r_\ell = \Delta\ell/2$.

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The angular component of the equation motion gives the well-known invariance of

$$r^2 \frac{\mathrm{d}\theta}{\mathrm{d}t} = \text{const} = b g_0, \tag{3}$$

and the radial component is given by

$$\frac{\mathrm{d}g_r}{\mathrm{d}t} = \frac{g_0^2 b_0}{r^2} \left(1 + \frac{b_0}{r} \tan^2\theta_0\right), \tag{4}$$

where $g_r \equiv \dot{r}$ denotes the radial velocity. The first term on the right hand side of Eq. (4) stands for the Coulomb force, and is much smaller for small angle scatterings, i.e. $\chi \ll 1$, than the second term which results from the conservation of angular momentum Eq.(3), since, at the closest point $r = r_{\min}$, we have

$$\frac{b_0 \tan^2\theta_0}{r_{\min}} \simeq \frac{2}{\chi} \gg 1.$$

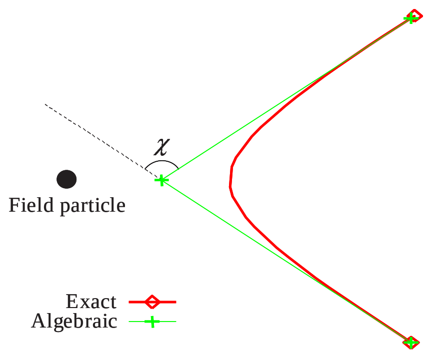

Fig. 2 Algebraic trajectory (broken line) and exact trajectory (curved line). A Field particle is on the left.

Thus the main force on the particle is not a Coulomb force, but the one due to the conservation of angular momentum. As a consequence, the exact hyperbolic trajetory Eq. (1) for the particle can be approximated as a broken line with an impulse force of

$$\mu\Delta\boldsymbol{g} = 2\mu g_0 \cos\theta_0 \boldsymbol{e}_x$$

at the closest point $r = r_{\min}$. With this in mind, we have approximated a multibody problem to a series of binary deflections at their closest point as shown in Fig. 2, in which a test particle starts at the lower-right point, and its final point is at the upper-right point due to the interaction with a field particle.

In the following we assume that all the filed particles are at rest and their spatial distribution is almost uniform with a spacing of the average inter particle separation, $\Delta\ell \equiv n^{-1/3}$, where n stands for the number density as shown in Fig. 3.

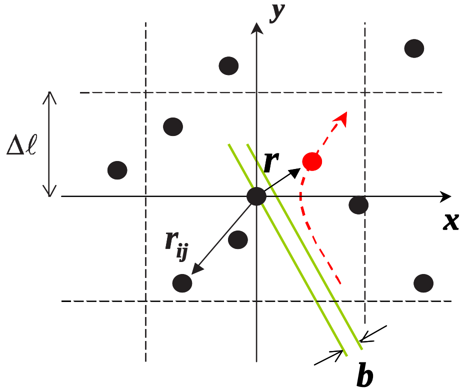

Fig. 3 A gray circle (or red in color) is a test particle at $\boldsymbol{r}$, and almost-uniformly-distributed black circles are field particles at $\boldsymbol{r}_{ij} = (i\Delta\ell + \delta x)\,\boldsymbol{e}_x + (j\Delta\ell + \delta y)\,\boldsymbol{e}_y$, $-N \le i, j \le N$.

First we seek for a field particle that gives the test particle an impulse force at the earliest time. The test particle moves along a strait line with a velocity of $(0, g)$ in the (ξ, η) coordinate system, in which η-axis is in the direction of the velocity vector $\boldsymbol{g}$ of the test particle. It is the field particle with $\eta = \eta_{\min}$, the minimum $|\eta|$, that we need. When the test particle moves to the position of $(0, \eta_{\min})$,

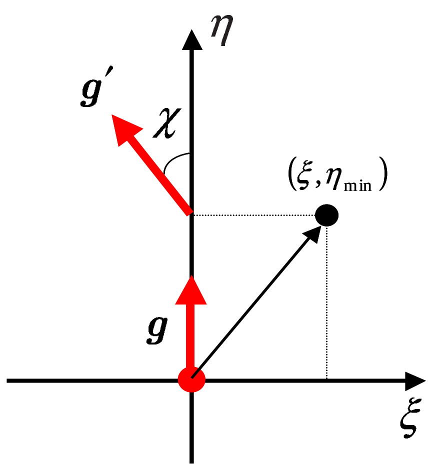

Fig. 4 Coordinates transform.

it changes the velocity $\boldsymbol{g}$ by $\Delta\boldsymbol{g}$ as shown in Fig. 4. This procedure will be repeated until the test particle leaves the prescribed interaction region, i.e. $r < \Delta\ell/2$ as depicted in Fig. 1.

