ISSN 0915-633X

NATIONAL INSTITUTE FOR FUSION SCIENCE

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(Received – Oct. 17, 1989)

NIFS-6 Feb. 1990

RESEARCH REPORT
NIFS Series

NAGOYA, JAPAN

Stochastic Properties of the Plasma Wave Heating Map

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Abstract

Diffusion coefficient for stochastic ion motion in a lower hybrid wave is derived analytically by use of the characteristic function method. The renormalization calculation is carried out successfuly to account for effects of the higher order correlation. Numerical observation of the diffusion process confirms the expectation that the renormalized diffusion coefficient describes correctly the stochastic properties of the systems even in the region of the small stochastic parameter, except where the accelerator modes manifest their influence upon the chaotic orbits.

Keywords :
stochastic ion heating, lower hybrid wave, area preserving 2D map, characteristic function, renormalized diffusion coefficient, higher order correlation, accelerator mode

§ 1. Introduction

Stochastic properties of the nonlinear wave-particle interaction plays critical role in further heating of plasma by waves. Under the action of external magnetic field, even a single mode of plasma wave induces stochastic behaviour of plasma particles, and thus gives rise to heating of plasma. Studies of such stochastic heating process have been carried out by Fukuyama et.al [1] and Karney et.al [2]$\sim$[4] for the lower hybrid wave. In particular, Karney has examined nonlinear dynamics of plasma particle under the action of wave by the method of nonlinear mapping, and has shown by numerical observation that the diffusion coefficient in the velocity space oscillates around the value predicted by the quasi-linear theory. Antonsen et.al [5] have obtained theoretical expression of the diffusion coefficient by applying the Fourier path integral method. Karney et.al [6] carried out analysis of the correlation function and reduced the diffusion coefficient with correction terms for the quasi-linear term. Their theoretical analysis, however, is valid only for asymptotically larger wave amplitude.

Now, in the recent years, statistical properties of the two dimensional area preserving map have been investigated extensively for the case of the standard map [7]$\sim$[12]. Meiss et.al [9]$\sim$[10] have carried out renormalization calculation for principal terms of the characteristic function. Ichikawa et.al [12] have confirmed that the renormalized diffusion coefficient reproduce the numerically observed result fairly well even in the small stochastic parameter region, except in the sharp region where the accelerator modes give rise to anomalous enhanced deviation from the theoretical value.

In the low dimensional nonlinear dynamical systems, coexistence of the regular motion and chaos violates the perfect ergodicity of the phase space. Transport process in such system has been the central problem in analyzing stochastic properties of the low dimensional nonlinear dynamics. We have been studying the interplay of the accelerator orbits and the chaotic motions of particles in systems described by the standard map.

Although the lower hybrid wave heating method has been well established as one of the major technology to heat plasmas up to the level of Lawson's criterion [13], still it stands as one of the basic problem in the theoretical physics view point. The purpose of present paper is to carry out the renormalization analysis to provide the theoretical expression of diffusion coefficient, which is valid in the small wave amplitude region, and to compare the result with numerical observation. Presenting the plasma wave heating map in the second section, we will carry out the renormalization analysis in the third section. We compare our theoretical result with numerical observation in the fourth section and identify the enhanced contribution of the accelerator modes. The last section present concluding remarks.

§ 2. The Lower Hybrid Wave Heating Map

We consider the electro-static wave

$$\vec{E} = Eo\ \hat{y} \cos(k_{\perp} y - \omega t) \tag{1}$$

travelling perpendicular to the externally applied static magnetic field $\vec{B} = B_0\hat{z}$ in the z-direction. Motion of particle with mass M and charge q is described by a Hamiltonian

$$H = 1/2 \cdot r^2 - \alpha \cdot \sin(y - \nu t) \qquad 2)$$

where r, α and ν are the normalized Larmor radius, amplitude and frequency of wave, given as

$$r = \left(\frac{k_\perp}{\Omega_i}\right)^2 (v_x^2 + v_y^2) \;, \quad \alpha = \frac{k_\perp E_0}{\Omega_i B_0} \;, \quad \nu = \frac{\omega}{\Omega_i} \qquad 3)$$

respectvely. The length and time are normalized by $k_\perp^{-1}$ and the ion cyclotron frequency $\Omega_i = qB_0/M$.

Firstly, following Karney's analysis, we reduce the lower hybrid wave heating map to describe the present dynamical system. Introducing normalized magnetic moment I and gyrophase of particle w as

$$I = \frac{1}{2} r^2 \;, \qquad w = \sin^{-1}\left(\frac{y}{r}\right) \qquad 4)$$

we can express the Hamiltonian 2) as

$$H = I - \alpha \sum_{m=-\infty}^{\infty} J_m(\sqrt{2I}) \sin(mw - \nu t) \qquad 5)$$

where $J_m(x)$ is the m-th order first kind of Bessel function. Choosing n as the closest integer to the wave frequency ν, we carry out a canonical transformation with a generating function

$$F = (nw - \nu y)\, I \tag{6}$$

which gives rise to the transformed Hamiltonian

$$\tilde{H} = -\delta \tilde{I} - \alpha \sum_{m=-\infty}^{\infty} J_m\left(\sqrt{2n\tilde{I}}\right) \sin\left(\left(\frac{m}{n}\right)\tilde{w} - \left(1 - \frac{m}{n}\right)\nu t\right) \tag{7}$$

where $\delta \equiv \nu - n$ measures shift of the wave frequency from the cyclotron higher harmonics frequency. Considering the high frequency range of $n \gg 1$, we will examine the dominant contribution from the neighborhood of resonance $m \approx n$. Recalling an asymptotic expansion of the Bessel function

$$J_m(m \sec\alpha) \sim \frac{1}{\left(\frac{1}{2} m\pi \tan\alpha\right)^{1/2}} \cos\left[m\left(\tan\alpha - \alpha\right) - \frac{\pi}{4}\right] \tag{8}$$

we obtain an approximate Hamiltonian

$$\tilde{H} \simeq -\delta \tilde{I} - \alpha \sqrt{\frac{2}{\pi}} \left(r^2 - \nu^2\right)^{-1/4} \sum_{k=-\infty}^{\infty} \cos\left(R - k\phi - \delta t\right) \cdot \sin\left(\tilde{w} + kt\right) \tag{9}$$

where k is an integer, $k = m - n \ll n$, and

$$r = \sqrt{2n\tilde{I}} \tag{10.a}$$

$$R \equiv \left(r^2 - \nu^2\right)^{1/2} - \nu \cos\left(\frac{\nu}{r}\right) - \frac{\pi}{4} \tag{10.b}$$

$$\phi = \cos^{-1}\left(\frac{\nu}{r}\right) \tag{10.c}$$

Let us introduce simple transformation of variables as

$$\rho = R - \nu\pi \tag{11.a}$$

$$\theta = n\pi - w \tag{11.b}$$

Assuming change of the Larmor radius is negligible during particle interacts with wave, we may expand I into a Taylor series, and can reduce eq.9) to

$$\tilde{H} \simeq -\delta\frac{r^2}{\nu}(r^2-\nu^2)^{-1/2}\rho - \alpha\sqrt{\frac{2}{\pi}}(r^2-\nu^2)^{-1/4} \sum_{k=-\infty}^{\infty}\cos(\rho-\nu\pi-k\phi-\delta\phi)\sin(n\pi-\theta-kt) \tag{12}$$

Finally, changing the scale of energy, we obtain the Hamiltonian

$$M = -\delta\rho - A\sum_{k=-\infty}^{\infty}\cos(\rho-\nu\pi-k\phi-\delta\phi)\sin(n\pi-\theta-kt) \tag{13}$$

with

$$A \equiv \sqrt{\frac{2}{\pi}}\,\frac{(r^2-\nu^2)^{1/2}}{r^2}\,\nu\,\alpha \tag{14}$$

Here, it should be noticed that change in the Larmor radius is accounted for only through ρ, and the quantity r in the parameter A is regarded as a constant. Furthermore, the summation over k is extended over infinite sum under the assumption of smallness of nonresonnant contribution. The dynamical variable ρ is an effective Larmor radius, and the variable θ is the phase of wave at the particle gyrophase $w=\pi$.

