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Abstract

Even though our primary interest is concerned with stochastic properties of the low dimensional nonlinear dynamical systems, identification of the fixed points and analysis of the periodic orbits are necessary and unavoidable step in the studies of chaotic behaviour. For the case of 2-D area preserving mapping, the symmetry properties of the map porvides critical information on the island structure of the mappings. In order to illustrate usefulness of knowledge of the symmetry properties, the periodic orbits of the standard map are discussed in detail by constructing the families of symmetry curves.

Keywords :
2D area preserving mapping,
reversible dynamical system,
involution decomposition, periodic orbits,
symmetry curves, standard map,
Poincare-Birkhoff period-q islands,

§1. Introduction

Research frontier of the low dimensional nonlinear dynamical systems are expanding to the vast range of subjects in statistical mechanics[1], celestial mechanics[2], accelerator physics[3], plasma physics[4], fluid dynamics and chemical physics[5]. Restriction to the conservative Hamiltonian system is not successful to squeeze the list of relevant references down to countable numbers[6]. The method of Poincare's surface of section, or the method of iterative nonlinear map have been proved to be useful to characterize the low dimensional nonlinear dynamical systems.

In order to study motion of the non-integrable Hamiltonian systems with the aid of the above mentioned methods, we admit that numerical and graphical analysis with computer aid is inevitable approach. Yet, we emphasize that certain analytical information gives rise to critical knowledge to resolve confusing aspect of numerical and graphical analysis.

The purpose of present paper is to examine the symmetry property of the two dimensional area-preserving map, and to discuss structures of the islands of the standard map as an illustrative example. In the second section, we present general discussion in determining the symmetry curves of the two dimensional area-preserving map. We apply the method to the standard map in the third section. The last section presents concluding discussion referring to generic case of the two-dimensional area preserving map.

§2. Symmetry of Two Dimensional Reversible Map

In the studies of nonlinear dynamical systems, even our interests are focused on chaotic behaviour, analysis of periodic orbits has the primary importance to understand the global properties of the dynamical systems. Applying Birkhoff's symmetry analysis, de Vogelaere[7] developed a systematic analysis of periodic orbits in the conservative dynamical system with two degrees of freedom. Pina and Lara[8] presented explicit analysis of the symmetry lines of the standard map. Extending their analysis, we have carried out systematic analysis of stochasticity of the standard map referring to the symmetry structure[9].

Now, Quispel and Roberts[10] call our attention to the fact that reversible dynamical systems can bear certain symmetry even if the system is dissipative. A mapping T is called reversible if there is a symmetric I_0 such that

$$T \cdot I_0 \cdot T = I_0 \qquad 1)$$

and I_0 is an involution

$$I_0 \cdot I_0 = 1 \qquad 2)$$

Equations 1) and 2) lead to the relation that T is the product of two involutions.

$$T = I_1 \cdot I_0 \qquad I_0 \cdot I_0 = I_1 \cdot I_1 = 1 \qquad 3)$$

where $I_1 = T \cdot I_0$. The inverse transformation T^{-1} is expressed as

$$T^{-1} = I_0 \cdot T \cdot I_0 \quad 4)$$

For the j-th iteration of mapping T, we define

$$I_j = T^j \cdot I_0 \;, \quad j = \text{integers} \quad 5)$$

then, we have

$$I_j \cdot I_j = 1 \quad 6)$$

which confirms that I_j is also an involution. For arbitrary integers j and k, an ensemble of I_j and T^k forms a discrete infinite group with the following relationships.

$$T^j \cdot I_k = I_{j+k} \quad 7)$$

$$I_j \cdot I_k = T^{j-k} \quad 8)$$

$$I_j \cdot T^k = I_{j-k} \quad 9)$$

For a vector R, with the minimum N, if

$$T^N R = R \quad 10)$$

valid, R represents the period-N orbit.

The j-th order symmetry curve S_j consists of ensemble of the fixed points of involution I_j, i.e.

$$S_j : \{ R \mid I_j R = R \} \tag{11}$$

Hence, eq.8) defines that the intersection of S_j and S_k determines the periodic orbits of T, whose period N divides $|j - k|$.

For a 2-D area preserving map of the form given as

$$\begin{pmatrix} X' \\ P' \end{pmatrix} = T \begin{pmatrix} X \\ P \end{pmatrix}, \quad T = \begin{pmatrix} 1+h & , & 1 \\ h & , & 1 \end{pmatrix} \tag{12}$$

if h(x) is anti-symmetric function, the following factorization is well known[11),

$$T \begin{pmatrix} X \\ P \end{pmatrix} = I_1 \cdot I_0 \begin{pmatrix} X \\ P \end{pmatrix} \tag{13}$$

with

$$I_0 \begin{pmatrix} X \\ P \end{pmatrix} = \begin{pmatrix} -1 & , & 0 \\ h & , & 1 \end{pmatrix} \begin{pmatrix} X \\ P \end{pmatrix} = \begin{pmatrix} -X \\ P + h(X) \end{pmatrix} \tag{14}$$

$$I_1 \begin{pmatrix} X \\ P \end{pmatrix} = \begin{pmatrix} -1 & , & 1 \\ 0 & , & 1 \end{pmatrix} \begin{pmatrix} X \\ P \end{pmatrix} = \begin{pmatrix} -X+P \\ P \end{pmatrix} \tag{15}$$

A factorization into two involutions of the nonlinear map is not unique. In order to analyze the dense distribution of periodic orbits in the standard map, Tanikawa and Yamaguchi[12] have used the following decomposition

$$T\begin{pmatrix} X \\ P \end{pmatrix} = J_1 \cdot J_0 \begin{pmatrix} X \\ P \end{pmatrix} \tag{16}$$

with

$$J_0\begin{pmatrix} X \\ P \end{pmatrix} = \begin{pmatrix} X - P \\ -P \end{pmatrix} \tag{17}$$

$$J_1\begin{pmatrix} X \\ P \end{pmatrix} = \begin{pmatrix} X - 2P + h(X-P) \\ -P + h(X-P) \end{pmatrix} \tag{18}$$

for which we have

$$J_0 \cdot J_0 = 1 \quad \text{and} \quad J_1 \cdot J_1 = 1 \tag{19}$$

We notice here that the above decomposition holds even through h(x) is not anti-symmetric function.

Applying the above factorizations for the standard map, we have discussed statistical properties of the standard map[9]. Here, we describe some details of construction of the symmetry curves for the 2-D area preserving map. For the first kind of involution factorization, applying eq.5), we can write

$$I_n\begin{pmatrix} X \\ P \end{pmatrix} = \begin{pmatrix} -X + nP + G_n \\ P + F_n \end{pmatrix} \tag{20}$$

where

$$F_n = F_{n-1} + h\,[\,-X + (n-1)P + G_{n-1}\,] \qquad 21.a)$$

$$G_n = G_{n-1} + F_n \qquad 21.b)$$

with

$$F_0 = h(X)\ ,\ G_0 = 0 \qquad 22.a)$$

$$F_1 = 0\ ,\ \ G_1 = 0 \qquad 22.b)$$

$$F_2 = h(-X+P)\ ,\ \ G_2 = h(-X+P) \qquad 22.c)$$

The n-th order symmetry curve Γ_n is determined from

$$I_n\begin{pmatrix} X \\ P \end{pmatrix} = \begin{pmatrix} X \\ P \end{pmatrix} \qquad 23)$$

which gives rise to

$$nP - 2X + G_n = 0\ ,\ \ F_n = 0 \qquad 24)$$

With eq.21.b), we can reduce eq.24) as

$$nP - 2X + G_{n-1} = 0 \qquad 25)$$

Now, the requirement of $F_n=0$ with eq.21.a) and eq.25) leads to

$$F_{n-1} = -h\ [\ -X + (n-1)P + G_{n-1}\ ] = -h(X-P) \qquad 26)$$

If the transformation function h(X) is anti-symmetric, i.e., if

$$h(X) = -\ h\ (-X) \qquad 27)$$

valid, we have

$$F_{n-1} = h(-X+P) = F_2 = G_2 \qquad 28)$$

Using eq.21.b) again, we get

$$G_{n-1} = F_2 + G_{n-2} \qquad 29)$$

Thus, for the expression of Γ_n, we get

$$nP - 2X + G_2 + G_{n-2} = 0 \qquad 30)$$

For further reduction of eq.30), it is worth to notice that eq.30) can be decomposed as