The *exact* calculation hereafter refers to that obtained by solving the following equation of motion for the test particle:

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{g} \tag{5}$$

$$\mu\frac{\mathrm{d}\boldsymbol{g}}{\mathrm{d}t} = \frac{e^2}{4\pi\varepsilon_0}\sum_{i=-N}^{N}\sum_{j=-N}^{N}\frac{\boldsymbol{r}-\boldsymbol{r}_{ij}}{\left|\boldsymbol{r}-\boldsymbol{r}_{ij}\right|^3}, \tag{6}$$

where the field particles positions $\boldsymbol{r}_{ij}$

$$\boldsymbol{r}_{ij} = (i\Delta\ell + \delta x)\,\boldsymbol{e}_x + (j\Delta\ell + \delta y)\,\boldsymbol{e}_y \quad (-N \le i, j \le N)$$

using the 5-th order Runge-Kutta-Fehlberg method known as the RKF56.

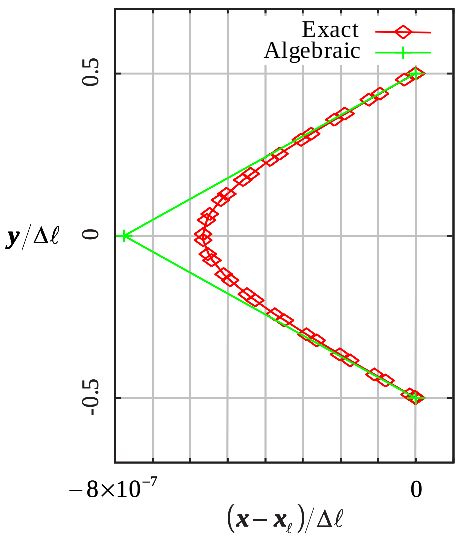

Fig. 5 Comparison of algebraic trajectory and exact trajectory in the case of binary Coulomb collision ($N = 0$) with an impact parameter $b = 0.2\Delta\ell$.

Figure 5 compares trajectories of the algebraic approximation and the exact hyperbola in the case of the pure binary Coulomb interaction, i.e. $N = 0$ in Eq. (6), with an impact parameter $b = 0.2\Delta\ell$. The only one field particle is at the origin in this case; $\boldsymbol{r}_{00} = \boldsymbol{0}$. The test particle starts at the lower right point goes through the closest point and ends at the upper right point in the figure. In the case of multiple field particles, we have assumed that there are nearly uniformly distributed 21×21 field particles at rest. In the algebraic approximation to multiboby problems ($N > 1$), as explained earlier, after a coordinate tranformation $(x, y) \to (\xi, \eta)$, where

$$\eta_{ij} = \frac{\left(\boldsymbol{r}_{ij} - \boldsymbol{r}\right)\cdot\boldsymbol{g}}{g}, \tag{7}$$

we find the field particle with minmum $\left|\eta_{ij}\right|$.

Figures 6, 7, and 8 are three examples out of 10^5 Monte Carlo calculations for an impact parameter $b =$

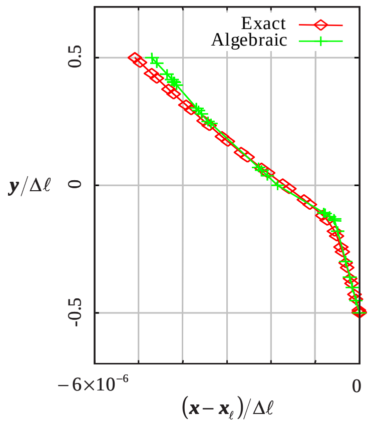

Fig. 6 Comparison of algebraic trajectory and exact trajectory in the case of multibody Coulomb collisions with an impact parameter $b = 0.2\Delta\ell$. This is an example of small angle scatterings.