Lastly, it is convenient to introduce auxiliary variables u and v defined as

$$u = \theta - \rho \qquad , \quad v = \theta + \rho \tag{15}$$

The Hamiltonian equation for the variables u and v are

$$\frac{d}{dt} v = \frac{\partial}{\partial u} \bar{M} \quad , \quad \frac{d}{dt} u = -\frac{\partial}{\partial v} \bar{M} \tag{16}$$

with

$$\begin{aligned} \bar{M} = 2M = \delta u + A \sin\left(u + (\pi - \phi)\delta\right) \sum_{k=-\infty}^{\infty} \cos k(t - \phi) \\ - \delta v + A \sin\left(v + (\pi - \phi)\delta\right) \sum_{k=-\infty}^{\infty} \cos k(t + \phi) \end{aligned} \tag{17}$$

Thus, we can immediately write down the Poincare map on the local surface of section at the $t = (2j-1)\pi$ as

$$u_{j+1} = u_j + 2\pi\delta - 2\pi A \cos v_j \tag{18.a}$$

$$v_{j+1} = v_j + 2\pi\delta + 2\pi A \cos u_{j+1} \tag{18.b}$$

with

$$\rho_j \equiv 1/2\ (v_j - u_j) \tag{19.a}$$

$$\theta_j \equiv 1/2\ (v_j + u_j) \tag{19.b}$$

The two dimensional nonlinear map eqs.18.a) and b) were firstly introduced by Karney to study the stochastic heating by the lower hybrid wave. The ranges of validity of eqs.18.a) and b) are specified by

$$\nu \gg 1\,, \quad \Upsilon - \nu \gg \left(\frac{1}{2}\nu\right)^{1/3}, \quad A \ll \frac{(\Upsilon^2 - \nu^2)^{3/2}}{\Upsilon^2} \tag{20}$$

The lower hybrid wave heating map eqs.18.a) and b) has peculiar structure that it is not an explicit map for the dynamical variables (ρ , θ) but a map for the auxiliary variables (u,v). The map depends on the stochastic parameter A and the parameter δ. Comparing with the standard map, we find that the both variables of u and v are subject to nonlinear evolution. In spite of such complex strdcture, however, it is straight forward to confirm that the lower hybrid wave heating map eqs.18.a) and b) is area-preserving. We have simply,

$$\frac{\partial(u_{j+1}, v_{j+1})}{\partial(u_j, v_j)} = \begin{vmatrix} 1 & 2\pi A \sin v_j \\ -2\pi A \sin u_{j+1} & 1 - 4\pi^2 A^2 \sin u_{j+1} \sin v_j \end{vmatrix} = 1 \tag{21}$$

Furthermore, orbits for the regular motion of eqs.18.a) and b) are determined by

$$u_1 - u_0 = -2n\pi \quad , \quad v_1 - v_0 = +2m\pi \tag{22}$$

which determine the coordinates of the stationary points (u_0, v_0) at

$$\delta + A\cos u_0 = m \quad , \quad \delta - A\cos v_0 = -n \tag{23}$$

Hence, the dynamical variable ρ_1 is given as

$$\rho_1 = \rho_0 + (m+n)\pi \equiv \rho_0 + s\pi \tag{24}$$

The value s=0 determines the fixed point ρ_0 , while when $s \neq 0$ the variable ρ increases with the amount $s\pi$ at each step of the mapping. Namely, ρ_0 with s=0 is the fundamental accelerator mode.

Stabibility of these regular motions are specified by the tangential map of eqs.18.a) and b). We obtain

$$\begin{pmatrix} \Delta u_{j+1} \\ \Delta v_{j+1} \end{pmatrix} = \Delta T \begin{pmatrix} \Delta u_j \\ \Delta v_j \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2\pi A \sin v_0 \\ -2\pi A \sin u_0 & 1 - 4\pi^2 A^2 \sin u_0 \sin v_0 \end{pmatrix} \begin{pmatrix} \Delta u_j \\ \Delta v_j \end{pmatrix} \tag{25}$$

of which residue R is defined as

$$R = 1/2 - 1/4 \text{ trace } (\Delta T) \tag{26}$$

The stability of the point (u_0, v_0) is determined by the condition of $0<R<1$. We can determine the stability region of the stationary orbits (u_0, v_0) as

$$\mathrm{Max}\ (|m-\delta|,\ |n+\delta|) < A < A_u \tag{27.a)}$$

$$A_u \equiv \left[\frac{1}{2}\left\{(m-\delta)^2+(n+\delta)^2+\sqrt{\left((m-\delta)^2-(n+\delta)^2\right)+\frac{4}{\pi^4}}\right\}\right]^{1/2} \tag{27.b)}$$

As for the fundamental accelerator mode, even if s is specified, combination of m and n leads to occurence of an infinite number of the fundamental accelerator mode in the vicinity of

$$A = N \pm \delta \tag{28)}$$

where N is the natural number. This is due to the fact that both variables u and v are subject to the nonlinear mapping. Eq.27.a) and b) indicate that the stability range is getting narrower for the larger value of m and n, so the lower (m, n) mode will manifest its dominant effect upon the diffusion process.

§ 3. Correlation Function and Diffusion Corfficient

Now, assuming that the entire phase space is ergodic, we investigate stochastic properties of the two dimensional nonlinear area-preserving map, eqs.18.a) and b). Regarding the auxiliary variables (u,

v) as probabilistic variables, we define the diffusion coefficient in the velocity space as

$$D \equiv \lim_{T\to\infty} \frac{1}{2T} \left\langle \left(\rho_T - \rho_0 \right)^2 \right\rangle \tag{29}$$

where < > is an ensemble average of particles in the entire phase space. Since the map eqs.18.a) and b) is periodic in the both variables u and v, we can define the average < > as

$$\langle F \rangle \equiv \frac{1}{4\pi^2} \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} F(u_0, v_0)\, du_0\, dv_0 \tag{30}$$

Now, one iteration of mapping gives rise to the increment $\Delta\rho_j$, given as

$$\Delta\rho_j = \frac{1}{2}\left(\Delta u_j - \Delta v_j\right) = \pi A \left(\cos u_{j+1} + \cos v_j\right) \tag{31}$$

The diffusion coefficient eq.27), is expressed in terms of $\Delta\rho_j$ as

$$\begin{aligned} D &= \frac{1}{2} C_0 + \lim_{T\to\infty} \sum_{\tau=1}^{T-1} \left(1 - \frac{\tau}{T}\right) C_\tau \\ &\simeq \frac{1}{2} C_0 + \sum_{\tau=1}^{\infty} C_\tau \end{aligned} \tag{32}$$

with

$$C_\tau \equiv \left\langle \Delta\rho_{T_0}\, \Delta\rho_{T_0+\tau} \right\rangle \tag{33}$$