$$-X + (n-2)P + G_{n-2} = -(-X+2P+G_2) \qquad 31)$$

Using eq.21.a) and eq.31), we can reduce eq.28) to

$$F_{n-2} = G_2 + h(-X+2P+G_2) \tag{32}$$

Therefore, we can reduce eq.30) to

$$nP - 2X + 2G_2 + h(-X+2P+G_2) + G_{n-3} = 0 \tag{33}$$

Here, it would be worth to illustrate the construction of symmetry curves. With the definition of eq.14), we get simply

$$\Gamma_0 : X = 0 \tag{34.a}$$

Applying eq.25) with n=1, 2 and 3, we get

$$\Gamma_1 : P - 2X = 0 \tag{34.b}$$

$$\Gamma_2 : 2P - 2X = 0 \tag{34.c}$$

$$\Gamma_3 : 3P - 2X + h(-X+P) = 0 \tag{34.d}$$

Now, for n=4, in applying eq.25), we need to express G_3 in terms of G_2. Hence, using eq.30), we obtain

$$\Gamma_4 : 4P - 2X + 2h(-X+P) = 0 \tag{35}$$

while in the previous report[9)] we have presented an expression for Γ_4 as

$$4P-2X+2h(-X+P)+h(-X+2P+h(-X+P)) = 0 \qquad 36)$$

which is nothing but the expression we get from eq.33) with n=4. However, for n=4, eq.32) gives

$$h(-X+2P+G_2) = 0 \qquad 37)$$

Therefore, eq.36) is confirmed to be consistent with eq.35).

Taking n=5 in eq.33), we get for Γ_5 the following expression,

$$\Gamma_5 : 5P-2X+3h(-X+P)+h(-X+2P+h(-X+P)) = 0 \qquad 38)$$

while in the previous report[9], basing on the recurrence formulae constructed by the expression of Γ_2, Γ_3 and Γ_4 with the redundant term, we have used the following expression,

$$5P-2X+3h(-X+P) + 2h(-X+2P+h(-X+P)) +$$

$$h\{-X+3P+2h(-X+P)+h[-X+2P+h(-X+P)]\} = 0 \qquad 39)$$

We notice here eq.33) can be formally reduced to the following expression

$$nP-2X+3G_2+2h(-X+2P+G_2)+h(-X+3P+2G_2+h(-X+2P+G_2))+G_{n-4}=0 \qquad 40)$$

by using eq.32) with eqs.21.a) and b). Here, we have made use of a relationship of

$$F_{n-3} = G_2+h(-X+2P+G_2) + h(-X+3P+2G_2+h(-X+2P+G2)) \qquad 41)$$

Taking n=5 in eq.40), we reproduce eq.39). However, at the same time, eq.41) with n=5 gives rise to the identity

$$h(-X+3P+2G_2+h(-X+2P+G_2)) = -h(-X+2P+G_2) \qquad 42)$$

hence eq.39) is reduced to eq.38).

For n=6, eq.40) leads to the following expression

$$6P-2X+4h(-X+P) + 2h(-X+2P+h(-X+P)) +$$

$$h\Big(-X+3P+2h(-X+P) + h\big(-X+2P+h(-X+P)\big)\Big) = 0 \qquad 43)$$

Now, for this case, eq.41) with $F_3 = F_2 + h(-X+2P+G_2)$ gives rise to

$$h(-X+3P+2G_2 + h(-X+2P+G_2)) = 0 \qquad 44)$$

Hence, the last redundant term of eq.43) is shown to be identically zero, and the expression for Γ_6 is reduced to

$$\Gamma_6 : 6P-2X+4h(-X+P)+2h(-X+2P+h(-X+P)) = 0 \qquad 45)$$

For n=7, eq.40) leads to

$$7P-2X+3h(-X+P)+2h(-X+2P+G_2)+h(-X+3P+G_3)+G_3=0 \qquad 46)$$

while

$$G_3 = G_2+F_3 = 2G_2+h(-X+2P+G_2) \qquad 47)$$

Thus, we get

$$\Gamma_7 : 7P-2X+5h(-X+P)+3h(-X+2P+h(-X+P))$$

$$+h(-X+3P+2h(-X+P)+h(-X+2P+h(-X+P)))=0 \qquad 48)$$

Expressing the left hand sides of equations for the n-th order symmetry curves by Γ_{2N} and Γ_{2N+1}, we can write down the recurrence formula as

$$\Gamma_{2N} = 2\Gamma_{2N-1} - \Gamma_{2N-2} \qquad 49.a)$$

$$\Gamma_{2N+1} = 2\Gamma_{2N} - \Gamma_{2N-1} + h(1/2\Gamma_{2N}) \qquad 49.b)$$

Now, for the negative integer -n, n>0, using eq.9) with j=0, we can construct a general expression

$$I_{-n}\begin{pmatrix} X \\ P \end{pmatrix} = \begin{pmatrix} -X-nP-\tilde{G}_n \\ P+\tilde{F}_n \end{pmatrix} \qquad 50)$$

with the recurrence relations of

$$\widetilde{G}_n = \widetilde{G}_{n-1} + \widetilde{F}_{n-1} \tag{51.a}$$

$$\widetilde{F}_n = \widetilde{F}_{n-1} + h(X+nP+\widetilde{G}_n) \tag{51.b}$$

We have

$$\widetilde{G}_1 = h(X) \tag{52.a}$$

$$\widetilde{F}_1 = \widetilde{G}_1 + h(X+P+\widetilde{G}_1) \tag{52.b}$$

The symmetry curves Γ_{-n} is determined by

$$I_{-n}\begin{pmatrix} X \\ P \end{pmatrix} = \begin{pmatrix} X \\ P \end{pmatrix} \tag{53}$$

as

$$nP + 2X + \widetilde{G}_n = 0 \ , \quad \widetilde{F}_n = 0 \tag{54}$$

Equation 54) with eq.51.b) gives rise to

$$\widetilde{F}_{n-1} = -h(X+nP+\widetilde{G}_n) = -h(-X) = h(X) \tag{55}$$

so that the -n-th symmetry curve is determined as

$$\Gamma_{-n} : nP + 2X + h(X) + \widetilde{G}_{n-1} = 0 \qquad 56)$$

Using eq.51.a), for n>2, we can reduce eq.56) to

$$nP+2X+h(X)+\widetilde{G}_{n-2}+\widetilde{F}_{n-2} = 0 \qquad 57)$$

Since eq.56) gives

$$X + (n-1)P + \widetilde{G}_{n-1} = -X-P-h(X) \qquad 58)$$

eq.55) is reduced to

$$\widetilde{F}_{n-2} = \widetilde{G}_1 - h(X+(n-1)P+\widetilde{G}_{n-1}) = \widetilde{G}_1+h(X+P+h(X)) = \widetilde{F}_1 \qquad 59)$$

For n=1, eq.54) gives

$$\Gamma_{-1} : P + 2X + h(X) = 0 \qquad 60)$$

and F_1 vanishes identically. For n=2, eq.56) gives

$$\Gamma_{-2} : 2P + 2X + 2h(X) = 0 \qquad 61)$$

We can verify the condition $\widetilde{F}_2=0$ as

$$\widetilde{F}_2 = \widetilde{F}_1+h(X+2P+\widetilde{G}_2) = \widetilde{F}_1+h(X+2P+2\widetilde{G}_1+h(X+P+\widetilde{G}_1))$$

$$= \widetilde{G}_1+h(X+P+h(X))+h(-X) = 0 \quad 62)$$

For n=3, eq.57) with eq.59) gives

$$\Gamma_{-3} : 3P+2X+3h(X)+h(X+P+h(X)) = 0 \quad 63)$$

It is straight forward to confirm $\widetilde{F}_3=0$.

Now, for n=4, eq.57) with eq.59) gives

$$\Gamma_{-4} : 4P+2X+h(X)+\widetilde{G}_2+\widetilde{G}_1+h(X+P+h(X)) = 0 \quad 64)$$

where $\widetilde{G}_2$ is

$$\widetilde{G}_2 = \widetilde{G}_1+\widetilde{F}_1 = 2h(X)+h(X+P+h(X)) \quad 65)$$

Hence, eq.64) is reduced to

$$\Gamma_{-4} : 4P+2X+4h(X)+2h(X+P+h(X)) = 0 \quad 66)$$

In our previous report[9]), we have presented an expression with the redundant term, which fails to account correctly the condition of $\widetilde{F}_4=0$.

