Physical data assessment for energy confinement scaling laws

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The principle of *data adaptive planning* (DAP) is applied to estimate the information gained by a single datum from a given set of data about the parameters of interest. The information gain is thereby quantified by an information measure. By using DAP one is capable to express the importance of a single datum. A W7-AS confinement data set was analysed with respect to different energy confinement scaling laws, the importance of the data points is assessed with respect to the respective scaling law. The physical question (i.e. the scaling law constraints) thereby shows significant impact on the expected information gain of single data points.

Keywords: energy confinement, scaling laws, Connor-Taylor model, Bayesian experimental design, data adaptive planning

1 Introduction

Fusion experiments like stellarators today are complex and also expensive devices. To gain a maximum of output by acceptable effort and costs a purposive design and planning of such experiments is necessary. For this, it is also important to assess the available data from previous measurements with respect to the experimental goals.

During the past years, data bases like the International Stellarator Confinement Data Base (ISCDB) [1] containing the experimental results of different fusion machines have grown and are widely used as reference and for inter-machine comparison (see, e.g., [2, 3]). Current work also includes the assignment of specific physical models to the data (or sets of data), using model comparison techniques [4, 5].

Given a set of measured data, it is useful to estimate the information gain of a single data point, e.g., to identify the most informative datum, with respect to a certain physical model describing the experimental situation. It seems reasonable that in this case the underlying model should have a significant influence on that evaluation. Also, the impact of the other data has to be taken into account. A possible method for the validation of a given data set, basing on the concept of Bayesian experimental design, is presented in this paper. With this method, the assessment of data from fusion experiments with respect to different physical models is possible.

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1.1 Scaling laws

Because the detailed dependencies between plasma parameters (electron density n, heating power P), machine parameters (minor radius a, magnetic field strength B) and quantities describing the energy confinement (confined energy W, energy confinement time τ_E) are not completely known, semi-empirical scaling laws are used for comparison of different experiments. Also, the design of future experiments is possible by extrapolation of the scaling relation.

A typical approach is given by a power scaling law, e.g. the International Stellarator Scaling 2004 (ISS04 [3]), which was found by studies of data from different stellarator experiments:

$$\begin{aligned}\tau_E^{ISS04} &= 0.134 \cdot a^{2.28} \cdot R^{0.64} \\ &\quad \cdot P_{tot}^{-0.61} \cdot n^{0.54} \cdot B^{0.84} \cdot \iota_{2/3}^{0.41}.\end{aligned}$$

A different approach was introduced by Connor and Taylor [6]: Here, the basic assumption is that scaling invariance properties of the particular plasma model lead to constraints of scaling exponents. Plasma models are assumed to be derived from some basic equations like:

- the Fokker-Planck equation describing collisions,
- the Maxwell equations describing the influence of β,
- and the continuity equation, momentum equation and the energy equation for the MHD description of the plasma.

Following this ansatz, the Connor-Taylor scaling model for the confined energy can be rephrased for volume averaged quantities:

$$\frac{W}{na^4B^2} \propto \left(\frac{P}{na^4B^3}\right)^{\alpha_1} \left(\frac{a^3B^4}{n}\right)^{\alpha_2} \left(\frac{1}{na^2}\right)^{\alpha_3} \tag{1}$$

Here, the α_i are the scaling parameters. Depending on the physical plasma model, these parameters may be dependent of each other, or even zero. E.g., for the case of a collisional low-β plasma one obtains $\alpha_3 = 0$, whereas α_1 and α_2 have to be estimated.

The scaling laws can be linearised by taking the logarithm. Then the linearised power law reads

$$\ln W = \alpha_c + \alpha_a \ln a + \alpha_P \ln P + \alpha_n \ln n + \alpha_B \ln B.$$

1.2 Data adaptive planning

The approach of Bayesian experimental design (BED) offers a mathematically consistent way for the optimisation of diagnostics as well as for the design of experiments and experimental campaigns using a physical question, expressed by a set of parameters of interest, as the design criterion. It bases on the maximisation of a utility function, namely the Kullback-Leibler distance, which is a measure for the information gain by a measurement. The method was proposed by Lindley [7], a review on different applications can be found in [8], the design of plasma diagnostics is studied in [9, 10].