The correlation function of acceleration C_τ is independent of the initial time To as far as the phase space is ergodic. In deriving the last expression of eq.30), use has been made of an assumption that C_τ decays rapidly with τ. The correlation function C_τ can be calculated from the characteristic function

$$\chi_{J,K}(m_0, m_1, \cdots m_J ; n_0, n_1, \cdots n_K) \equiv \left\langle \exp\left(\sum_{j=0}^{J} i\, m_j\, u_{T_0+j} + \sum_{k=0}^{K} i\, n_k\, v_{T_0+k} \right) \right\rangle \qquad 34)$$

as

$$C_0 = 1/2\pi^2 A^2 \mathrm{Re}\, [2\chi_{0,0}(0;0) + 2\chi_{0,0}(1;1) + \chi_{0,0}(2;0)$$

$$+ \chi_{0,0}(0;2) + \chi_{0,0}(1;-1) + \chi_{0,0}(-1;1)\,] \qquad 35)$$

$$C = 1/2\pi^2 A^2 \mathrm{Re}\, [\chi_{\tau,\tau}(1,0,\cdots,0,1;0,\cdots,0)$$

$$+ \chi_{\tau,\tau}(1,0,\cdots 0,-1;0,\cdots 0) + \chi_{\tau,\tau}(0,\cdots,0;1,0,\cdots,0,1)$$

$$+ \chi_{\tau,\tau}(0,\cdots 0;1,0,\cdots 0,-1) + \chi_{\tau,\tau}(1,\cdots,0;0,\cdots 0,1)$$

$$+ \chi_{\tau,\tau}(1,0,\cdots 0;0,\cdots,0,-1) + \chi_{\tau,\tau}(0,\cdots 0,1;1,0,\cdots 0)$$

$$+ \chi_{\tau,\tau}(0,0,\cdots,-1;1,0,\cdots 0)] \qquad 36)$$

The characteristic function specifies statistical properties of the system described by the given nonlinear map.

Since the probability variables (u_J, v_J) are related each other by the nonlinear map of eqs.18.a) and b), we can reduce the rank of the characteristic function. Using eq.18.b), we can express $\chi_{J,J}$ as

$$\chi_{J,J}(m_0,\cdots,m_J;n_0,\cdots,n_J)$$

$$= \sum_{\ell=-\infty}^{\infty} J_\ell(2\pi n_J A)\exp i(2\pi n_J \delta + 1/2\ell\pi)$$

$$\cdot \chi_{J,J-1}(m_0,\cdots m_J+\ell;n_0,\cdots n_{J-2},n_{J-1}+n_J) \qquad 37)$$

then using eq.18.a), we can reduce $\chi_{J,J-1}$ as

$$\chi_{J,J-1}(m_0,\cdots,m_J;n_0,\cdots,n_{J-1})$$

$$= \sum_{k=-\infty}^{\infty} J_k(2\pi m_J A)\exp i(2\pi m_J \delta - 1/2k\pi)$$

$$\cdot \chi_{J-1,J-1}(m_0,\cdots,m_{J-2},m_{J-1}+m_J;n_0,\cdots,n_{J-1}+k) \qquad 38)$$

Combining eqs.35) and 36), we obtain the recurrence formula

$$\chi_{J,J}(m_0,\cdots,m_{J-1},m_J;n_0,\cdots,n_{J-1},n_J)$$

$$= \sum_{\ell=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} J_l(2\pi n_J A) J_k(2\pi(m_J+\ell)A)\ \exp i\ [2\pi(n_J+m_J+\ell)\delta + 1/2(\ell-k)\pi]$$

$$\cdot\ \chi_{J-1,J-1}(m_0,\cdots,m_{J-2},m_{J-1}+m_J+\ell;n_0,\cdots,n_{J-2},n_{J-1}+n_J+k) \qquad 39)$$

In order to calculate the correlation function C_τ, applying eq.39) repeatedely, we obtain

$$\chi_{\tau,\tau}(l_{\tau+1},0,\cdots,0,l_0;k_{\tau+1},0,\cdots,0,k_0)$$

$$=\sum_{l_1=-\infty}^{\infty}\cdot\cdot\sum_{l_\tau=-\infty}^{\infty}\sum_{k_1=-\infty}^{\infty}\cdot\cdot\sum_{k_\tau=-\infty}^{\infty}\cdot\prod_{n=0}^{\tau-1}J_{-l_n+l_{n+1}}(2\pi k_n A)$$

$$J_{-k_n+k_{n+1}}(2\pi l_{n+1}A)\exp i\ [2\pi\sum_{j=0}^{\tau-1}(l_{j+1}+k_j)\delta$$

$$+\pi/2(l_\tau-l_0-k_\tau+k_0)]\chi_{0,0}(l_{\tau+1}+l_\tau;k_{\tau+1}+k_\tau) \qquad 40)$$

Now, the characteristic function $\chi_{0,0}(m_0,n_0)$ is evaluated as

$$\chi_{0,0}(m_0,0)=\delta_{m_0,0}\ \delta_{n_0,0} \qquad 41)$$

upon the use of the definition of eq.30) for eq.34). $\delta_{j,k}$ is the Kronecker's δ symbol. Thus, we get the following restrictions for the indicies l_s and k_t,

$$l_\tau=-l_{\tau+1},\qquad k_\tau=-k_{\tau+1} \qquad 42)$$

We can show explicitly that eq.41) corresponds to the random phase approximation with respect to the initial value (u_0, v_0). Equation 35) gives rise to the value of $C_0=\pi^2A^2$. The diffusion coefficient in the random phase approximation D_Q is

$$D_Q = 1/2\ C_0 = 1/2\ \pi^2 A^2 \tag{43)}$$

which is nothing but the quasi-linear diffusion coeficient. The next order correlation C_1 is reduced from the characteristic function

$$\chi_{1,1}(l_2, l_0; k_2, k_0) = J_{-\ell_0-\ell_2}(2\pi k_0 A) J_{-k_0-k_2}(-2\pi l_2 A)$$

$$\exp i[2\pi(k_0 - l_2)\delta + \pi/2(k_0 + k_2 - l_0 - l_2)] \tag{44)}$$

as

$$D_1 \equiv C_1 = \pi^2 A^2 [\ J_0(2\pi A)\cos 2\pi\delta - J_1{}^2(2\pi A)\sin^2 2\pi\delta\ ] \tag{45)}$$

which is nothing but the result obtained by Antonsen et.al [5] and Karney et.al [6]. Similarly, we can calculate the correlation C_2, assuming that triple product of the Bessel functions are sufficiently small,as follows,

$$D_2 = C_2 \approx \pi^2 A^2 [\ J_0{}^2(2\pi A)\cos 4\pi\delta - J_1{}^2(2\pi A)\sin^2 2\pi\delta\ ] \tag{46)}$$

Equation 46) is the principal term of eq.12) in the reference 5.

Now, let us turn to evaluate principal tems of the correlation C_τ with arbitrary values of τ. Here, since the Bessel functions are getting asymtotically small for the larger indicies as shown by eq.8), we identify the principal terms as the tersm reduced by constructing a factor of $J_0(0)$ as many as possible in the characteristic function in eq.40). We will examine contributions of the eight components of the characteristic function of C as given by eq.36).

i) $\chi^{(1)} \equiv X_{\tau,\tau}(1,0,\cdots,0,1;0,\cdots,0)$, $\ell_0 = \ell_{\tau+1} = 1$, $k_0 = k_{\tau+1} = 0$.