Expressing the left hand sides of equations for the -n-th order symmetry curves by Γ_{-2N} and $\Gamma_{-(2N+1)}$, we can write down the recurrence formula as

$$\Gamma_{-2N} = 2\Gamma_{-(2N-1)} - \Gamma_{-(2N-2)} \qquad 67.a)$$

$$\Gamma_{-(2N+1)} = 2\Gamma_{-2N} - \Gamma_{-(2N-1)} + h(1/2\Gamma_{-2N}) \qquad 67.b)$$

In the above construction of the symmetry curves, the use has been made fully the anti-symmetric property of the transformation function h(X). For a general function h(X), we have the second type of involution factorization as defined by eqs.17) and 18). We find that

$$J_n\begin{pmatrix} X \\ P \end{pmatrix} = \begin{pmatrix} X-(n-1)P+g_n \\ -P+f_n \end{pmatrix} \qquad 68)$$

with the recurrence relations

$$f_{n+1} = f_n + h(X-(n+1)P+g_n) \qquad 69.a)$$

$$g_{n+1} = g_n + f_{n+1} \quad , \qquad 69.b)$$

and

$$f_0 = g_0 = 0 \qquad 70.a)$$

$$f_1 = g_1 = h(X-P) \qquad 70.b)$$

The n-th order symmetry curve γ_{+n} is determined by

$$J_n \begin{pmatrix} X \\ P \end{pmatrix} = \begin{pmatrix} X \\ P \end{pmatrix} \qquad 71)$$

which gives rise to

$$-(n+1)P + g_n = 0 \qquad 72.a)$$

$$-2P + f_n = 0 \qquad 72.b)$$

We have for n=0 symmetry

$$\gamma_0 \; : P = 0 \qquad 73)$$

For n=1, eqs.72.a) and b) give rise to

$$\gamma_{+1} \; : -2P = h(X-P) \qquad 74)$$

For n=2, we have

$$-3P + g_2 = 0 \qquad 75.a)$$

$$-2P + f_2 = 0 \qquad 75.b)$$

Hence, we get

$$P = g_2 - f_2 = g_1 = h(X-P) \qquad 76)$$

namely,

$$\gamma_{+2} : 2P = 2h(X-P) \qquad 77)$$

Equation 75.a) with eqs.69.a) and b) leads to

$$3P = g_1 + f_1 + h(X-2P+h(X-P)) \qquad 78)$$

which turns out to be consistent with eq.77). For n=3, we have

$$-4P + g_3 = 0 \qquad 79.a)$$

$$-2P + f_3 = 0 \qquad 79.b)$$

Hence, we get

$$2P = g_3-f_3 = g_2 = g_1+f_2 = 2f_1+h(X-2P+f_1) \qquad 80)$$

namely,

$$\gamma_{+3} : 2P = 2h(X-P) + h(X-2P+h(X-P)) \qquad 81)$$

Equation 79.b) requires that the right hand side of eq.81) should be identical with f_3. Equation 69.a) gives

$$f_3=f_2+h(X-3P+g_2)=f_2+h(X-P)=2h(X-P)+h(X-P+h(X-P)) \qquad 82)$$

For n=4, we have

$$-5P + g_4 = 0 \qquad 83.a)$$

$$-2P + f_4 = 0 \qquad 83.b)$$

Hence, eqs.83.a) and b) lead to

$$3P = g_4 - f_4 = g_3 = g_2 + f_3 \qquad 84)$$

while eq.83.b) itself gives

$$2P = f_4 = f_3+h(X-4P+g_3) = f_3+h(X-P) = f_2+h(X-3P+g_2)+h(X-P)$$

$$= 2h(X-P) + h(X-2P+g_1) + h(X-3P+g_2) \qquad 85)$$

On the other hand,

$$h(X-3P+g_2) = h(X-f_3) = h(X-2P+g_1) \qquad 86)$$

Therefore, finally we get for γ_{+4} as

$$\gamma_{+4} : 2P = 2h(X-P) + 2h(X-2P+h(X-P)) \qquad 87)$$

Straight forward calculation for n=5 and n=6 gives

$$\gamma_{+5} \; : \; 2P = 2h(X-P) + 2h(X-2P+h(X-P)) + h(X-3P+2h(X-P) + h(X-2P+h(X-P))) \qquad 88)$$

$$\gamma_{+6} \; : \; 2P = 2h(X-P) + 2h(X-2P+h(X-P)) + 2h(X-3P+2h(X-P) + h(X-2P+h(X-P))) \qquad 89)$$

Turning to the negative integer -n, n>0, we obtain

$$J_{-n}\begin{pmatrix} X \\ P \end{pmatrix} = \begin{pmatrix} X + (n-1)P + \tilde{g}_n \\ -P - \tilde{f}_n \end{pmatrix} \qquad 90)$$

with the recurrence formula

$$\tilde{g}_n = \tilde{g}_{n-1} + \tilde{f}_{n-1} \qquad 91.a)$$

$$\tilde{f}_n = \tilde{f}_{n-1} + h(X+(n-1)P+\tilde{g}_n) \qquad 91.b)$$

and

$$\tilde{f}_1 = h(X) \qquad 92.a)$$

$$\tilde{g}_1 = 0 \qquad 92.b)$$

The -n-th order symmetry curve γ_{-n} is determined by

$$J_{-n}\begin{pmatrix} X \\ P \end{pmatrix} = \begin{pmatrix} X \\ P \end{pmatrix} \tag{93}$$

which gives rise to

$$(n-1)P + \tilde{g}_n = 0 \tag{94.a}$$

$$2P + \tilde{f}_n = 0 \tag{94.b}$$

For n=1, eqs.94.a) and b) give rise to

$$\gamma_{-1} \; : \; 2P + h(X) = 0 \tag{95}$$

For n=2, we have

$$P + \tilde{g}_2 = 0 \tag{96.a}$$

$$2P + \tilde{f}_2 = 0 \tag{96.b}$$

Hence, we get

$$P + h(X) = 0 \tag{97.a}$$

$$2P + h(X) + h(X+P+h(X)) = 0 \tag{97.b}$$

namely,

$$\gamma_{-2}:\ 2P + 2h(X) = 0 \qquad 98)$$

For n=3, we have

$$2P + \widetilde{g}_3 = 0 \qquad 99.a)$$

$$2P = \widetilde{f}_3 = 0 \qquad 99.b)$$

Equation 99.b) is reduced to

$$2P + \widetilde{f}_2 + h(X+2P+\widetilde{g}_3) = 0 \qquad 100)$$

hence, it is reduced to the expression for γ_{-3} as

$$\gamma_{-3}\ :\ 2P + 2h(X) + h(X+P+h(X)) = 0 \qquad 101)$$

For n=4, we have

$$3P + \widetilde{g}_4 = 0 \qquad 102.a)$$

$$2P + \widetilde{f}_4 = 0 \qquad 102.b)$$

Equation 102.b) is reduced to

$$2P + \tilde{f}_3 + h(X+3P+\tilde{g}_4) = 0 \tag{103}$$

Making use of eq.102.a), we get

$$2P + h(X) + \tilde{f}_2 + h(X+2P+\tilde{g}_3) = 0 \tag{104}$$

Explicit form of eq.102.a) gives, however,

$$3P + \tilde{g}_3 + \tilde{f}_2 + h(X+2P+\tilde{g}_3) = 0 \tag{105}$$

Combining eqs.104) and 105), we get

$$P + \tilde{g}_3 - h(X) = 0 \tag{106}$$

which gives rise to the expression for γ_{-4} as

$$\gamma_{-4} \;:\; 2P + 2h(X) + 2h(X+P+h(X)) = 0 \tag{107}$$

Carrying out similar analysis for n=5 and n=6, we can derive the expressions for γ_{-5} and γ_{-6} as follows,

$$\gamma_{-5} \;:\; 2P+2h(X)+2h(X+P+h(X))+h(X+2P+2h(X)+h(X+P+h(X)))=0 \tag{108}$$

and

$$\gamma_{-6} : 2P+2h(X)+2h(X+P+h(X))+2h(X+2P+2h(X)$$

$$+h(X+P+h(X)))=0 \qquad 109)$$

To conclude the present section, we notice that the set of symmetry curves $\gamma_{\pm n}$ exists regardless whether the transformation function h(X) is anti-symmetric or not.

§3. Island Structure of the Standard Map

In the course of extensive investigations of statistical properties of the 2-D Hamiltonian systems, it has been well recognized that understanding of regular motions is unavoidable. Recent advancement of studies of transport in Hamiltonian systems emphasizes important effect of the island structure. Since the symmetric properties of 2-D area preserving map characterize the periodic orbits, applying the results obtained in the preceeding section, we investigate island structure of the standard map.

