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T. Hatori

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Magnetohydrodynamic Cellular Automata

Tadatsugu Hatori
National Institute for Fusion Science,
Nagoya,464-01,Japan

Abstract

There has been a renewal of interest in cellular automata, partly because they give an architecture for a special purpose computer with parallel processing optimized to solve a particular problem. The lattice gas cellular automata are briefly surveyed, which are recently developed to solve partial differential equations such as hydrodynamics or magnetohydrodynamics. A new model is given in the present paper to implement the magnetic Lorentz force in a more deterministic and local procedure than the previous one.

Keywords ;
Magnetohydrodynamics, cellular automaton, lattice gas method,

1 Introduction

The computer is now a day the indispensable tool for modern science. As large computers emerge year after year, number of solvable problems increase explosively. There remains however a large number of problems which are well beyond what existing computers can give answer. As one increases the size of a given problem by increasing grid points or traces the evolution of the system over a longer time, the capacity and the speed of the computer required to solve it grows much faster than one's naive expectation.

New technologies for the element of computer hardware are being developed, but a new computer architecture,i.e. the parallel processing, offers a more immediate solution. A familiar example of the use of parallel processing is the vector super computer of general purpose, but a new trend in science is to build the special purpose computer (SPC) with parallel processing. The architecture of a general purpose computer (GPC)is not optimized for any particular problem. For the special class of problems, it is found to be possible to design and construct a SPC, at modest cost as compared with GPC, which runs faster than the general purpose commercial super computers. The many processors, which are assigned to different parts of the task, are applied to a single problem in SPC, so that SPC could compete in the computational speed with the super computer. The processor in SPC should be simpler and cheaper than the processor in the super computer, but it can match the super computer performance because of a great number of processors. Once SPC is constructed, it is available full time to a single user. Construction of SPC has been realized in some areas which include quantum chromodynamics, Monte Calro simulation of the Ising model, molecular dynamics of classical many-particle system with short-range interactions, lattice gas hydrodynamics and neural networks.

Especially the lattice gas hydrodynamics has made progress recently [1)] during the last few years. It is based on cellular automata which was first introduced by Von Neumann and Ulam, and has been taken a renewed interest. The hydrodynamics is modeled with a system of many particles interacting via artificial microscopic process which can be described totally by Boolian logic. The algorithm is quite different from using the traditional finite-difference method to solve the partial differential equations. As for the present state of arts of the lattice gas methods for partial differential equations, one should refer to Ref.2.

The Lattice gas and the real gas have quite different microscopic structures, but have the same macroscopic behaviour, governed by the Navier-Stokes equation. The

fictitious microscopic process, which is however convenient for the computational process, is found to converge to the physical fluid dynamics in the macroscopic limit. In other words, the form of resulting differential equation is insensitive to the microscopic process. By the lattice gas model based upon cellular automata(CA in short), we mean the discrete analogue of molecular dynamics. Definition of CA due to S. Wolfram[3] is the following. A cellular automaton consists of a regular lattice of sites. Each site takes on finite number of possible values and is updated in discrete time steps according to a rule that depends on the value of sites in some neighborhood around it. The underlying discreteness of CA model in space, time and variables make it particularly suitable for digital computation.

The most important point to construct the hydrodynamic lattice gas model is the choice of lattice system. For the two dimensional hydrodynamics, for instance, the square lattice as investigated by Hardy, de Pazzis and Pomeau[4] leads to a partial differential equation different from the Navier-Stokes equation. The true choice of lattice system for the two dimensional Navier-Stokes equation is found to be the hexagonal lattice (see Fig. 1) by Frisch, Hasslacher and Pomeau[5]. The form of macroscopic equation reduced from particular CA with particular lattice system is determined by symmetry properties. Define a tensor $\overleftrightarrow{E}^{(n)}$ of rank n as

$$\overleftrightarrow{E}^{(n)} = \sum_a \overbrace{\hat{e}_a \hat{e}_a \cdots \hat{e}_a}^{n}, \tag{1.1}$$