Equation 42) gives $\ell_\tau = -1$, and $k_\tau = 0$. We can express $\chi^{(1)}$ as

$$\chi^{(1)} = \sum_{\ell_1=-\infty}^{\infty}\cdots\sum_{\ell_{\tau-1}=-\infty}^{\infty}\sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_{\tau-1}=-\infty}^{\infty} J_{-1+\ell_1}(0)\ J_{k_1}(2\pi\ell_1 A)$$

$$\cdot\ J_{-\ell_{\tau-1}-1}(2\pi k_{\tau-1}A)J_{-k_{\tau-1}}(-2\pi A)\cdot\prod_{n=1}^{\tau-2} J_{-\ell_n+\ell_{n+1}}(2\pi k_n A)$$

$$J_{-k_n+k_{n+1}}(2\pi\ell_{n+1}A)\exp i[2\pi\sum_{j=0}^{\tau-1}(\ell_{j+1}+k_j)\delta-\pi] \qquad 47)$$

The first factor in eq.47) determines $\ell_1=+1$. Remaining indicies (ℓ_2, $\cdots$, $\ell_{\tau-1}$) and ($k_1,\cdots,k_{\tau-1}$) are set equal to zero as possible as many, while the other indicies are assinged to be ±1. In general, if we set k_n and ℓ_n or k_n and ℓ_{n+1} to be zeto at the same time, eq.47) vanishes identically, so we have the following restrictions

$$\text{if } k_n=0,\ \text{then } \ell_n=\ell_{n+1}=0$$
$$\text{if } \ell_{n+1}=0,\ \text{then } k_n=k_{n+1}=0 \qquad 48)$$

Hence, among the indicies ℓ_n, we set all the indicies between $\ell_{n'+1}$ to $\ell_{n''}$ equal to zero, and remaining indicies k equal to zero and thus we obtain

$$\ell_{n_1+1} = \cdots = \ell_{n_2} = 0 \longrightarrow k_{n_1} = \cdots = k_{n_2} = \pm 1$$
$$k_1 = \cdots = k_{n_1-1} = 0 \longrightarrow \ell_1 = \cdots = \ell_{n_1} = 1$$

$$k_{n_2} = \cdots = k_{\tau-1} = 0 \longrightarrow \ell_{n_2+1} = \cdots = \ell_{\tau-1} = -1 \qquad 49)$$

so that total of (τ-2) Bessel functions are reduced to $J_0(0)=1$. Various choice of n_1 and n_2 gives rise to (τ-1)(τ-2) ways of combinations of eq.49). Thus, the principal term $\chi^{(1)}{}_{P1}$ is expressed as

$$\chi^{(1)}{}_{P1} \equiv -J_1{}^4(2\pi A)J_0{}^{(\tau-3)}(2\pi A)[1+e^{-i(\tau-1)4\pi\delta}+(\tau-3)\sum_{n=0}^{\tau-1} e^{-in4\pi\delta}]\, e^{i(\tau-1)2\pi\delta}$$

$$=J_1{}^3(2\pi A)\frac{d}{d(2\pi A)}[J_0(2\pi A)]^{(\tau-2)}\frac{\sin(2\pi\tau\delta)}{\sin(2\pi\delta)}$$

$$+J_1{}^4(2\pi A)J_0{}^{(\tau-3)}(2\pi A)\frac{\sin(2\pi(\tau-2)\delta)}{\sin(2\pi\delta)} \qquad 50)$$

with $\tau \geq 3$.

Another possible choice is to assign ±1 to the indicies of ℓ_n. Then, having

$$k_1 = \cdots = k_{n-1} = 0 \longrightarrow \ell_1 = \cdots = \ell_n = 1$$
$$k_{n+1} = \cdots = k_{\tau-1} = 0 \longrightarrow \ell_{n+1} = \cdots = \ell_{\tau-1} = -1$$
$$-\ell_n + \ell_{n+1} = -2 \longrightarrow k_n = \pm 1 \qquad 51)$$

we can set (τ-2) Bessel functions to be $J_0(0)=1$. We have 2(τ-1) ways of combination, obtaining the corresponding principal term $\chi^{(1)}{}_{P2}$ as

$$\chi^{(1)}{}_{P2} = -J_2(2\pi A)J_1{}^2(2\pi A)J_0{}^{(\tau-2)}(2\pi A)[1+e^{-i4\pi(\tau-1)\delta}$$

$$+ 2\sum_{n=1}^{\tau-2} e^{-i4\pi n\delta}]e^{i2\pi(\tau-1)\delta}$$

$$= -J_2(2\pi A)J_1^2(2\pi A)J_0^{(\tau-2)}(2\pi A)2\cot(2\pi\delta)\sin[2\pi(\tau-1)\delta] \qquad 52)$$

The principal term of $\chi^{(1)}$ is composed of $\chi^{(1)}{}_{P1}$, eq.50) and $\chi^{(1)}{}_{P2}$, eq.52).

ii) $\chi^{(2)} \equiv \chi_{\tau,\tau}(1,0,\cdots,0,-1,\cdots,0)$, $\ell_0=-1$, $\ell_{\tau+1}=1$, $k_0=k_{\tau+1}=0$.

Having $\ell_\tau=-1$ and $k_\tau=0$, we can express $\chi^{(2)}$ as

$$\chi^{(2)} = \sum_{\ell_1=-\infty}^{\infty} \cdots \sum_{\ell_{\tau-1}=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_{\tau-1}=-\infty}^{\infty} J_{1+\ell_1}(0) J_{k_1}(2\pi \ell_1 A)$$

$$J_{-\ell_{\tau-1}-1}\cdot(2\pi k_{\tau-1}A) J_{-k_{\tau-1}}(-2\pi A)\cdot \prod_{n=1}^{\tau-2} J_{-\ell_n+\ell_{n+1}}\cdot(2\pi k_n A)$$

$$J_{-k_n+k_{n+1}}(2\pi \ell_{n+1} A)\exp i[2\pi \sum_{j=0}^{\tau-1}(\ell_{j+1}+k_j)\delta] \qquad 53)$$

which determines $\ell_1=-1$, In this case, setting

$$\ell_1 = \cdots = \ell_{\tau-1} = -1 \text{ and then } k_1 = \cdots = k_{\tau-1} = 0 \qquad 54)$$

we reduce $(\tau-1)$ Bessel functions to $J_0(0)=1$. If we takd $\ell_j=0$ or $\ell_j=1$, we obtain only $\tau-2$ factors of $J_0(0)=1$. Hence the principal term of $\chi^{(2)}$ is reduced to

$$\chi^{(2)}{}_p = (J_0(2\pi A)e^{-i2\delta})^\tau \qquad 55)$$

iii) $\chi^{(3)} \equiv \chi_{\tau,\tau}(0,\cdots,0\ ;\ 1,0,\cdots,0,1)$, $\ell_0=\ell_{\tau+1}=0$, $k_0=k_{\tau+1}=1$

Equation 42) gives $\ell_\tau=0$ and $k_\tau=-1$. We get

$$\chi^{(3)} = \sum_{\ell_1=-\infty}^{\infty} \cdots \sum_{\ell_{\tau-1}=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_{\tau-1}=-\infty}^{\infty} J_{\ell_1}(2\pi A) J_{-1+k_1}(2\pi \ell_1 A)$$

$$J_{-l_{\tau-1}}(2\pi k_{\tau-1}A) J_{-k_{\tau-1}-1}(0) \cdot \prod_{n=1}^{\tau-2} J_{-\ell_n+\ell_{n+1}}(2\pi k_n A)$$

$$J_{-k_n+k_{n+1}}(2\pi \ell_{n+1}A) \exp i[2\pi \sum_{j=1}^{\tau-1} (\ell_{j+1}+k_j)\delta+\pi] \qquad 56)$$

which is nothing but the expression obtained from eq.47) by the replacement of $\ell_j \rightarrow -k_{\tau-j}$ and $k_j \longrightarrow -\ell_{\tau-j}$. Hence the principal term of eq.56) is given as the sum of $X^{(1)}{}_{P1}$, eq.50) and $\chi^{(1)}{}_{P2}$, eq.52).

iv) $\chi^{(4)} \equiv \chi_{\tau,\tau}(0,\cdots,0\ ;\ 1,0,\cdots,0,-1)$; $\ell_0=\ell_{\tau+1}=0$, $k=-1$, $k_{\tau+1}=1$