The BED approach can also be utilised for the assessment of a data point. For this, the expected information gain for this particular measurement is calculated with respect to

- the physical question of interest,
- the experimental configuration,
- the measurement error and error statistics and
- other data available.

If existing data sets are implemented, the Kullback-Leibler distance as the utility function for the assessment reads:

$$\begin{aligned} U_{KL}(D, \boldsymbol{d}, \boldsymbol{\xi}) &= \int \mathrm{d}\boldsymbol{\alpha}\, p(\boldsymbol{\alpha}|\, D, \boldsymbol{d}, \boldsymbol{\xi}) \\ &\quad \cdot \ln \left[\frac{p(\boldsymbol{\alpha}|\, D, \boldsymbol{d}, \boldsymbol{\xi})}{p(\boldsymbol{\alpha}|\, \boldsymbol{d})}\right] \end{aligned}$$

This expression is an measure for the information gained by the new datum D about the parameters of interest, $\boldsymbol{\alpha}$. It is given in *bit* if the base-2 logarithm is used. The probability density function (PDF) $p(\boldsymbol{\alpha}|\, D, \boldsymbol{d}, \boldsymbol{\xi})$ is called "posterior distribution", describing the knowledge about $\boldsymbol{\alpha}$ given the new datum D, the experimental configuration $\boldsymbol{\xi}$ of the new measurement, and the old data $\boldsymbol{d}$. The prior function $p(\boldsymbol{\alpha}|\, \boldsymbol{d})$, on the other hand, shows the knowledge about $\boldsymbol{\alpha}$ before the new measurement (given only the old data).

For Bayesian experimental design, the information measure has to be averaged over the expected values for D, described by the PDF $p(D|\boldsymbol{\xi}, \boldsymbol{d}) = \int \mathrm{d}\widetilde{\boldsymbol{\alpha}}\, p(\widetilde{\boldsymbol{\alpha}}|\, \boldsymbol{d})\, p(D|\, \widetilde{\boldsymbol{\alpha}}, \boldsymbol{\xi})$, to cover all possible outcome of the future experiment. Here, $\widetilde{\boldsymbol{\alpha}}$ is the result for the parameters of interest given only the old measurements $\boldsymbol{d}$. This leads to the Expected Utility (EU) function

$$\begin{aligned} EU(\boldsymbol{\xi}, \boldsymbol{d}) &= \int \mathrm{d}D\, p(D|\, \boldsymbol{d}, \boldsymbol{\xi}) \cdot U(D, \boldsymbol{d}, \boldsymbol{\xi}) \\ &= \int \mathrm{d}\widetilde{\boldsymbol{\alpha}}\, p(\widetilde{\boldsymbol{\alpha}}|\, \boldsymbol{d}) \int \mathrm{d}D\, p(D|\, \widetilde{\boldsymbol{\alpha}}, \boldsymbol{\xi}) \\ &\quad \int \mathrm{d}\boldsymbol{\alpha}\, p(\boldsymbol{\alpha}|\, D, \boldsymbol{d}, \boldsymbol{\xi}) \\ &\quad\quad \cdot \ln \left[\frac{p(\boldsymbol{\alpha}|\, D, \boldsymbol{d}, \boldsymbol{\xi})}{p(\boldsymbol{\alpha}|\, \boldsymbol{d})}\right]. \end{aligned}$$

For a linear physical problem $\boldsymbol{d} = \boldsymbol{X} \cdot \boldsymbol{\alpha}$ and $D = \boldsymbol{\xi}^T \cdot \boldsymbol{\alpha}$ (matrix $\boldsymbol{X}$ contains the experimental configurations of the old data) and assuming a Gaussian error statistics, the EU can be calculated analytically after some algebra:

$$EU(\boldsymbol{d}, \boldsymbol{\xi}) = \frac{1}{2} \left[\log(1+G) - \frac{G}{(1+G)^2}\right], \tag{2}$$

with

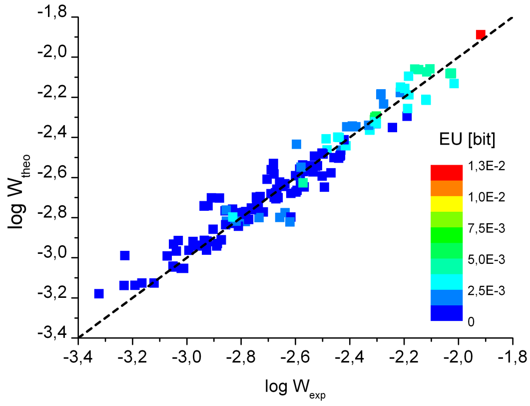
$$G = \frac{\boldsymbol{\xi}^T \left(\boldsymbol{X}^T \boldsymbol{C} \boldsymbol{X}\right)^{-1} \boldsymbol{\xi}}{\sigma^2}; \quad C_{ii} = 1/s_i^2. \tag{3}$$