where $\hat{e}_a$ means a unit vector in the direction of neighbouring cite from a particular cite, and the suffix a indicates an each neighbouring site, running from 1 to 6 in the hexagonal lattice. We impose a priori a restriction, which will be derived in §2, that $\overleftrightarrow{E}^{(n)}$ should be isotropic for any $n \leq 4$. The tensor $\overleftrightarrow{E}^{(n)}$ must always be invariant under the discrete symmetry group characterized by a given lattice system. It should be invariant also under the full continuous rotation group. In the two dimensional space, the hexagonal lattice gives in fact isotropic tensors $\overleftrightarrow{E}^{(n)}$ up to $n = 4$. In the three dimensional space, however, it is impossible to make $\overleftrightarrow{E}^{(4)}$ isotropic by any crystallographic lattices. But, fortunately, in the four dimensional space, it is found that there exists a system called the face-centered-hyper-cubic lattice with required symmetry. Projection from the four dimensional space to the three dimensional space gives the required space.

Dynamical rules for artificial particles of the hydrodynamical CA are as follows. Particles located at a particular cite can have only g different velocities oriented to neighbouring cites, where g is number of neighbouring cites. All velocities have the

same modulus as the distance of lattice size, so that after a unit time step each particle streams to one of neighbouring cites. To describe the algorithm in the Boolian logic, we require an exclusion principle that not more than one particle is to be found at a given time and a cite, moving in a given direction $\hat{e}_a$. When more than one particles arrive at a cite, we proceed them under a prescribed collision rule which conserve the total particle number and momentum.

A magnetohydrodynamical(MHD) CA in the two dimensional space has recently been proposed by Montgomery and Doolen.[6] In the present article, a new MHD · CA model is addressed which implements the magnetic Lorentz force in a more deterministic and local procedure as compared with the previous one. It is applicable to the Strauss equation in the three dimensional space.

In §2, a MHD · CA in two dimensional space(2D · MHD · CA) is described in detail, and reduction of the magnetohydrodynamics from 2D · MHD · CA is given. In §3, the correlation function formalism is applied to transport coefficients, viscosity and magnetic diffusivity for 2D · MHD · CA. §4 describes a new MHD · CA, and §5 contains some remarks.

2 Magnetohydrodynamic lattice gas model

Many properties of 2D · MHD · CA[6] are shared by the two dimensional Navier-Stokes CA. The lattice system is hexagonal. The MHD particles reside at the cites in one of six discrete states of velocity, $\hat{e}_a = (\cos 2\pi a/6, \sin 2\pi a/6), a = 1, 2, \cdots 6$. In every integer time step, they move to the centers of the adjacent hexagons toward which they are directed. They carry an additional index σ corresponding to the z-component of the microscopic magnetic vector potential A_z, σ taking the values $\pm$ 1,0. Thus the allowed states for a single site is 18(= 6 × 3) bits. A 2D · MHD particle is a rod having a " spin ". The Lorentz force $\vec{J} \times \vec{B}$ which is given by $-\nabla A_z \Delta A_z$ in the present case is introduced by imparting the requisite momentum per unit volume by randomly flipping the microscopic distribution of $\hat{e}_a$ proportional to $-\nabla A_z \Delta A_z$. When different particles come into a single site, they undergo collisions under a rule which conserves total momentum and " spin ".

It is possible to derive a macroscopic differential equation in a form,

$$n(\frac{\partial}{\partial t}\vec{u} + \vec{u} \cdot \nabla \vec{u}) = -\nabla \overleftrightarrow{P} + \vec{J} \times \vec{B} + n\nu\nabla^2 \vec{u}, \tag{2.1}$$

$$(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla) A_z = \eta \nabla^2 A_z, \tag{2.2}$$

$$\frac{\partial}{\partial t}n + \nabla n\vec{u} = 0, \tag{2.3}$$

$$\nabla \cdot \vec{u} = 0, \tag{2.4}$$

that is an incompressible two dimensional MHD system. The conventional symbols $\vec{u}, \vec{J}, \vec{B}$ mean the flow velocity, the electric current, and the magnetic field. ν and η are the kinematic viscosity and the magnetic diffusivity. The pressure tensor $\overleftrightarrow{P}$ is given by

$$\overleftrightarrow{P} = \frac{n}{2}\overleftrightarrow{1}\,(1 - \frac{u^2}{2}) - \frac{n}{2}\vec{u}\vec{u}. \tag{2.5}$$

A kinetic theory to derive the equations(2.1) through(2.5) is described in the following.