Equation 42) gives $\ell_\tau=0$ and $k_\tau=-1$. We have

$$\chi^{(4)} = \sum_{\ell_1=-\infty}^{\infty} \cdots \sum_{\ell_{\tau-1}=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_{\tau-1}=-\infty}^{\infty} J_{\ell_1}(-2\pi A) J_{1+k_1}(2\pi \ell_1 A)$$

$$J_{-\ell_{\tau-1}}(2\pi k_{\tau-1}A) J_{-k_{\tau-1}-1}(0) \cdot \prod_{n=1}^{\tau-2} J_{-\ell_n+\ell_{n+1}}(2\pi k_n A)$$

$$J_{-k_n+k_{n+1}}(2\pi \ell_{n+1}A) \exp i[2\pi \sum_{j=1}^{\tau-1} (\ell_{j+1}+k_j)\delta \qquad 57)$$

which determines $k_{\tau-1}=-1$. In this case, setting

$$k_1 = \cdots = k_{\tau-1} = -1 \text{ and then } \ell_1 = \cdots = \ell_{\tau-1} = 0 \qquad 58)$$

we reduce $(\tau-1)$Bessel functions to $J_0(0)=1$, and obtain the principal term of $\chi^{(4)}$ as given by eq.55),

$$\chi^{(4)}{}_P = \chi^{(2)}{}_P \qquad 59)$$

v) $\chi^{(5)} \equiv \chi_{\tau,\tau}(1,0,\cdots,0\ ;\ 0,\cdots,0,1)$; $\ell_0=0,\ \ell_{\tau+1}=1,\ k=1,\ k_{\tau+1}=0$

Equation 42) gives $\ell_\tau=-1$ and $k_\tau=0$. We have

$$\chi^{(5)} = \sum_{\ell_1=-\infty}^{\infty}\cdots\sum_{\ell_{\tau-1}=-\infty}^{\infty}\ \sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_{\tau-1}=-\infty}^{\infty} J_{\ell_1}(2\pi A)J_{-1+k_1}(2\pi\,\ell_1 A)$$

$$J_{-\ell_{\tau-1}-1}(2\pi k_\tau\ A)J_{-k_{\tau-1}}(-2\pi A)\prod_{n=1}^{\tau-2} J_{-\ell_n+\ell_{n+1}}(2\pi k\ A)$$

$$J_{-k_n+k_{n+1}}(2\pi\,\ell_{n+1}A)\exp i[2\pi\sum_{j=0}^{\tau-1}(\ell_{j+1}+k_j)\delta] \qquad 60)$$

Observing the indicies of four Bessel functions outside of the product terms, we can reduce (τ-1) Bessel functions to $J_0(0)=1$ by the choice of

$$\ell_1 = \cdots = \ell_{\tau-1} = 0 \text{ and, then } k_1 = \cdots = k_{n-1} = 1$$

$$k_n = \cdots = k_{-1} = 0 \text{ and, then } \ell_n = \cdots = \ell_{\tau-1} = -1 \qquad 61)$$

Since there is τ ways of this combination, the principal term of $\chi^{(5)}$ is given as

$$\chi^{(5)}{}_P = -J_1{}^2(2\pi A)J_0{}^{(\tau-1)}(2\pi A)e^{i2\pi(\tau-1)\delta}\sum_{n=0}^{\tau-1} e^{-in4\pi\delta}$$

$$= -J_1{}^2(2\pi A)J_0{}^{(\tau-1)}(2\pi A)\ \frac{\sin(2\pi\tau\delta)}{\sin(2\pi\delta)} \qquad 62)$$

vi) $\chi^{(6)} \equiv \chi_{\tau,\tau}(1,0,\cdots,0\ ;\ 0,\cdots,0,-1)$; $\ell_0=0,\ \ell_{\tau+1}=1,\ k_0=-1,\ k_{\tau+1}=0$. Having $\ell_\tau=-1$ and $k_\tau=0$ from eq.42), we get

$$\chi^{(6)} = \sum_{l_1=-\infty}^{\infty} \cdots \sum_{\ell_{\tau-1}=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_{\tau-1}=-\infty}^{\infty} J_{\ell_1}(-2\pi A) J_{1+k_1}(2\pi \ell_1 A)$$

$$J_{-l_{\tau-1}-1}(2\pi k_{\tau-1} A) J_{-k_{\tau-1}}(-2\pi A) \cdot \prod_{n=1}^{\tau-2} J_{-\ell_n+\ell_{n+1}}(2\pi k_n A)$$

$$J_{-k_n+k_{n+1}}(2\pi \ell_{n+1} A) \exp i[2\pi \sum_{j=0}^{\tau-1} (\ell_{j+1}+k_j)\delta - \pi] \tag{63}$$

From the structure of indicies of the four Bessel functions, we can reduce $(\tau-1)$ Bessel functions to $J_0(0)=1$. Having τ ways of combination, we can write down the principal ferm of $\chi^{(6)}$ as.

$$\chi^{(6)}{}_P = \tau J_1{}^2(2\pi A) J_0{}^{(\tau-1)}(2\pi A) e^{-i2\pi(\tau+1)\delta}$$

$$= -J_1(2\pi A) \frac{d}{d(2\pi A)} (J_0(2\pi A))^{\tau} e^{-i2\pi(\tau+1)\delta} \tag{64}$$

viii) $\chi^{(7)} \equiv \chi_{\tau,\tau}(0,\cdots,0,1\ ;\ 1,0,\cdots,0)$; $\ell_0=1$, $\ell_{\tau+1}=0$, $k_0=0$, $k_{\tau+1}=1$.

Equation 42) gives $\ell_\tau=0$ and $k_\tau=-1$. We have

$$\chi^{(7)} = \sum_{\ell_1=-\infty}^{\infty} \cdots \sum_{\ell_{\tau-1}=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_{\tau-1}=-\infty}^{\infty} J_{-1+\ell_1}(0) J_{k_1}(2\pi \ell_1 A)$$

$$J_{-\ell_{\tau-1}}(2\pi k_{\tau-1} A) J_{-k_{\tau-1}-1}(0) \cdot \prod_{j=0}^{\tau-1} J_{-\ell_n+\ell_{n+1}}(2\pi k_n A)$$

$$J_{-k_n+k_{n+1}}(2\pi \ell_{n+1} A) \exp i[2\pi \sum_{j=0}^{\tau-1} (\ell_{j+1}+k_j)\delta] \tag{65}$$

which determines $\ell_1=+1$ and $k_{\tau-1}=-1$. Eq.65) is further reduced to

$$\chi^{(7)} = \sum_{\ell_1=-\infty}^{\infty} \cdots \sum_{\ell_{\tau-1}=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_{\tau-1}=-\infty}^{\infty} J_{-1+\ell_2}(2\pi k_1 A) J_{k_1}(2\pi A)$$

$$J_{-1_{\tau-1}}(2\pi A) J_{-k_{\tau-2}-1}(2\pi \ell_{\tau-1} A) \cdot \prod_{n=1}^{\tau-3} J_{-\ell_n+\ell_{n+1}}(2\pi k_{n+1} A)$$

$$J_{-k_n+k_{n+1}}(2\pi \ell_{n+1} A) \exp i[2\pi \sum_{j=0}^{\tau-2} (\ell_{j+1}+k_{j+1})\delta] \qquad 66)$$

which is nothing but the expression of $\chi_{\tau-1,\tau-1}(1,0,\cdots,0\ ;\ 0,\cdots,0,1)$ with the relacement of $\ell_n \longrightarrow k_n$ and $k_n \longrightarrow \ell_{n+1}$. Therefore, referring to eq.2), we get

$$\chi^{(7)}{}_P = -J_1{}^2(2\pi A) J_0{}^{(\tau-2)}(2\pi A)\ \frac{\sin(2\pi(\tau-1)\delta)}{\sin(2\pi\delta)} \qquad 67)$$

viii) $\chi^{(8)} \equiv \chi_{\tau,\tau}(0,\cdots,0,-1,\ ;\ 1,0,\cdots,0)$; $\ell_0=-1, \ell_{\tau+1}=0,\ k_0=0,\ k_{\tau+1}=1.$