The measurement error of the new datum is thereby given with σ, the error of the old data are encoded with $\boldsymbol{s}$.

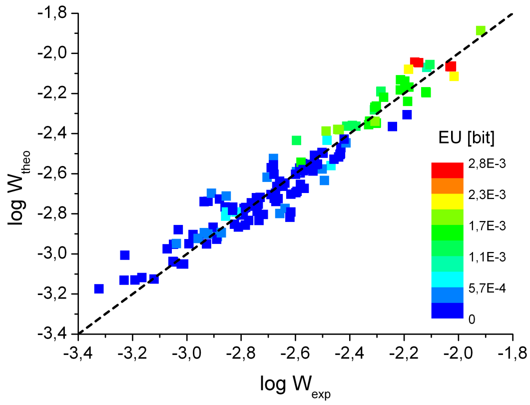
2 Results

For the study presented here, the Expected Utility of every datum from a given set was calculated according to linearised scaling laws. For this, the respective datum was removed from the data set and treated as "new datum". The experimental configuration (n, P, a and B) of this measurement then corresponds to $\boldsymbol{\xi}$, $\boldsymbol{X}$ and $\boldsymbol{d}$ are the configuration and the data outcome from the other data points in the set (see eqs. (2) and (3)).

The data set itself consists of 153 $\iota = 1/3$ data of W7-AS taken from ISDB. In this data base, the values for the configuration parameters n, P, a and B as well as for the measured confined energy are listed. Furthermore, the measurement errors for all quantities are given. For the data analysed here the collisional low-β model was identified to be the most probable one [5]; therefore, $\alpha_3 = 0$ was used in the Connor-Taylor scaling law. As a second model, a linearised power law was applied.



(a)



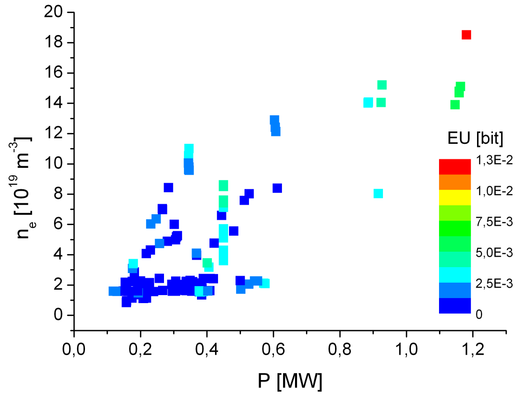
(b)

Fig. 1 Data set plotted against experimental and theoretical value of the confined energy. The theoretical value was calculated by a power scaling law (a) and the Connor-Taylor model for collisional low-β plasmas (b). The EU of the respective datum is color-coded.

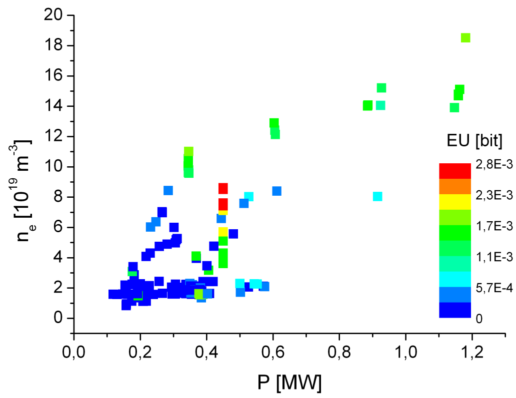
Figure 1 shows the data set plotted against the experimental value of the confined energy, W_{exp} and the theoretical value W_{theo} calculated by the respective scaling law. The EU is given as color scheme.

For both analysed scaling laws the data points with the highest expected information gain are found at high values of W. As a difference, the most informative point with respect to the low-β model is not the one at maximum W, which is the case for the power law model.