Consider an ensemble of above mentioned CA with different origins of space $\vec{x}$ and time t to define a one-body smooth distribution function, $f_{a,\sigma}(\vec{x},t)$ in $(\vec{x},t)$ space. The distribution function gives macrovariables, such as the density $n(\vec{x},t)$, the flow velocity $\vec{u}(\vec{x},t)$ and the vector potential $A_z(\vec{x},t)$,

$$n(\vec{x},t) = \sum_{a,\sigma} f_{a,\sigma}(\vec{x},t) \tag{2.6}$$

$$n\vec{u}(\vec{x},t) = \sum_{a,\sigma} \hat{e}_a f_{a,\sigma}(\vec{x},t) \tag{2.7}$$

$$nA_z(\vec{x},t) = \sum_{a,\sigma} \sigma f_{a,\sigma}(\vec{x},t) \tag{2.8}$$

It is reasonable to assume that $f_{a,\sigma}$ may be advanced according to the Markovian stochastic process,

$$f_{a,\sigma}(\vec{x},t) = \sum_{b,\tau} \int d\delta\vec{x} P(a,\sigma;\vec{x},t|b,\tau;\vec{x}-\delta\vec{x},t-1) f_{b,\tau}(\vec{x}-\delta\vec{x},t-1), \tag{2.9}$$

where $P(a,\sigma;\vec{x},t|b,\tau;\vec{x}-\delta\vec{x},t-1)$ means a transition probability from an old state (b,τ) at a position $\vec{x}-\delta\vec{x}$ and a time t to a new state (a,σ) at $\vec{x}$ and t. Note that P is implicitly a functional of f and some many-body distribution functions. In the case of present CA,

$$P(a,\sigma;\vec{x},t|b,\tau;\vec{x}-\delta\vec{x},t-1) = P_{a,\sigma;b,\tau}(\vec{x},t)\delta(\delta\vec{x}-\hat{e}_b). \tag{2.10}$$

Then eq.(2.9)becomes,

$$f_{a,\sigma}(\vec{x},t) = \sum_{b,\tau} P_{a,\sigma;b,\tau}(\vec{x},t) f_{b,\tau}(\vec{x}-\hat{e}_b,t-1). \tag{2.11}$$

If the process is collisionless, transition probability becomes trivially $P_{a,\sigma;b,\tau} = \delta_{a,b}\delta_{\sigma,\tau}$, however in the presence of collision,

$$P_{a,\sigma;b,\tau} = \delta_{a,b}\delta_{\sigma,\tau}(1 - \sum_{c,\mu} W_{c,\mu;b,\tau}) + W_{a,\sigma;b,\tau}, \tag{2.12}$$

where $W_{a,\sigma;b,\tau}$ is a transition probability density per unit time interval from a state (a,σ) to another state (b,τ). From eqs.(2,11) and (2.12), we obtain

$$\begin{aligned} f_{a,\sigma}(\vec{x},t) - f_{a,\sigma}(\vec{x}-\hat{e}_a,t-1) &= \sum_{b,\tau}(P_{a,\sigma;b,\tau} - \delta_{a,b}\delta_{\sigma,\tau})f_{b,\tau}(\vec{x}-\hat{e}_b,t-1) \\ &= -\sum_{b,\tau} W_{b,\tau;a,\sigma} f_{a,\sigma}(\vec{x}-\hat{e}_a,t-1) \\ &\quad + \sum_{b,\tau} W_{a,\sigma;b,\tau} f_{b,\tau}(\vec{x}-\hat{e}_b,t-1). \end{aligned} \tag{2.13}$$