Equation 42) gives $\ell_\tau=0$ and $k_\tau=-1$. We have

$$\chi^{(8)} = \sum_{\ell_1=-\infty}^{\infty} \cdots \sum_{\ell_{\tau-1}=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_{\tau-1}=-\infty}^{\infty} J_{1+\ell_1}(0) J_{k_1}(2\pi \ell_1 A)$$

$$J_{-1_{\tau-1}}(2\pi k_{\tau-1} A) J_{-k_{\tau-1}}(0) \cdot \prod_{n=1}^{\tau-2} J_{-\ell_n+\ell_{n+1}}(2\pi k_n A)$$

$$J_{-k_n+k_{n+1}}(2\pi \ell_{n+1} A) \exp i[2\pi \sum_{j=0}^{\tau-1} (\ell_{j+1}+k_j)\delta+\pi] \qquad 68)$$

which determines $\ell_1=-1$ and $k_{\tau-1}=-1$. Equation 68) is reduced further to

$$\chi^{(8)} = \sum_{\ell_1=-\infty}^{\infty} \cdots \sum_{\ell_{\tau-1}=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_{\tau-1}=-\infty}^{\infty} J_{1+\ell_2}(2\pi k_1 A) J_{k_1}(-2\pi A)$$

$$J_{-1_{\tau-1}}(-2\pi A) J_{-k_{\tau-2}-1}(2\pi \ell_{\tau-1} A) \cdot \prod_{n=1}^{\tau-3} J_{-\ell_{n+1}+\ell_{n+2}}(2\pi k_{n+1} A)$$

$$J_{-k_n+k_{n+1}}(2\pi \ell_{n+1}A)\exp i[2\pi \sum_{j=0}^{\tau-2} (\ell_{j+1}+k_{j+1})\delta+\pi] \qquad 69)$$

which is nothing but the expression of $\chi_{\tau-1,\tau-1}(1,0,\cdots,0\ ;\ 0,\cdots,0,-1)$ with the replacement of $\ell_n \longrightarrow k_n$ and $k_n \longrightarrow \ell_{n+1}$. Therefore, referring to eq.64), we get

$$\chi^{(8)}{}_P = -J_1(2\pi A)\ \frac{d}{d(2\pi A)}(J_0(2\pi A))^{\tau-1}e^{-i2\pi\tau\delta} \qquad 70)$$

Summerizing the results obtained in the above, we can determine the principal terms of the correlation C_τ from eq.36) and by carrying out the summation over $C\tau$ in eq.32), we obtain the renormalized diffusion coefficient as

$$\frac{D}{D_Q} = \Delta(A,\delta)^{-1}\{1-J_0{}^2(2\pi A)-4J_2(2\pi A)J_1{}^2(2\pi A)\cos 2\pi\delta\}$$

$$-\Delta(A,\delta)^{-2}\{4J_1{}^2(2\pi A)\sin^2 2\pi\delta$$

$$-4J_1{}^4(2\pi A)(J_0(2\pi A)\cos 2\pi\delta-\cos 4^2\delta)\} \qquad 71)$$

with the abbreviation of

$$\Delta(A,\delta) = 1-2J_0(2\pi A)\cos 2\pi\delta+J_0{}^2(2\pi A) \qquad 72)$$

where $D_Q=\pi^2A^2/2$ is the quasilinear diffusion coefficient. We may disregard the product terms with three Bessel functions in the numerator of eq.71), giving rise to the expression of

$$\frac{D}{D_Q} = \Delta(A,Q)^{-1}(1-J_0{}^2(2\pi A))-\Delta(A,Q)^{-2}4J_1{}^2(2\pi A)\sin^2 2\pi\delta \qquad 73)$$

When $2\pi A \gg 1$, assuming the smallness of the Bessel function, we may reduce eq.73) to the approximate expression of

$$\frac{D}{D_Q} \approx 1+2J_0(2\pi A)\cos 2\pi\delta+2J_0{}^2(2\pi A)\cos 4\pi\delta-4J_1{}^2(2\pi A)\sin^2 2\pi\delta \qquad 74)$$

which recovers the results obtained as eqs.43), 44) and 45). It should be noticed that eq.74) indicates critical improvement over the results given by Antonsen et.al [5] and Karney et.al [6] who gave the following expression for the diffusion coefficient,

$$\frac{D}{D_Q} \approx 1+2J_0(2\pi A)\cos 2\pi\delta-2J_1{}^2(2\pi A)\sin^2 2\pi\delta \qquad 75)$$

Even in the valid region of $2\pi A \gg 1$, eq.75) can not account correctly the diffusion process for the phase parameter of $\delta \approx 1/4$.

We conclude the present section by emphasizing that the renormalized diffusion coefficient given as eq.73) is expected to be valid down to the small wave amplitude region over the entire range of phase parameter δ.

§ 4. Comparison with Numerical Observation

In order to confirm our expectation that the renormalized diffusion coefficient, eq.73), describes correctly the statistical properties of the lower hybrid wave heating phenomena, we will carry out some detailed comparison of the numerical observation and the theoretical prediction. Karney [4] has reported the diffusion coefficient of the lower hybrid wave heating map on the basis of numerical observation of the correlation function C_τ. We carried out the direct measurement of the diffusion coefficient by distributing 100x100 particle uniformly over the region of $[-\pi,\pi]x[-\pi,\pi]$ in the (u,v) phase plane. We proceeded up 1000 time steps to measure the diffusion coefficient. We illustrate in Fig.1.a), b) and c) the observed diffusion coefficient for the phase parameter δ=0,11, 0.23 and 0.47, respectively. The real lines are the theoretical prediction of eq.73). The crosses indicate the estimated values from the data presented by Karney [4].

In these figures, we indicated by the vertical lines where the accelerator modes exist. In particular, the hatched regions indicate the region where the fundamental accelerator modes exist. In Fig.1.c), in the range of $0.53 \leq A \leq 0.60$, we can not determine the diffusion coefficient because of the contribution of accelerating particles. In the same region, Karney [4] observed sharp enhancement of the diffusion, of which numerical values are indicated in the outside of the frame. The numerical observation confirms our expectation that the renormalized diffusion coefficient is valid down to the relatively small value of the stochastic parameter A except for the small value of the phase parameter δ. We reserve ourselves to account for the discrepancy of the observation and the theoretical result at the δ=0.11. At the intermediate value of $\delta \approx 1/4$, the second term of eq.72) diminishes its contribution, and thus gives rise to suppression of the amplitude of deviation from the quasi-linear diffusion coefficient. In Fig.2, we indicate the asymptotic behaviour of the diffusion coefficient in the region of $2\pi A \gg 1$. Here, we observe that the case of $\delta=0.23$ shows asymmetric oscillation around the quasi-linear limit, which indicates effect of the of $J_0^2(2\pi A)$ term.

Generalizing method of Meiss et.al, we have obtained the renormalized diffusion coefficient in the lower hybrid wave heating process, which is valid in wide range of the wave amplitude and frequency. Comparison with the numerical observation indicates that the accelerator modes cause the anomalous enhancement of the diffusion process. We will discuss the important problem of the interplay of the accelerator modes and stochastic behaviors of the low dimensional nonlinear dynamical systems in the separate paper.

In conclusion, we are oblidged to Professor Cary and Dr. Karney for their stimulating discussions at the occasion of the US-Japan JIFT workshop on the Low-Dimensional Nonlinear Dynamics and Applications to Plasma Physics at Boulder, held during July 24-28, 1989.

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Captions of Figures

Fig.1. Comparison of numerical observation and theoretical values of the diffusion coefficient for a) $\delta=0.11$, b) $\delta=0.23$ and c) $\delta=0.47$. The black circles indicate the present results for 10^3 time steps evolution of 10^4 particles uniformly distributed in the (u,v) plane. The crosses are the converted points from Karney's result. The vertical arrows indicate the range where the stable accelerator mode exists.

Fig.2 Asymptotic behaviour of the diffusion coefficient in the region of $2\pi A \gg 1$.