We define the standard map eq.12) with the transformation function

$$h(X) = -\frac{K}{2\pi} \sin(2\pi X) \qquad 110)$$

Now, when the point (X_n, P_n) is mapped to a point (X_{n+1}, P_{n+1}), the neighborhood point $(X_n+\delta X_n, P_n+\delta P_n)$ is mapped to a neighborhood point

$(X_{n+1}+\delta X_{n+1},\ P_{n+1}+\delta P_{n+1})$. The tangent map δT transforms the displacement $(\delta X_n,\ \delta P_n)$ into a displacement $(\delta X_{n+1},\ \delta P_{n+1})$ by

$$\begin{pmatrix} \delta X_{n+1} \\ \delta P_{n+1} \end{pmatrix} = \delta T \begin{pmatrix} \delta X_n \\ \delta P_n \end{pmatrix}, \quad \delta T = \begin{pmatrix} 1+h', & 1 \\ h', & 1 \end{pmatrix} \qquad 111)$$

where h' denotes a differential coefficient with respect to its variable X. The eigen value of two dimensional matrix δT is determined as

$$\lambda = 1 - 2R \pm 2[R(R-1)]^{1/2} \qquad 112)$$

where the residue R is given by

$$R = 1/4 \ \{ 2\text{-Trace}(\delta T) \} \qquad 113)$$

For the 2-D aea preserving map of eq.12), we have

$$R = - 1/4 \ h' \qquad 114)$$

When $0<R<1$, the point $(X_n,\ P_n)$ is an elliptic (stable) point, and the tangential orbit $(\delta X_n,\ \delta P_n)$ encircle around this stable point. In this case, the eigenvalue λ is expressed as

$$\lambda = \exp(\pm i2\pi\rho), \quad \rho = 1/2\pi \ \cos^{-1}(1-2R) \qquad 115)$$

where ρ gives an average rotation number.

Nonlinear map T may possess the points which remain to be fixed upon the iterative application of T. For the standard map with eq.110), the fixed points are (0,0) and (1/2,0). The stability condition takes a form of

$$-4 < -K \cos (2\pi X_f) < 0 \qquad 116)$$

Hence, the point (0,0) is stable as long as K remains in a range of 0<K<4, while the point (1/2,0) is unstable. When the rotation number ρ takes a value p/q with p and q the prime integers, the Poincare-Birkhoff period-q islands are born around the stable fixed point (0,0). Eq.115) gives the threshold for this process as

$$K(p/q) = 4\sin^2 (\pi p/q) \qquad 117)$$

First, we illustrate a typical structure of regular as well as chaotic orbits of the standard map at K=1.300 in Fig.1. Eq.117) gives K(1/8)=0.5858, K(1/7)=0.7530, K(1/6)=1.000, K(1/5)=1.382. Therefore, at K=1.300, we expect to observe the period-8, period-7 and period-6 Poincare-Birkhoff chain of islands. In Fig.1, we observe the large six islands, the small fourteen islands and eight islands in the fringe of the central island. We superpose the family of symmetry curves Γ_n given in the previous section for $0<n\leq 7$ in Fig.2.

The intersections of symmetry curves Γ_j and Γ_k determines the $|j-k|$ periodic orbits. This is confirmed for the period-6 and two sets of the period-7 orbits. Here, we remark that the intersection of symmetry lines Γ_j and Γ_k determines both of the stable and unstable even-period orbits, yet the intersection of Γ_j and Γ_k determines only the stable odd-period orbits. The observed fourteen islands are classified to a group of period-7 orbits, which go through the stable intersection of $\Gamma_{\pm 7}$ and Γ_0 with P>0, and another group of period-7 orbits, which pass the stable intersection of $\Gamma_{\pm 7}$ and Γ_0 with P<0.

Now, we are led to a question. What kind of symmetry does determine unstable odd-periodic orbits. Figure 3 illustrates the family of symmetry curves $\gamma_{\pm n}$ with $0<n\leq 7$. Here, we observe these symmetry curves pass through unstable period-7 points. As for the even-periodic orbits, these symmetry curves pass through both of the stable and unstable points.

When we increase the stochastic parameter K, the Poincare-Birkhoff chain of islands with lower periodicity are born out of the fixed point at the origin. For K=1.60, we observe in Fig.4 that the period-6 islands are now merging into the sea of chaos, while thin 22 islands and clear ten islands are observed in the closed island. Superposing the family of symmetry curves $\Gamma_{\pm n}$, $0<n\leq 6$, as shown in Fig.5, we can confirm that the ten islands are two sets of the period-5 orbits with $P_5>0$ and $P_5<0$, where P_5 is the intersection of $\Gamma_{\pm 5}$ and Γ_0.

As for the thin 22 islands, we can identify that Γ_{+6} and Γ_{-5} intersect at the stable point in the first quadrungle and then pass through the stable point in the third quadrungle, while Γ_{+5} and Γ_{-6} intersect at the stable point in the fourth quadrungle and then pass through the stable point in the second quadrungle. Thus, we can recognize these 22 islands as two sets of period-11 orbits. For p=2, q=11, eq.117) gives rise to a value of K(2/11)=1.169, these period-11 orbits were born after the period-6 orbits and grew in size when K increases further. Although we can observe these intersections in Fig.2 at K=1.30, the resonance was not so strong to make up the visible islands. Figure 6 shows the family of the symmetry curves of $\gamma_{\pm n}$, $0<n\leq 6$, for K=1.60.

For much larger value of K, the central island is getting thinner along the symmetry line Γ_{+1}, and the sea of chaos spreads over the phase space. Figure 7 illustrates the case of K=3.10. We observe very small six spots around the stable fixed point at the origin, ten islands and fourteen islands at the edge of the central island. Here, we can illustrate that the symmetry curves provide critical information to define these obscure observation. In Fig.8, we superpose the symmetry curves Γ_n with $0<n\leq 7$. Starting at the X-axis to the clockwise direction in the first quadrungle, we can identify the intersections of pairs of (Γ_{-7}, Γ_0),(Γ_{-3}, Γ_{+4}),(Γ_{-6}, Γ_{+1}), (Γ_{-2}, Γ_{+5}),(Γ_{-5}, Γ_{+2}) and (Γ_{-1}, Γ_{+6}). Since eq.117) gives K(2/7)=2.445 for p=2 and q=7, we can identify these fourteen islands are two sets of the 7-2 resonance.

Similarly, for the ten islands, we can identify Γ_{+7} and Γ_{-3} intersect at the stable islands, while Γ_{+4} and Γ_{-6} pass through the unstable point, then the symmetry line Γ_{+1} passes through the stable island and Γ_{-2} passes through the unstable point. We see Γ_{-5} and Γ_{+5} intersect at the stable point. Since these symmetry curves intersect both at the stable and unstable points, we can confirm that these are even-periodic orbit. Now, as eq.117) gives K(3/10)=2.618 for p=3 and q=10, we can identify these ten periodic islands as the 10-3 resonance.

Increasing the stochastic parameter K to 3.30, we see in Fig.9 that the two sets of the period 3 islands grow up in their size. Since these symmetry curvses of Γ_n pass through only the stable points, we can confirm that these islands are not of the even period of 6 but that of the odd period of 3.

§4. Concluding Discussions

We have shown in the previous section that the analysis based on the symmetry curves provide critical information on the structure of the islands formed by the Poincare-Birkhoff multi-furcation around the stable fixed point. We have shown that not only the fundamental resonance periodic mode but also the higher resonance mode such as the 11-2 resonance, the 7-2 resonance and the 10-3 resonance were identified. With regard to these higher resonance mode, it would be worth to recall the properties noticed by Pina and Lara. They have shown that the symmetry curves are transformed by T^N into other symmetry curves, expressed as

$$T^N \Gamma_K = \Gamma_{2N+K} \qquad 118)$$

Here, making use of eq.118), we will determine the multi-periodicity of these higher resonance orbits.

As has been discussed in the section 2, the periodicity q is determined as $|j-k|$ from the intersection of Γ_j and Γ_k. Take $R_0(X_0, P_0)$ as the intersection of Γ_{j0} and Γ_{k0}. The point R_0 is mapped to the next point R_1 by a mapping T. According to eq.118), the symmetry curves Γ_{j0} and Γ_{-k0} are mapped to Γ_{j0+2} and Γ_{k0+2}. Hence, the point R_1 should be at the intersection of Γ_{j0+2} and Γ_{k0+2}.

Let us examine the period-10 orbits at K=3.10. Although we did not determine Γ_9, we can identify the point R_0 in the first quadrungle as the intersection of Γ_{+1} and Γ_{-9}, which is mapped to R_1 at Γ_{+3} and Γ_{-7}, leaving two points at the intersections of (Γ_{+5}, Γ_{-5}) and (Γ_{+9}, Γ_{-1}). The point R_1 is mapped to the intersection at (Γ_{+7}, Γ_{-3}), and so on. Since at each mapping the point proceed to the third point in the ahead, the multi-periodicity p is equal to 3.