In the present units of time and space, the time derivative and the spatial gradient of one-body distribution function are very small. Expanding the left-hand side of eq.(2.13), we thus obtain a basic transport equation for the magnetohydrodynamic CA model,

$$\begin{aligned} [(\frac{\partial}{\partial t} + \hat{e}_a\cdot\nabla) - \frac{1}{2}(\frac{\partial}{\partial t} + \hat{e}_a\cdot\nabla)^2 + \cdots]f_{a,\sigma}(\vec{x},t) &= -\sum_{b,\tau} W_{b,\tau;a,\sigma}(\vec{x},t) f_a(\vec{x}-\hat{e}_a,t-1) \\ &\quad + \sum_{b,\tau} W_{a,\sigma;b,\tau}(\vec{x},t) f_b(\vec{x}-\hat{e}_b,t-1) \\ &\equiv \Omega_{a,\sigma} \end{aligned} \tag{2.14}$$

which is the counterpart of ordinary master equation. $\Omega_{a,\sigma}$ is called the collision term. The prescribed rule of collisions satisfies the three non-trivial conservation laws,

$$\sum_{a,\sigma}\Omega_{a,\sigma} = 0 \tag{2.15}$$

$$\sum_{a,\sigma}\hat{e}_a\Omega_{a,\sigma} = 0 \tag{2.16}$$

$$\sum_{a,\sigma}\sigma\Omega_{a,\sigma} = 0 \tag{2.17}$$

which mean the conservation for particle number, momentum, and σ -index, respectively. The conservation of kinetic energy is trivial in the present model. Some moments of the transport equation(2.14) yield some balance equations for macrovariables,

$$\frac{\partial}{\partial t}n + \nabla\cdot n\vec{u} = 0 \tag{2.18}$$

$$\frac{\partial}{\partial t}n\vec{u} + \nabla\cdot\overleftrightarrow{\Pi} = \frac{1}{2}\sum_{a,\sigma}\hat{e}_a(\frac{\partial}{\partial t} + \hat{e}_a\cdot\nabla)^2 f_{a,\sigma} \tag{2.19}$$

$$\frac{\partial}{\partial t}nA_z + \nabla\cdot\vec{\phi} = \frac{1}{2}\sum_{a,\sigma}\sigma(\frac{\partial}{\partial t} + \hat{e}_a\cdot\nabla)^2 f_{a,\sigma} \tag{2.20}$$

where

$$\overleftrightarrow{\Pi} = \sum_{a,\sigma}\hat{e}_a\hat{e}_a f_{a,\sigma}(\vec{x},t), \tag{2.21}$$

$$\vec{\phi} = \sum_{a,\sigma} \sigma \hat{e}_a f_{a,\sigma}(\vec{x},t). \tag{2.22}$$

The right-hand side of eqs.(2.19) and (2.20) give artificial dissipation due to the effect of finite lattice size and time step.

As is the case of real gas, the collision term $\Omega_{a,\sigma}$ has an effect of driving the distribution function $f_{a,\sigma}$ to a local thermal equilibrium over length and time scales short compared to those over which macrovariables vary. This make it possible to apply the Chapman-Enskog expansion to eq.(2.14),

$$f_{a,\sigma} = f^{(0)}_{a,\sigma}(\vec{x},t) + f^{(1)}_{a,\sigma}(\vec{x},t) + \cdots. \tag{2.23}$$

Here $f^{(0)}_{a,\sigma}(\vec{x},t)$ is the local thermal equilibrium satisfying $\Omega_{a,\sigma}(f^{(0)}) = 0$, given by

$$f^{(0)}_{a,\sigma}(\vec{x},t) = \{1 + \exp[\alpha + \beta \hat{e}_a \cdot \vec{u}(\vec{x},t) + \gamma\sigma A_z(\vec{x},t)]\}^{-1}, \tag{2.24}$$