The absolute scales for the EU are different for both scaling laws. This results from the fact that two different physical questions are analysed here. The set of parameters of interest, the scaling exponents, are not identical. Therefore, the absolute values of the expected information gain for the different problems cannot be compared directly.



(a)

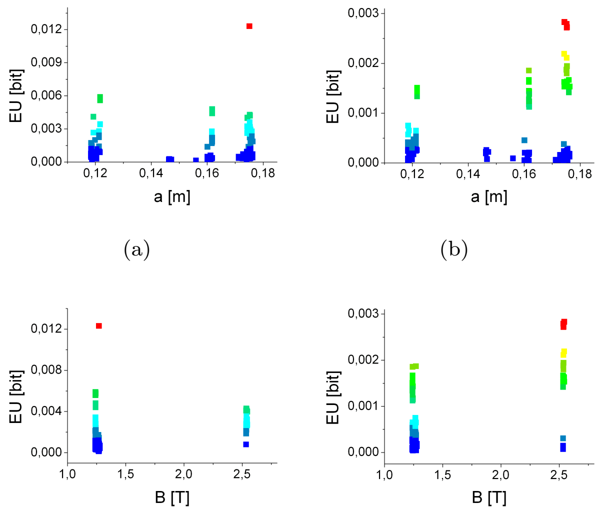


(b)

Fig. 2 Expected Utilities of the data set with respect to the experimental configuration parameters P and n, (a) for the power law model, (b) for the low-β model.

The behavior of the Expected Utility was then analysed with respect to the experimental configuration parameters a, n, B and P. To study the influence of these parameters, the data set is plotted in the n-P plane next (fig. 2). This plane is of interest because density and heating power can be varied. Again, one finds similarities between the different scaling laws: In both cases, data points in regions with a low sample rate (high n and P), far away from the most of the data, are very informative. In case of the power scaling law, the highest EU is given for the datum with maximum n and P. In contrast, this is not the case for the low-β scaling.

In order to study the impact of the further control parameters, the data set is plotted with respect to the minor radius a and the magnetic field B (fig. 3): In the case of the power law high values of the EU occur in all regions of a and B, for the Connor-Taylor model the highest values of the EU are found only at the



(a) (b) (c) (d)

Fig. 3 Expected Utilities of the data set with respect to the minor radius a (upper row) and the magnetic field strength B (lower row), for the power law model (left column) and for the low-β model (right column), respectively.

highest values of a and B.

These findings can be explained by taking into account the dependency on a and B in the low-β model: Both parameters are related to the α_1 and α_2 terms with high exponents (a^4, B^3, see eq. (1)), so high values of a and B will have a strong influence on the outcome of the model. In contrast, the parameters n and P enter the model only to the power of 1. Therefore, data points with high a and/or B will lead to higher information gain in case of the low-β model. This indicates that measurements at high a and B are more important for the validation of this model.

3 Conclusion

In this work, data adaptive planning was used to calculate the expected information gain of a datum with respect to a set of 153 $\iota = 1/3$ data from W7-AS. The capability of DAP to implement a physical question as an assessment criterion in a mathematical consistent way was demonstrated by applying the method to two different energy confinement scaling laws: a power law similar to the International Stellarator Scaling, and the Connor-Taylor scaling for collisional low-β plasmas. The expected information gain from the single data points turned out to be different for both models: Whereas for the power law the information gain of data points with values of low a and B are on a par data with values of high a and B, for the low-β model the data in the range of high a and B turn out to be more valuable. This can be explained by the dependence of the Connor-Taylor terms on high powers of a ($\propto a^4$) and B ($\propto B^3$). The Connor-Taylor terms are therefore dominated by these parameters at high values.

These results show the influence of the physical question on the Expected Utility: Different physical models will lead to different values of the information gain from a certain datum. DAP is a tool to quantify these differences and to estimate the contribution of individual data to the overall result. It is a new quality for the assessment of data going beyond qualitative arguments.

For future work, further analysis of the parameters influencing the value of the expected information gain have to be made. In particular, the measurement error and the error statistics may have significant impact (see [10]).

In addition, the DAP formalism can be used for the systematic planning of future experiments and experimental campaigns: The calculation of the expected information gain for a possible future measurement offers the possibility to estimate the most informative experimental condition for the next experiment.

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