Turning to the very thin fourteen islands at K=3.10, we start from the point R_0^+(P_0>0) at the interaction of (Γ_0, Γ_{-7}), which is mapped to R_1^+ at the intersecton of (Γ_{+2}, Γ_{-5}). Then R_1^+ is mapped to R_2^+ at the intersection of (Γ_{+4}, Γ_{-3}), apparently leaving three points between. However, we notice the two of them belong to the orbit started at R_0^- (P_0<0) at the intersection of (Γ_0, Γ_{-7}). Thus, in this case the multi-periodicity p is equal to 2.

The 22 islands observed at K=1.60 were identified as two sets of period-11 orbits. Taking R_0^+ at the intersection of (Γ_{+6}, Γ_{-5}) in the first quadrungle, we can find that R_0^+ is mapped to R_1^+ through which Γ_{-3} passes, leaving three points between them. Note that two of them belong to the orbit started at R_0^- in the third quadrungle, which is reflection of R_0^+ with respect to the origin. Therefore, the multi-periodicity of this orbit is p=2.

Now, we consider the difference between odd and even periodic orbits. Concerning the odd periodic orbit, we start from the point R_0^+

with P>0 at the intersection of (Γ_0, Γ_k) with k odd. Then, applying the mapping with $|k|$ iterations, we can obtain a complete set of stable points. However, reflecting the R_0^+ with respect to the origin, we always find R_0^- with P<0 which belongs to another set of stable points. We can regard R_0^- as the intersection of (Γ_k, Γ_0) which is the replacement of (Γ_0, Γ_k). Therefore, intersections of Γ_j and Γ_k determine two sets of stable points of odd-period. As for the even periodic orbit, we first consider the intersection of (Γ_j, Γ_k) with j and k odd in the first quadrungle, which determines a stable point R_0^+. By iterating the mapping by $|j-k|$, we can obtain a set of even periodic points. Now, the reflection of R_0^+ with respect to the origin gives the point R_0^- in the third quadrungle. We can also regard R_0^- as the intersection of (Γ_k, Γ_j) which is the replacement of (Γ_j, Γ_k). Unlike to the odd period case, R_0^- belongs to the set of periodic points generated from R_0^+. The difference of odd and even periodic orbits are as follows :
the replacement of $(\Gamma_{even}, \Gamma_{odd})$ to $(\Gamma_{odd}, \Gamma_{even})$ gives rise to the reflection with respect to the origin and determines a point belonging to another set of stable points. But the replacement of $(\Gamma_{odd}, \Gamma_{odd})$ only causes the reflection and does not generate another pair of stable points. Same things hold for the intersections of $(\Gamma_{even}, \Gamma_{even})$. Therefore, intersections of Γ_{even} and Γ_{odd} determine two sets of stable points of odd period, while intersections of Γ_{odd} and Γ_{odd} determine a set of stable points of even period and those of Γ_{even} and Γ_{even} define a set of unstable points of even period. These distinctive natures come from the anti-symmetric property of the transformation function h(X).

The analysis described in the above is sufficient to convince us the usefulness of the information on symmetry curves in the studies of island structure of the 2-D area preserving maps.

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Caption of Figures

Fig.1 Regular and chaotic orbits of the standard map at K=1.30

Fig.2 The symmetry curves $\Gamma_{\pm n}$ of the standard map at K=1.30 with $0<n\leq 7$

Fig.3 The symmetry curves $\gamma_{\pm n}$ of the standard map at K=1.30 with $0<n\leq 7$

Fig.4 Regular and chaotic orbits of the standard map at K=1.60

Fig.5 The symmetry curves $\Gamma_{\pm n}$ of the standard map at K=1.60 with $0<n\leq 6$

Fig.6 The symmetry curves $\gamma_{\pm n}$ of the standard map at K=1.60 with $0<n\leq 6$

Fig.7 Regular and chaotic orbits of the standard map at K=3.10

Fig.8 The symmetry curves $\Gamma_{\pm n}$ of the standard map at K=3.10 with $0<n\leq 7$

Fig.9 The symmetry curves $\Gamma_{\pm n}$ of the standard map at K=3.30 with $0<n\leq 6$