which is the Fermi-Dirac distribution function in form, but depends on the space and time variables through $n(\vec{x},t)$, $\vec{u}(\vec{x},t)$, and $A_z(\vec{x},t)$. The deviation $f^{(1)}_{a,\sigma}(\vec{x},t)$ from the local equilibrium represents the effect of the first order spatial gradient or the time derivative of the fluid macrovariables. The smallness parameter of expansion is therefore the ratio of collision mean free path to the characteristic length scale for the variation of the moments. The equation for $f^{(1)}$ is governed by

$$(\frac{\partial}{\partial t} + \hat{e}_a \cdot \nabla) f^{(0)}_{a,\sigma}(\vec{x},t) = [\overleftrightarrow{C}^{(0)} f^{(1)}]_{a,\sigma} \equiv \Omega^{(1)}_{a,\sigma}. \tag{2.25}$$

The right-hand side of(2.25) is a symbolical expression for the linearized version of $\Omega_{a,\sigma}(f)$ about $f^{(0)}$ in powers of $f^{(1)}$. As for the explicit forms for $\Omega_{a,\sigma}$ and $\Omega^{(1)}$ refer to Ref.7. To obtain transport fluxes $\overleftrightarrow{\Pi}$ and $\vec{\phi}$ defined by

$$\overleftrightarrow{\Pi}^{(1)} = \sum_{a,\sigma} \hat{e}_a \hat{e}_a f^{(1)}_{a,\sigma}, \tag{2.26}$$

$$\vec{\phi}^{(1)} = \sum_{a,\sigma} \sigma \hat{e}_a f^{(1)}_{a,\sigma}, \tag{2.27}$$

the inversion of 18×18 matrix $\overleftrightarrow{C}^{(0)}$ is required to express $f^{(1)}_{a,\sigma}$ in terms of $(\frac{\partial}{\partial t} + \hat{e}_a \cdot \nabla) f^{(0)}_{a,\sigma}$. It is found [7] that the matrix $\overleftrightarrow{C}^{(0)}$ can be written as the sum of four direct products of 6×6 matrices in $\hat{e}_a$-space and 3×3 matrices in σ-space as

$$\overleftrightarrow{C}^{(0)} = \sum_{j=1}^{4} \overleftrightarrow{\omega}^{(j)} \otimes \overleftrightarrow{\tau}^{(j)}, \tag{2.28}$$

where explicitly

$$[\overleftrightarrow{\omega}^{(j)} \bigotimes \overleftrightarrow{\tau}^{(j)}]_{a,b;\sigma,\lambda} = \omega^{(j)}_{a,b}\tau^{(j)}_{\sigma,\lambda} \tag{2.29}$$

The $\overleftrightarrow{\omega}^{(j)}$ $(j = 1, \cdots 4)$ are all circulant matrices but $\overleftrightarrow{\tau}^{(j)}$ are not all circulant. The inversion of 18×18 matrix $\overleftrightarrow{C}^{(0)}$ is now made possible by reducing it to that of 3×3 matrix, with use of the theorem that all circulant matrices of a given dimension have the same right eigen vectors.

It should be noted that since the function $f^{(1)}$ contains the dyads $\hat{e}_a\hat{e}_a$ the transport flux $\overleftrightarrow{\Pi}^{(1)}$ defined by (2.26) involves the quantity $\sum_a \hat{e}_a\hat{e}_a\hat{e}_a\hat{e}_a$ which is denoted by $\overleftrightarrow{E}^{(4)}$ in eq.(1.1). In order that the collisional transport term $\nabla\cdot\overleftrightarrow{\Pi}^{(1)}$ should give the isotropic kinematic viscosity, $\nu n\Delta\vec{u}$, $\overleftrightarrow{E}^{(4)}$ is required to be invariant under the continuous rotation group. The choice of hexagonal lattice is necessary in the case of two dimensional magnetohydrodynamics.