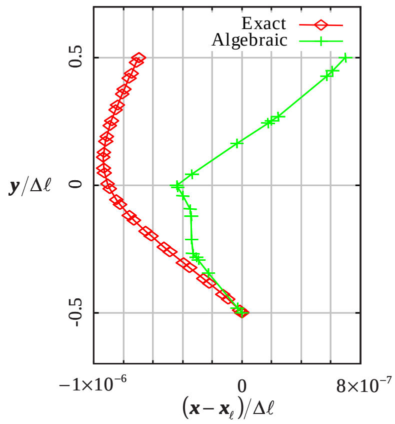

Fig. 8 Comparison of algebraic trajectory and exact trajectory in the case of multibody Coulomb collisions with an impact parameter $b = 0.2\Delta\ell$. The discrepancy is as small as 10^{-13} meter in typical fusion plasmas.

$0.2\Delta\ell$, and compare trajectories of the algebraic and the exact trajectories in the case of the multiple Coulomb interaction, i.e. $N = 10$ [1] in Eq. (6). The indivisual approximation is good in most cases as shown in Figs. 6 and 7, while 8 is one of few example which the approximation is bad. The algebraic trajectory in Fig. 7 seems to deviate from the exact one, however, the deviation is as small as $\Delta\ell \times 10^{-6} \sim 10^{-13}$ meter in typical fusion plasmas.

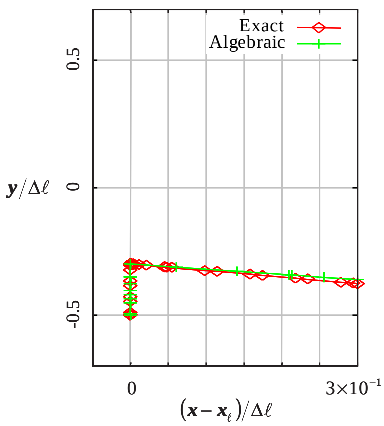

Fig. 7 Comparison of algebraic trajectory and exact trajectory in the case of multibody Coulomb collisions with an impact parameter $b = 0.2\Delta\ell$. This is an example of large angle scatterings.

Finally we conducted the above calcution for different impact parameters $0 < b < r_\ell$. Figure 9 shows the accumulated variance of velocity change, $\left\langle (\Delta \boldsymbol{g})^2 \right\rangle$, which is in proportion to the conventional momentum transfer cross section, σ_{m}, as

$$\sigma_{\mathrm{m}} = 4\pi b_0^2 \ln \frac{b_{\mathrm{max}}}{b_0} \tag{8}$$

The error in σ_{m} due to the algebraic calculation is seen to be quite small for both the binary ($N = 0$) and multibody ($N = 10$) cases, where there is only one field particle and there are 21×21 field particles, respectively. It should be noted that in the binary interactions the cross section σ_{m} is converged at $b \ll \Delta\ell$ which is far less than the Debye length λ_{D}. The CPU time for the algebraic approximation is only around 20 minutes on a PC, while the exact analysis needs 15 hours to integrate the whole set of multibody equations of motion.

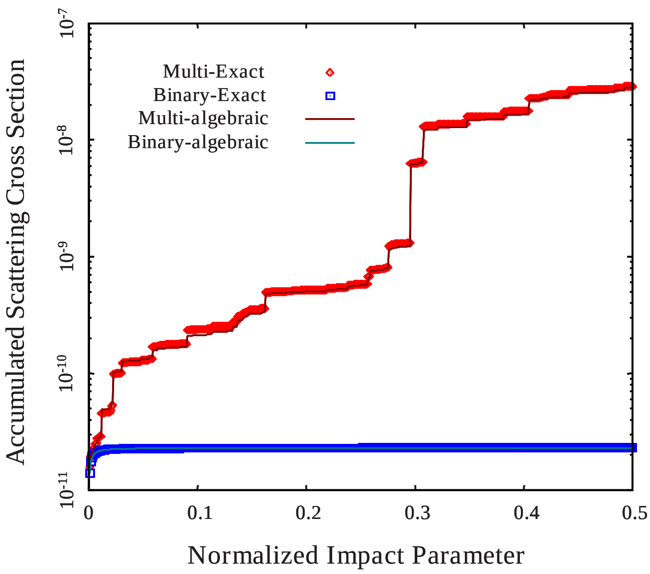

Fig. 9 Accumulated scattering cross section $\sigma_{\mathrm{m}} = \sigma_{\mathrm{m}}\left(\bar{b}\right)$ vs normalized impact parameter $\bar{b} = b/\Delta\ell$.

In the future, we apply this method to three dimensional multibody collision.

[1] H. Funasaka, Master thesis, Graduate School of Engineering, Hokkaido University (2007).