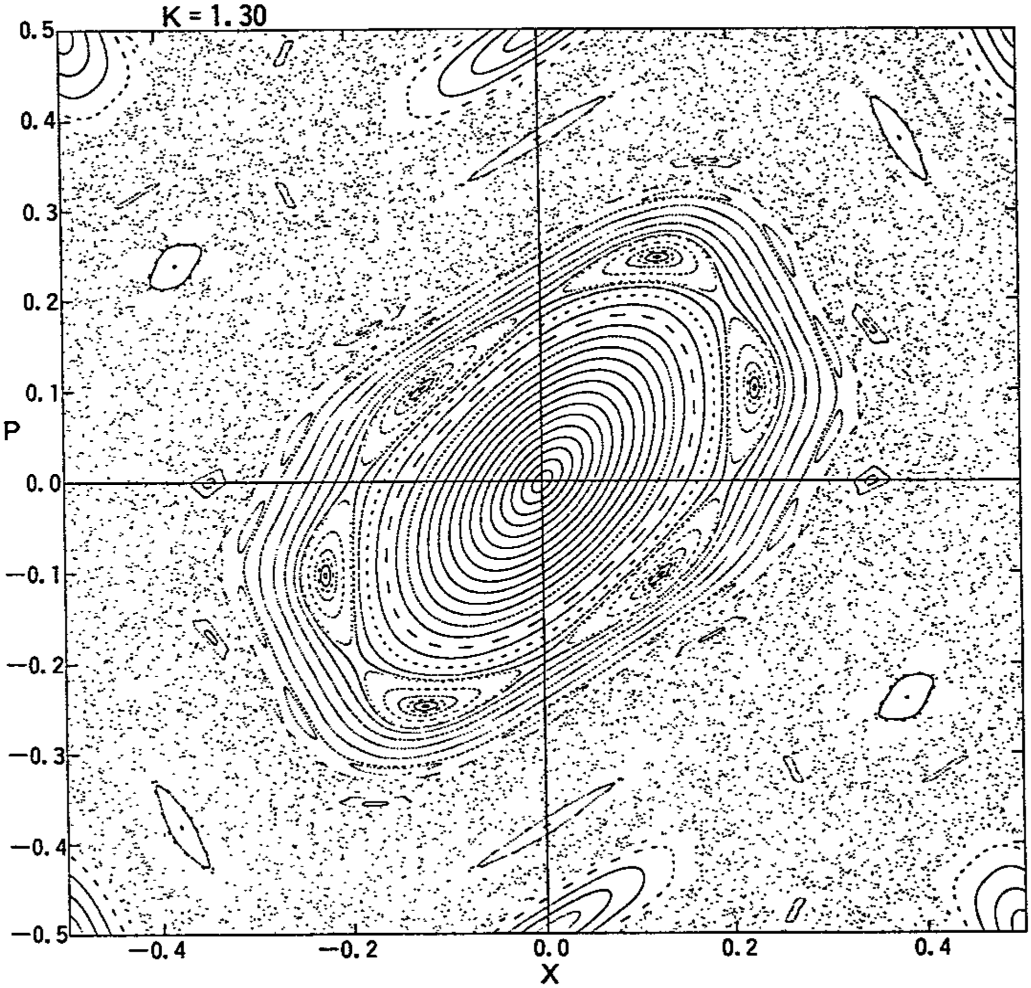
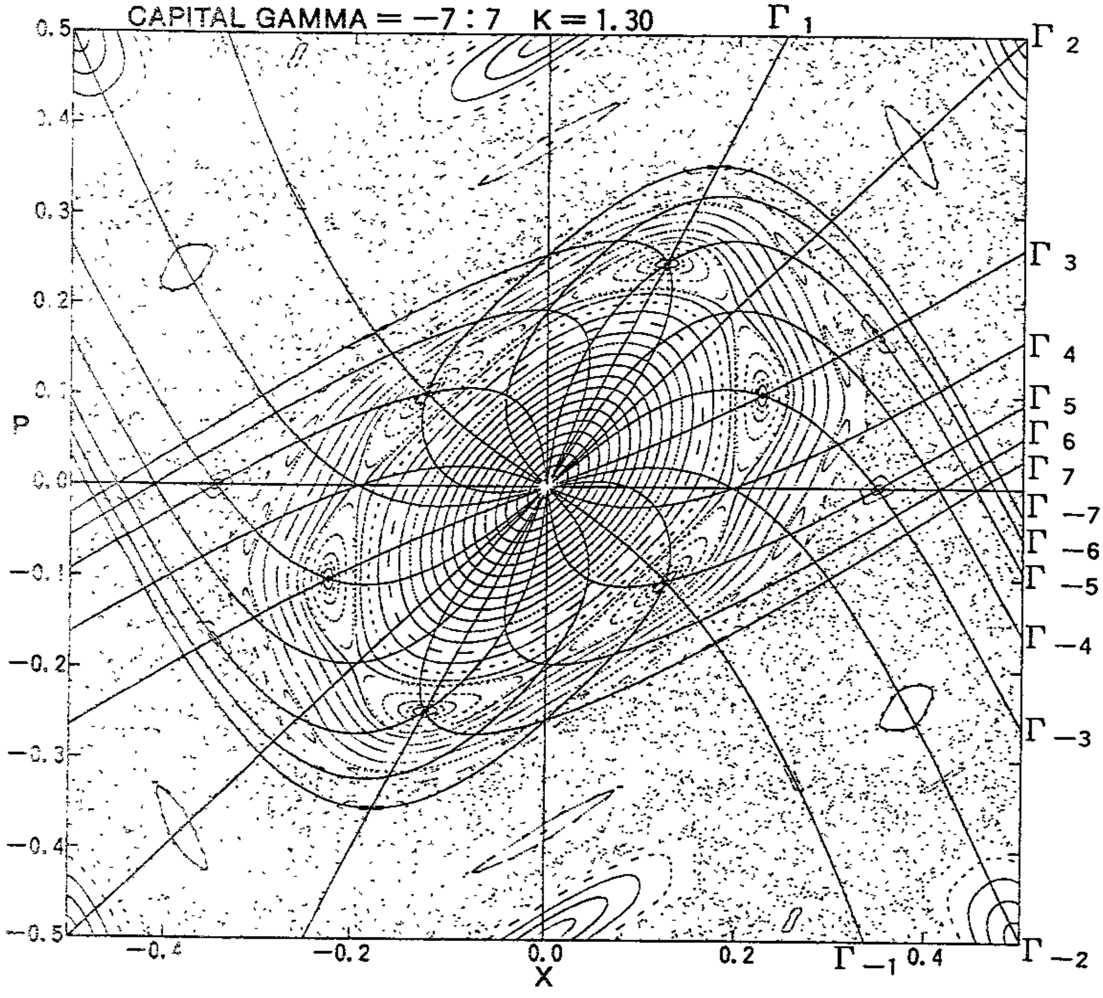
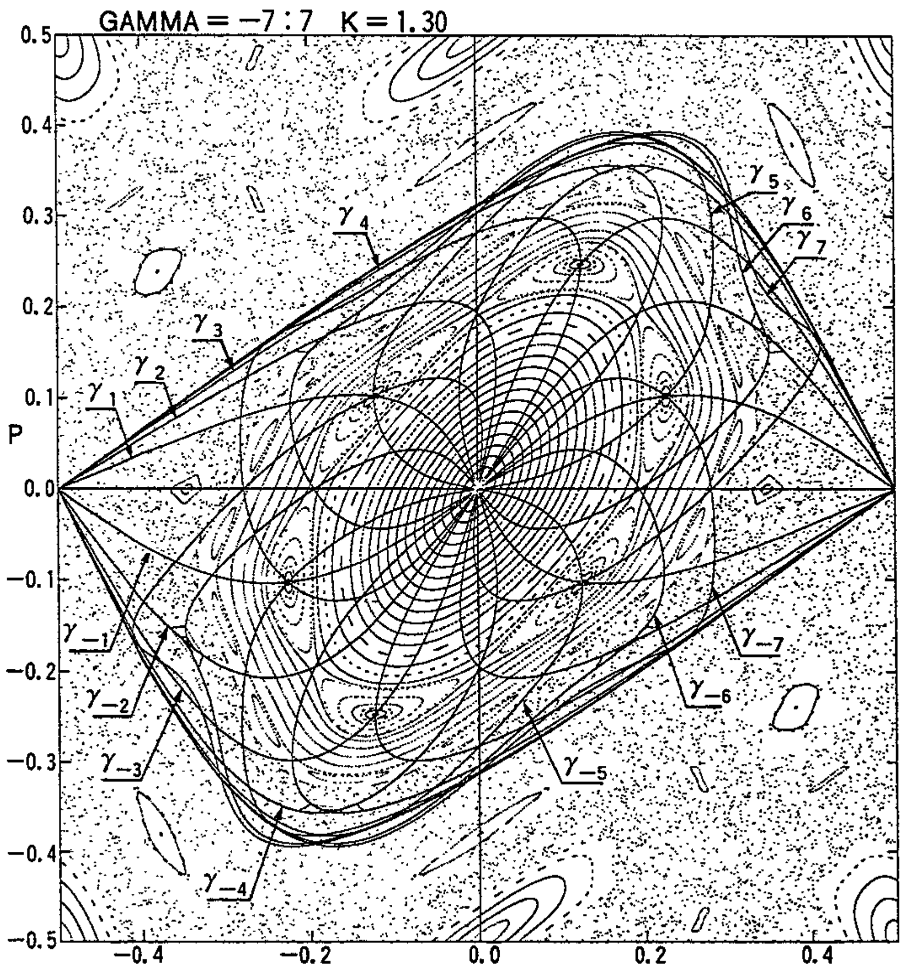


Fig. 1



CAPITAL GAMMA = −7 : 7 K = 1.30
Γ 1
Γ 2
Γ 3
Γ 4
Γ 5
Γ 6
Γ 7
Γ −7
Γ −6
Γ −5
Γ −4
Γ −3
Γ −2
Γ −1
P
X
0.5
0.4
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.4
−0.2
0.0
0.2
0.4

Fig. 2



GAMMA = −7 : 7 K = 1.30
0.5
0.4
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
−0.4
−0.5
P
−0.4
−0.2
0.0
0.2
0.4
X
γ_1
γ_2
γ_3
γ_4
γ_5
γ_6
γ_7
γ_{-1}
γ_{-2}
γ_{-3}
γ_{-4}
γ_{-5}
γ_{-6}
γ_{-7}