Without giving detailed calculation, we summarize the resultant expressions for $\overleftrightarrow{\Pi}$ and $\vec{\phi}$ as follows;

$$\nabla\cdot\overleftrightarrow{\Pi} = \nabla\cdot\overleftrightarrow{\Pi}^{(0)} + \nabla\cdot\overleftrightarrow{\Pi}^{(1)}, \tag{2.30}$$

$$\overleftrightarrow{\Pi}^{(0)} = \frac{n}{2}[1 - \frac{9-n}{18-n}u^2]\overleftrightarrow{1} - \frac{9}{18-n}n\vec{u}\vec{u}, \tag{2.31}$$

$$\nabla\cdot\overleftrightarrow{\Pi}^{(1)} = n\nu\Delta\vec{u} \tag{2.32}$$

$$\nabla\cdot\vec{\phi} = \nabla\cdot\vec{\phi}^{(0)} + \nabla\cdot\vec{\phi}^{(1)}, \tag{2.33}$$

$$\vec{\phi}^{(0)} = n\vec{u}\frac{2(9-n)}{18-n}A_z, \tag{2.34}$$

and

$$\nabla\cdot\vec{\phi}^{(1)} = n\sigma\Delta A_z. \tag{2.35}$$

The kinematic viscosity for arbitrary density is given by

$$\nu = \frac{1}{2}\frac{1}{(1-\frac{n}{18})^{15}} \tag{2.36}$$

and the magnetic diffusivity σ is

$$\sigma = \frac{n}{2}\frac{1}{(3h+9h^2)(1-\frac{n}{18})^{16}}, \tag{2.37}$$

with $h = n/(18-n)$. If we take the limit of $n \to 0$, we recover the value of kinematic viscosity 1/2 for the 2D · Navier-Stokes · CA model.

3 Correlation function formalism for transport coefficients

R. Kubo[8] derived a quantum-mechanical formula for the electrical conductivity in terms of the correlation function of equilibrium fluctuations. H. Mori[9,10] demonstrated a partial equivalence between the Kubo-type and the Chapman-Enskog formulae, for Maxwellian molecules. H. S. Green[11] established the equivalence of both by deriving the Kubo-type transport coefficient based on classical mechanics.

The correlation function formalism for the MHD lattice gas model is developed in this section. We start with the Liouville equation for M-body distribution $P(S,t)$. Let the total number of cites be G, then the total number of the states of the MHD · CA becomes $18G$. The vector S has $18G$ components, and each component of S Takes a value, 0 or 1. Number of components with finite value in S is the total particle number,M. The components of S is denoted by $S_{\vec{x},a,\sigma}$ where $\vec{x}$ is the coordinate of cite, a and σ the orientation of velocity and the spin. The Liouville equation for $P(S,t)$ is

$$P(S_{\vec{x},a,\sigma},t) = W_{a,\sigma;b,\lambda}(t)P(S_{\vec{x}-\hat{e}_b,b,\lambda},t-1), \tag{3.1}$$

where the evolution operator $W_{a,\sigma;b,\lambda}(t)$ has a function to change from an old state (b,λ) to a new state (a,σ). The M -body distribution function is assumed to be a homogeneous equilibrium at $t=-\infty$,

$$P(S,-\infty) = P^{(0)}(S_{\vec{x},a,\sigma};n^{(0)},\vec{u}^{(0)},A_z^{(0)}) \tag{3.2}$$

where $n^{(0)}$,$u^{(0)}$ and $A_z^{(0)}$ are constant parameters. The equilibrium distribution is

$$P^{(0)}(S) \quad = \prod_{\vec{x}}\prod_{a}\prod_{\sigma} N_{a,\sigma}^{S_{\vec{x},a,\sigma}}(1-N_{a,\sigma})^{1-S_{\vec{x},a,\sigma}} \tag{3.3}$$

and the function N is the Fermi-Dirac distribution,

$$N_{a,\sigma} = \{1+\exp[\gamma(n^{(0)}+\hat{e}_a\cdot\vec{u}^{(0)}+\sigma A_z^{(0)}]\}^{-1} \tag{3.4}$$