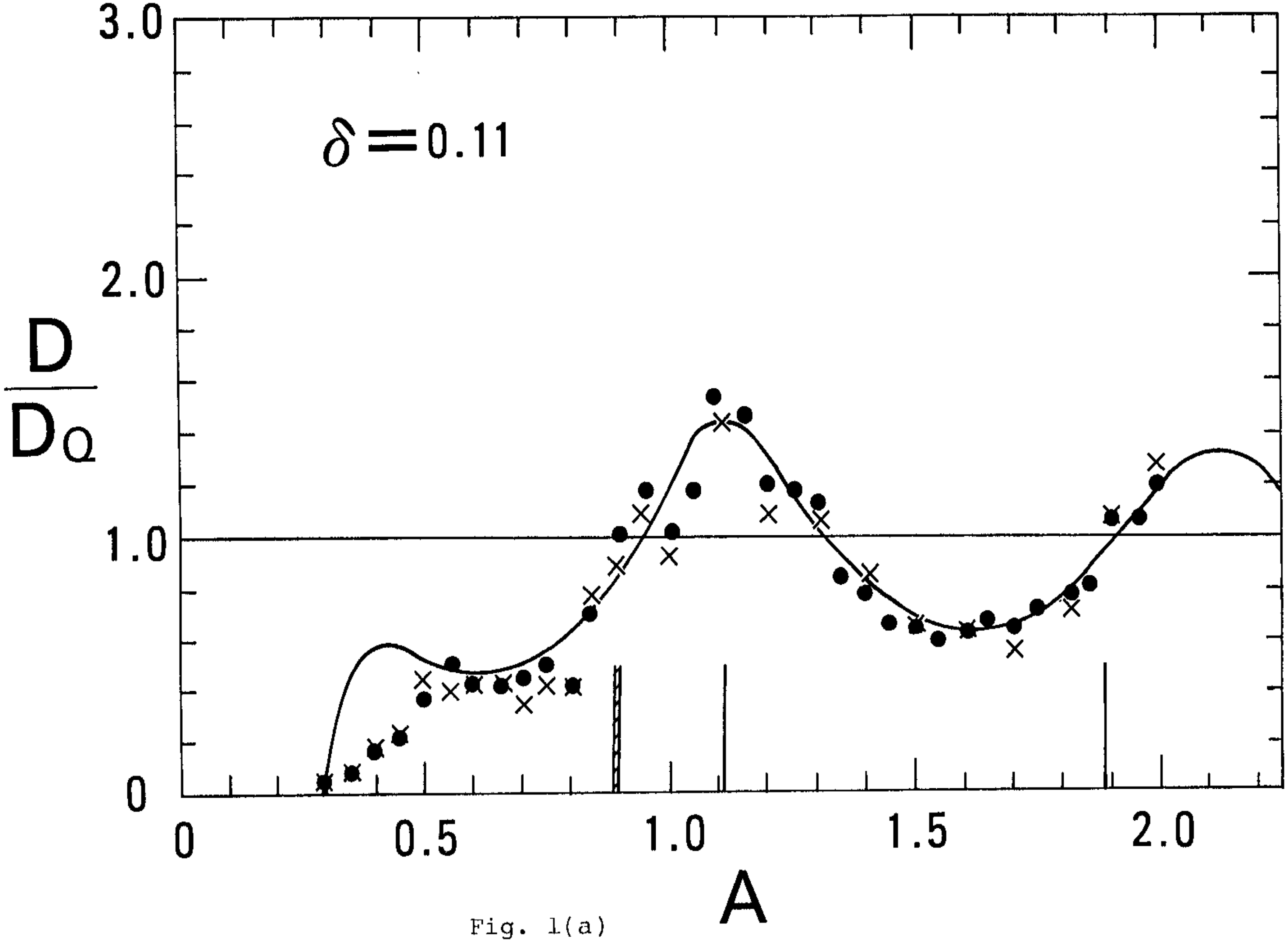

Fig. 1(a)

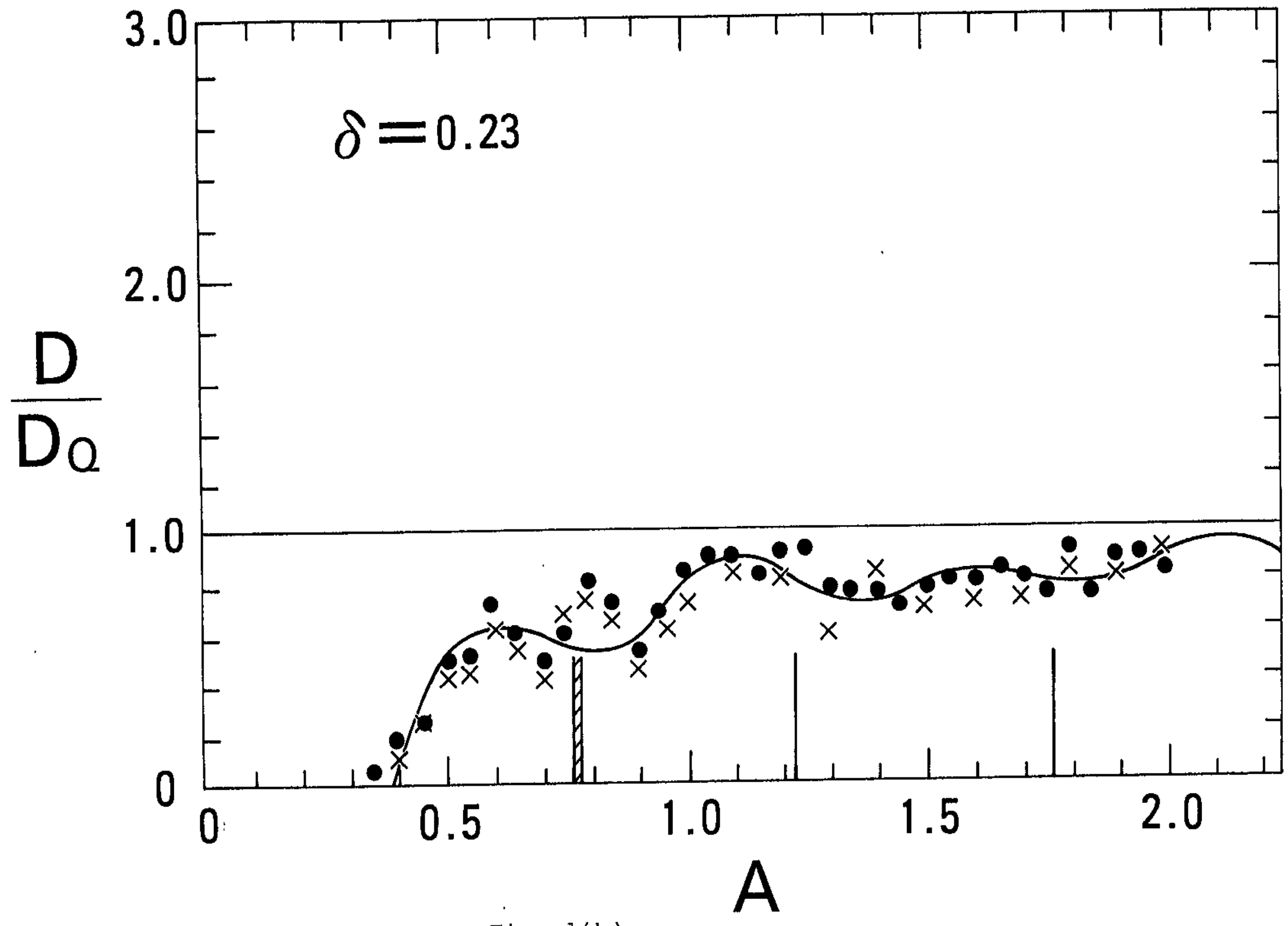

Fig. 1(b)