Fig. 3

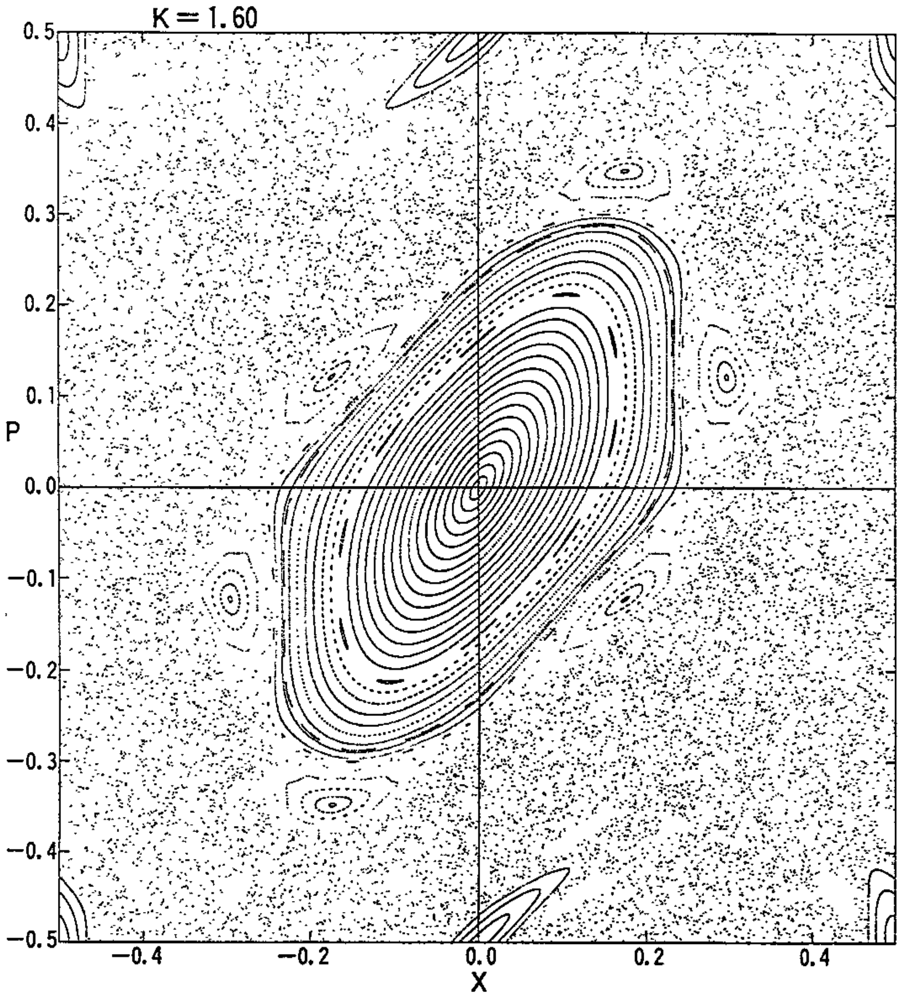


Fig. 4

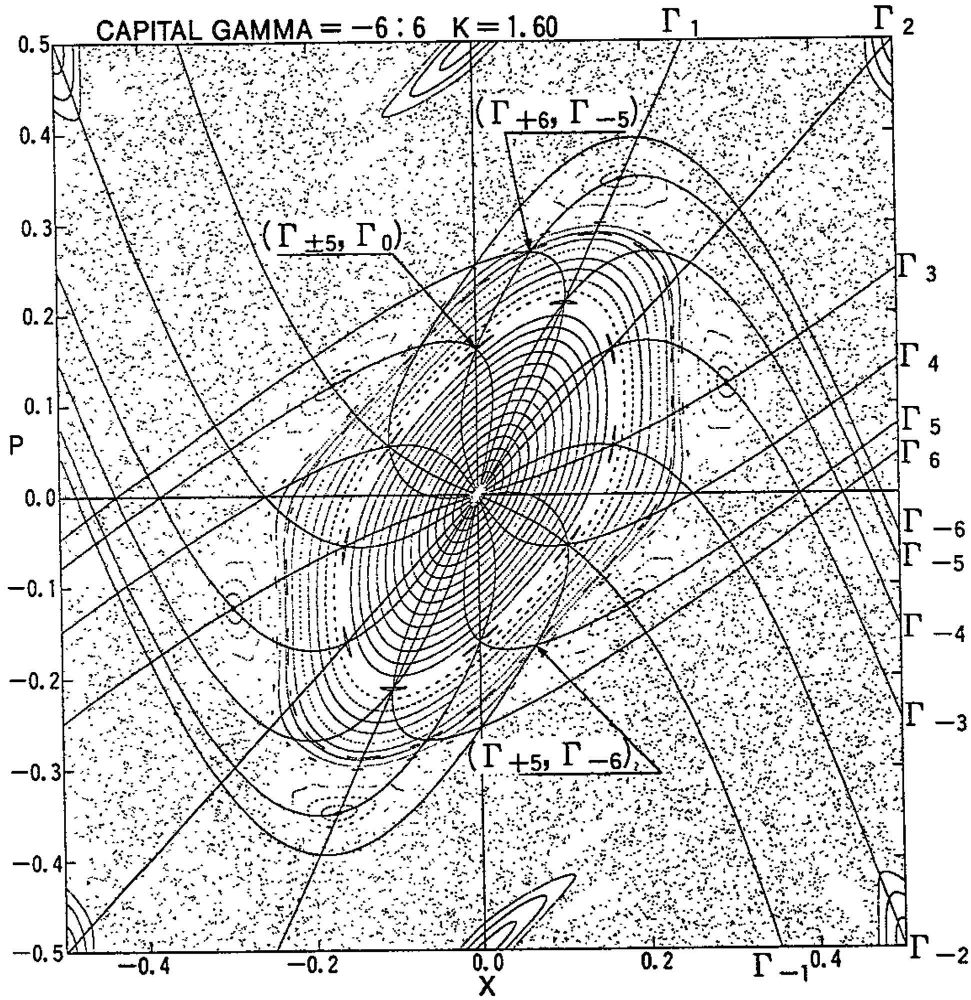

CAPITAL GAMMA = −6 : 6 K = 1.60
Γ 1
Γ 2
(Γ +6, Γ −5)
(Γ ±5, Γ 0)
Γ 3
Γ 4
Γ 5
Γ 6
Γ −6
Γ −5
Γ −4
Γ −3
(Γ +5, Γ −6)
Γ −2
Γ −1
P
X
0.5
0.4
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.4
−0.2
0.0
0.2
0.4