Frisch et al [1] proved that the distribution (3.3) in fact satisfies the steady state of the Liouville equation (3.1). Consider infinitesimal inhomogeneous perturbations, $\delta n(\vec{x})$, $\delta\vec{u}(\vec{x})$ and $\delta A_z(\vec{x})$,at time $t-T$, then subsequently n, $\vec{u}$, and A_z evolve in space and time, governed by a macroscopic differential equation. At present time t, the distribution P is composed of two parts, the local thermal equilibrium,

$$P^{(0)}(S_{\vec{x},a,\sigma};n(\vec{x},t),\vec{u}(\vec{x},t),A_z(\vec{x},t)), \tag{3.5}$$

plus the first-order deviation $P^{(1)}$ with respect to the spatial and time derivatives of macrovariables,

$$P = P^{(0)} + P^{(1)}. \tag{3.6}$$

It should be noted that if the local thermal equilibrium (3.5) is substituted to the evolution term in the right-hand side of eq.(3.1), then there remain only the effects of spatial and temporal derivatives of macrovariables, $n(\vec{x},t)$, $\vec{u}(\vec{x},t)$ and $A_z(\vec{x},t)$. To obtain an expression for $P^{(1)}$, after plugging eq.(3.6) into eq.(3.1) and Taylor-expanding $P(S_{\vec{x}-\hat{e}_b,b,\lambda},t-1)$ in the right-hand side of eq.(3.1), we obtain

$$P^{(1)}(S_{\vec{x},a,\sigma},t) = W_{a,\sigma;b,\lambda}(t)P^{(1)}(S_{\vec{x},b,\lambda},t) - W_{a,\sigma;b,\lambda}(t)(\frac{\partial}{\partial t} + \hat{e}_b\cdot\nabla)P^{(0)}(S_{\vec{x},b,\lambda};n,\vec{u},A_z), \tag{3.7}$$

where the temporal and spatial derivatives in (3.7) operate only on n, $\vec{u}$ and A_z. Successive use of eq.(3.7) yields

$$P^{(1)}(S_{\vec{x},a,\sigma},t) = -\sum_{j=1}^{T} W_{a,\sigma;b,\lambda}(t|t-j)(\frac{\partial}{\partial t} + \hat{e}_b\cdot\nabla)P^{(0)}(S_{\vec{x},b,\lambda};n,\vec{u},A_z), \tag{3.8}$$

where the physical meaning of $W_{a,\sigma;b,\lambda}(t|t-j)$ is an operator transferring from the state (b,λ) at $t-j$ to the state (a,σ) at t. More explicitly it is given by

$$W_{a,\sigma;b,\lambda}(t|t-j) = W_{a,\sigma;b_1,\lambda_1};(t)W_{b_1,\lambda_1;b_2,\lambda_2}(t-1)\cdots W_{b_j,\lambda_j;b,\lambda}(t-j+1). \tag{3.9}$$

Thus the transport fluxes $\vec{\phi}^{(1)}$ and $\overleftrightarrow{\Pi}^{(1)}$ are obtained as

$$\vec{\phi}^{(1)}(\vec{x},t) = \sum_{a,\sigma}\hat{e}_a\sigma\int dS' P^{(1)}(S_{\vec{x},a,\sigma}=1,S',t), \tag{3.10}$$

$$\overleftrightarrow{\Pi}^{(1)}(\vec{x},t) = \sum_{a,\sigma}\hat{e}_a\hat{e}_a\int dS' P^{(1)}(S_{\vec{x},a,\sigma}=1,S',t), \tag{3.11}$$

where the integrals in (3.10) and (3.11) mean summation over Boolian variables S except $S_{\vec{x},a,\sigma}$. From the definition of transport coefficients, $\overleftrightarrow{\sigma}$ and $\overleftrightarrow{\eta}$,

$$\vec{\phi}^{(1)} = -n\,\overleftrightarrow{\sigma}\,\nabla A_z \tag{3.12}$$

$$\overleftrightarrow{\Pi}^{(1)} = -n\,\overleftrightarrow{\eta}\,\nabla\vec{u}, \tag{3.13}$$

we finally arrive at the Kubo-style expressions,

$$n\,\overleftrightarrow{\sigma} = \frac{3}{2}\sum_{j=1}^{\infty}\langle\hat{e}(0)\sigma(0)\hat{e}(j)\sigma(j)\rangle, \tag{3.14}$$

$$n\,\overleftrightarrow{\eta} = 2\sum_{j=1}^{\infty}\langle\hat{e}(0)\hat{e}(0)\hat{e}(j)\hat{e}(j)\rangle. \tag{3.15}$$