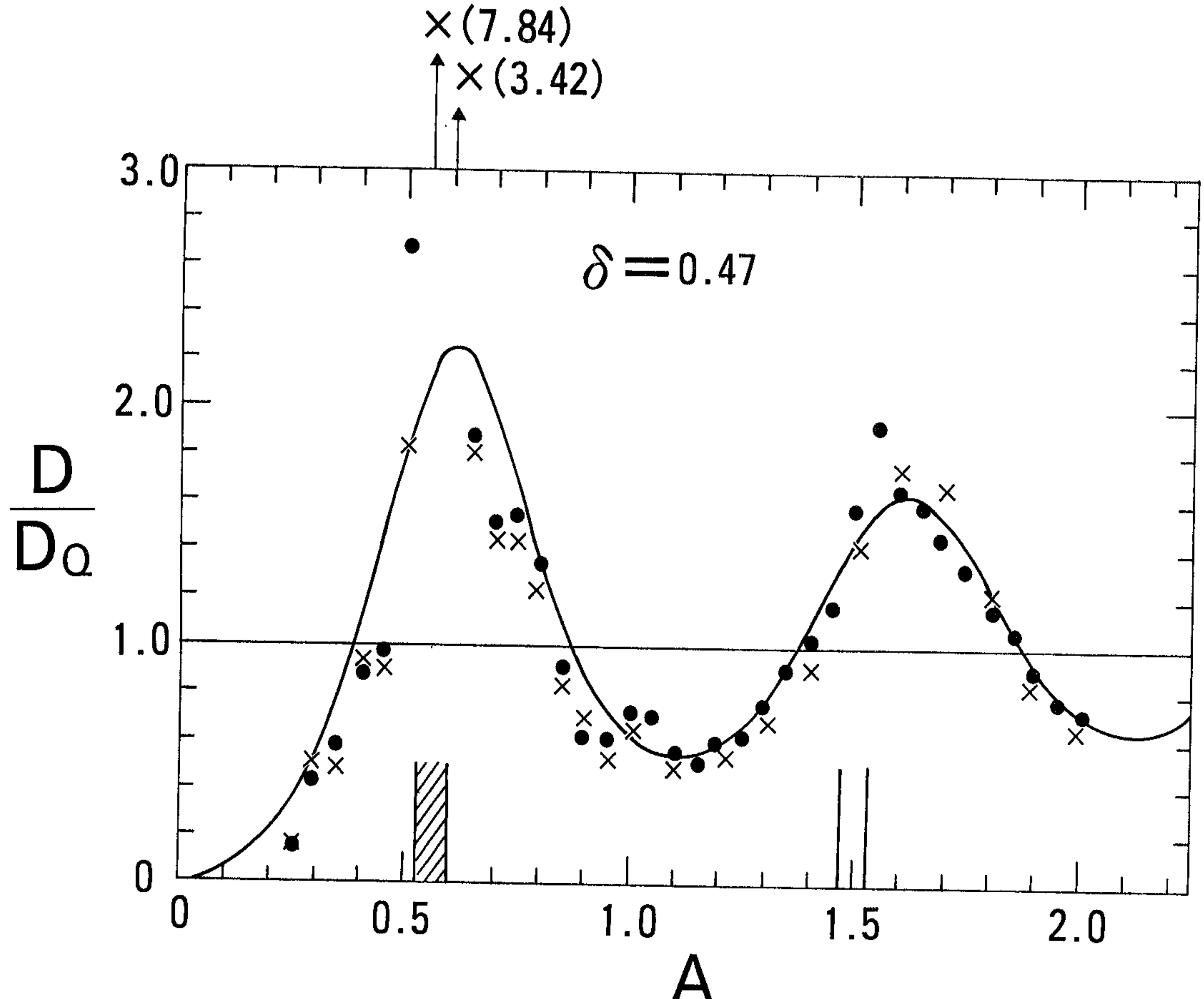

Fig. 1(c)

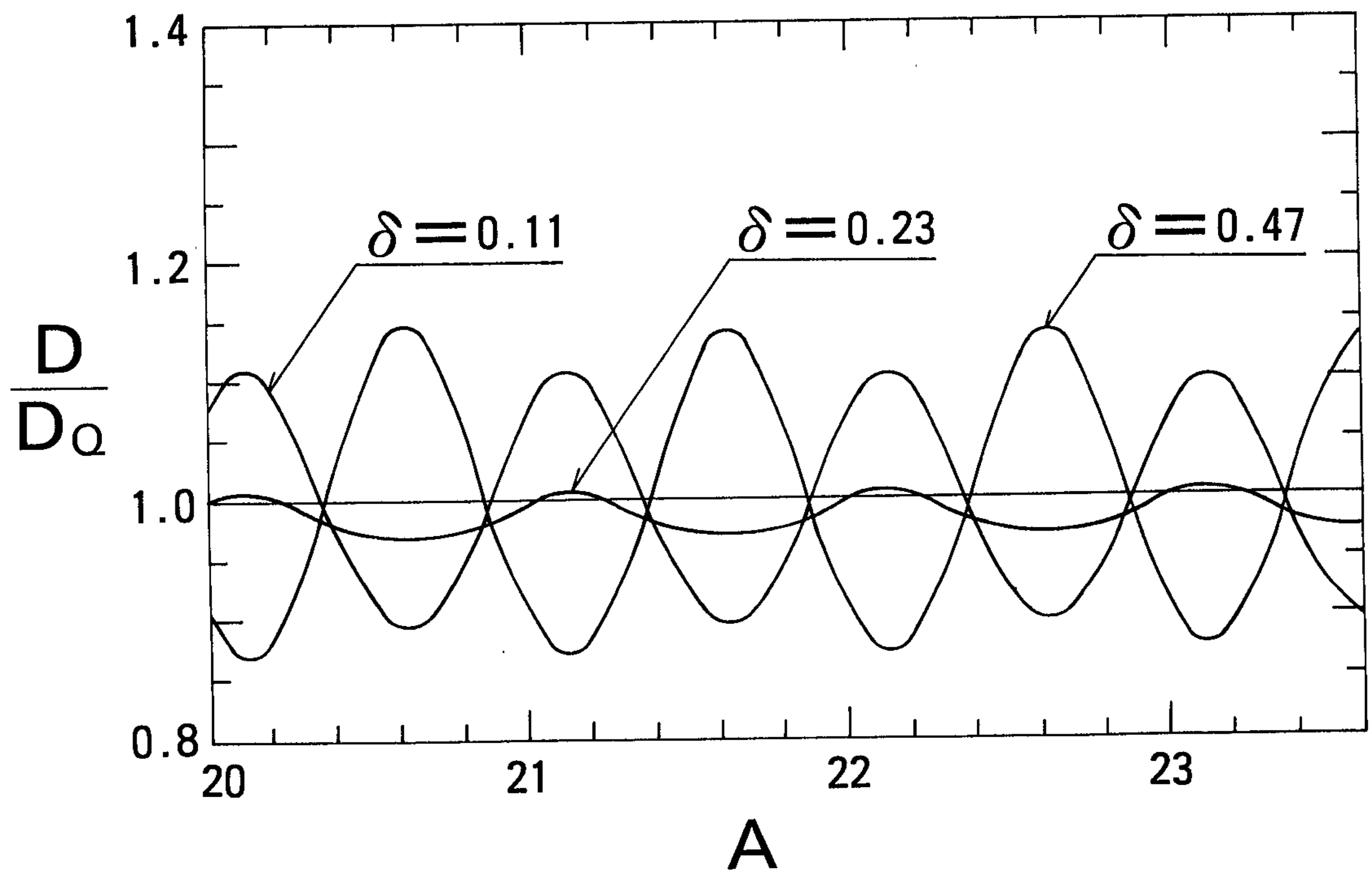

Fig. 2