Fig. 5

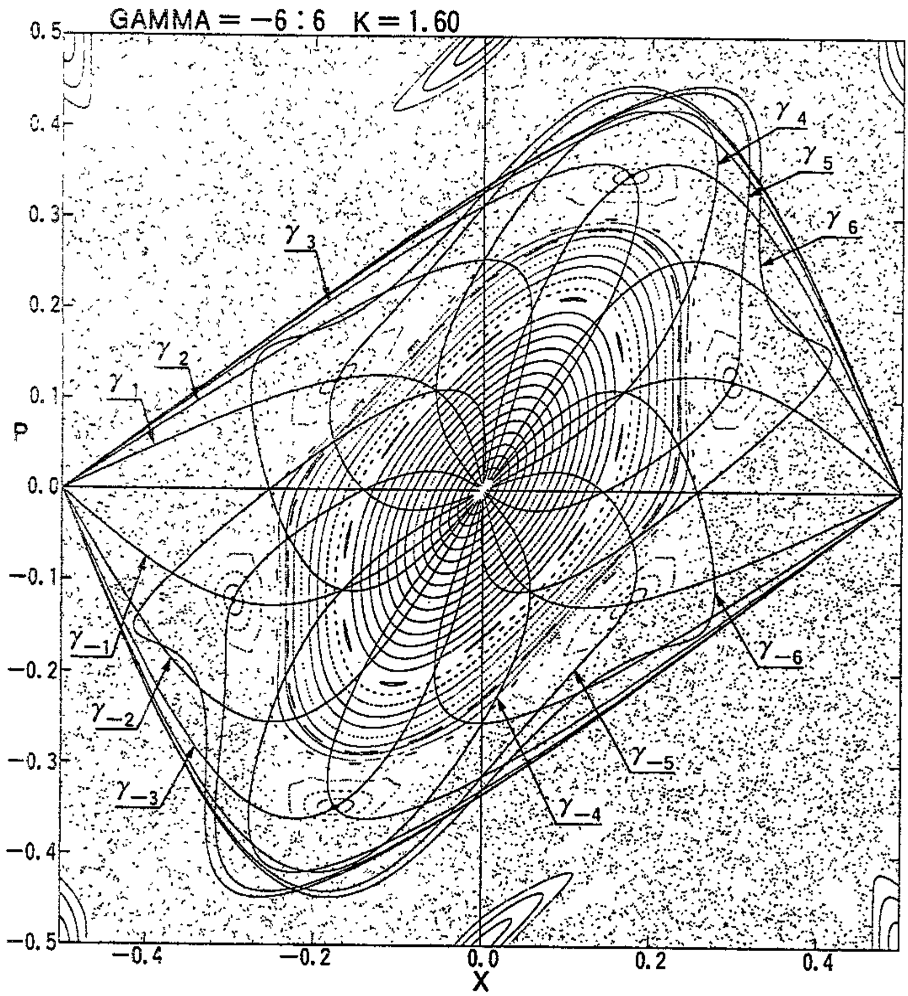


Fig. 6

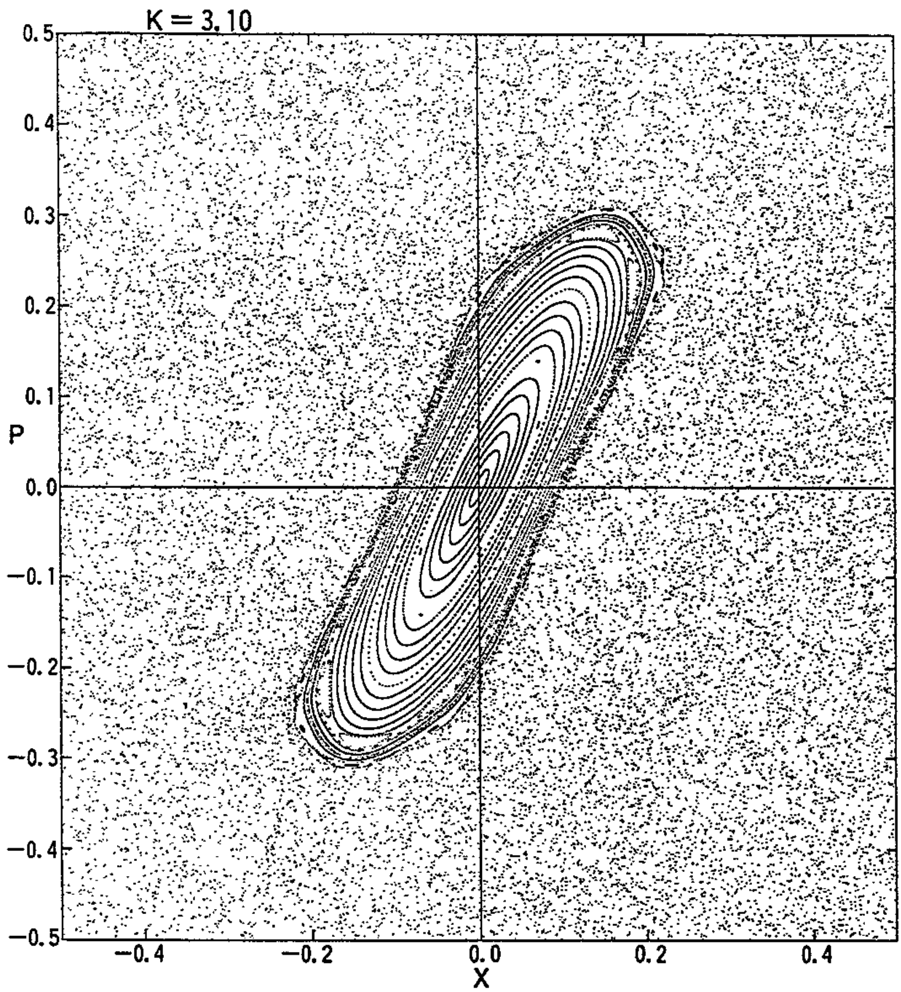


Fig. 7

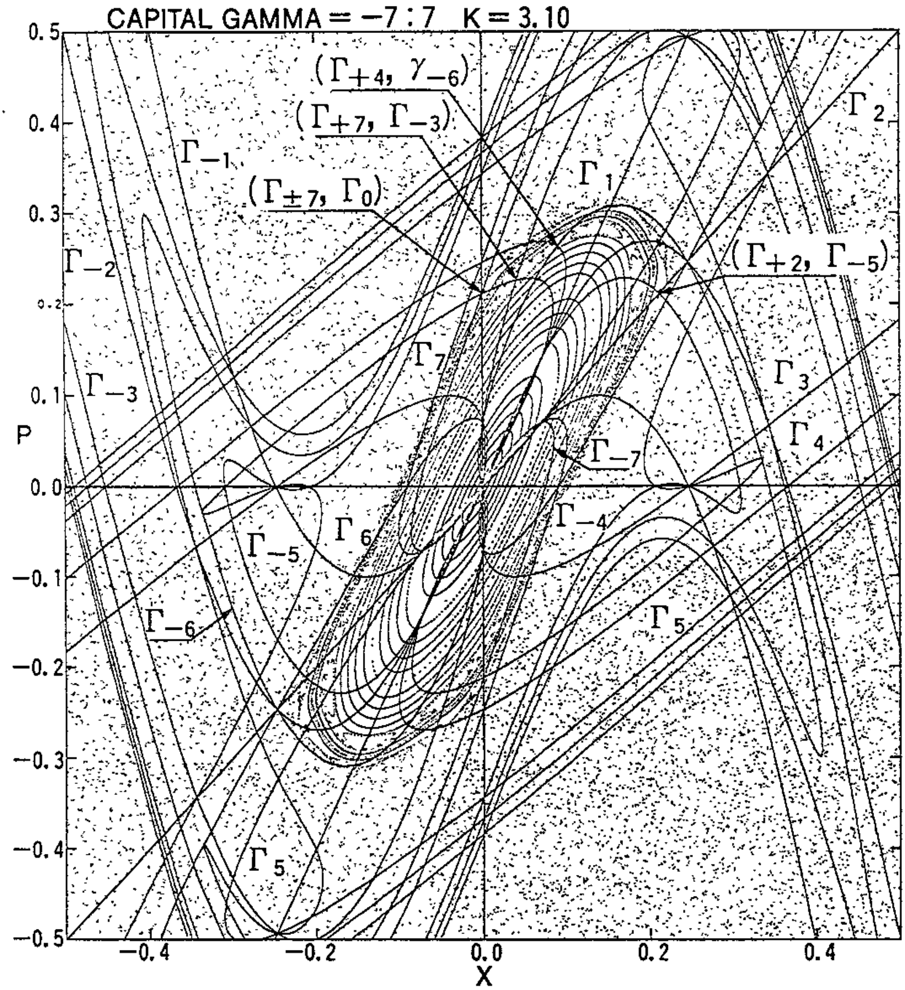

CAPITAL GAMMA = −7 : 7 K = 3.10
P
X
0.5
0.4
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.4
−0.2
0.0
0.2
0.4
(Γ+4, γ−6)
(Γ+7, Γ−3)
(Γ±7, Γ0)
(Γ+2, Γ−5)
Γ−1
Γ−2
Γ−3
Γ−5
Γ−6
Γ1
Γ2
Γ3
Γ4
Γ5
Γ6
Γ7
Γ−7
Γ−4
Γ5

Fig. 8

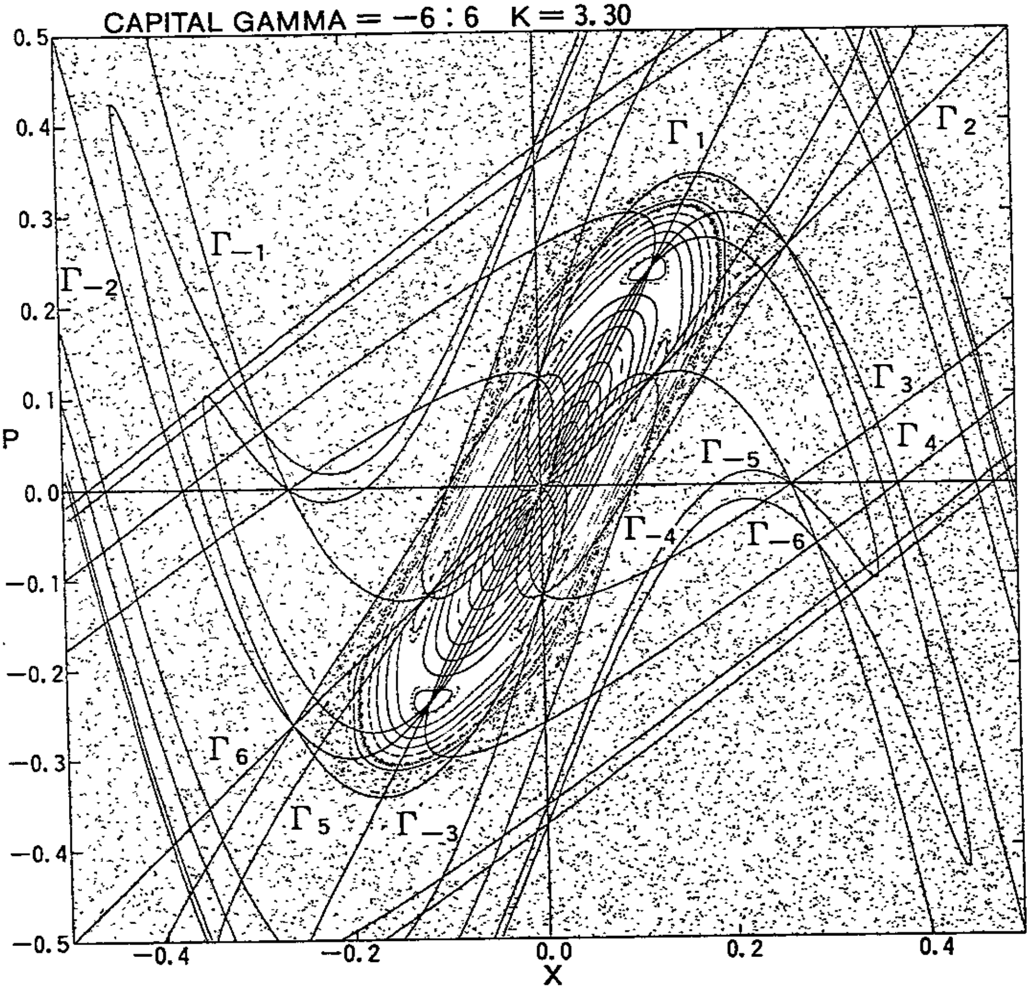

CAPITAL GAMMA = −6 : 6   K = 3.30
0.5
0.4
0.3
0.2
0.1
P
0.0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.4
−0.2
0.0
0.2
0.4
X
Γ1
Γ2
Γ−1
Γ−2
Γ3
Γ4
Γ−5
Γ−4
Γ−6
Γ6
Γ5
Γ−3

Fig. 9