Definition of time correlation function for arbitrary variable $\mu_{a,\sigma}$ is

$$\langle \mu(0)\mu(j) \rangle = \sum_{a,\sigma} \mu_{a,\sigma} W_{a,\sigma;b,\lambda}(t|t-j)\mu_{b,\lambda} N_{b,\lambda}. \tag{3.16}$$

4 Generalized magnetohydrodynamic cellular automata

Before describing the generalization[12], a fundamental invariant of MHD, what is called the Alfvenic invariant, is derived as a starting point of this section. Consider an arbitrary surface element in the plasma and define the magnetic flux by the amount of magnetic field passing through the surface,

$$\Psi = \int d\vec{S} \cdot \vec{B}. \tag{4.1}$$

When the magnetic field is changing and the surface together with the closed contour spanned by the surface element is deforming with flow velocity $\vec{v}$, each differential length $d\vec{l}$ of the contour sweeps out an area $d\vec{l} \times \vec{v} \cdot dt$ in a differential interval dt. The total rate of change of flux Ψ is then given by

$$\frac{d}{dt}\Psi = \int d\vec{S} \cdot \frac{\partial \vec{B}}{\partial t} - \oint d\vec{l} \times \vec{v} \cdot \vec{B}. \tag{4.2}$$

Applying the Ohm's law,

$$\vec{E} + \vec{v} \times \vec{B} = \eta \vec{J}, \tag{4.3}$$

and the Farady's law to eq.(4,2), we get

$$\frac{d}{dt}\Psi = 0, \tag{4.4}$$

if the resistivity of plasma η vanishes. In other words, the magnetic flux Ψ is frozen in the moving element of infinitely conductive plasma.

4.1 Generalized MHD particle

If MHD system can be modeled with the many particle system, the MHD particle must be characterized by the Alfvenic invariant frozen in itself, as well as by its coordinate and velocity. The variable indices (a, σ) of a single MHD particle correspond to the six directions of velocity and three states of magnetic flux, up, down and nothing. The generalized MHD particle [12] is characterized also by four parameters, r_1, r_2, l and b.

new particle is depicted in Fig.2. It is an annulus with inner radius r_1, outer radius r_2 and length l. Inside the particle, the radial profile of azimuthal magnetic field, the axial current density and the z -component of magnetic vector potential are shown in Fig.3.

4.2 Macrovariables

For the convenience in the following, define a Boolian function $n_{a,\sigma}(\vec{\rho}_*)$ as

$$n_{a,\sigma}(\vec{\rho}_*) = \begin{cases} 1, & \text{if the state (a,}\sigma\text{) at } \vec{\rho}_* \text{ is occupied,} \\ 0, & \text{otherwise.} \end{cases} \tag{4.5}$$

To observe macrovariables, we introduce a super-cell which is composed of many unit cells as illustrated in Fig. 4. It is now straightforward to define macrovariables, such as number density n, particle flux $n\vec{u}$, and magnetic field $\vec{B}$,

$$n = \sum_{\vec{\rho}_*} \sum_{a,\sigma} n_{a,\sigma}(\vec{\rho}_*)/S_0, \tag{4.6}$$

$$n\vec{u} = \sum_{\vec{\rho}_*} \sum_{a,\sigma} \hat{e}_a n_{a,\sigma}(\vec{\rho}_*)/S_0. \tag{4.7}$$

$$\vec{B} = \sum_{\vec{\rho}_*} \sum_{a,\sigma} \int \hat{B}^\sigma_\theta(\vec{\rho} - \vec{\rho}_*) d\vec{\rho} n_{a,\sigma}(\vec{\rho}_*)/S_0. \tag{4.8}$$

Here the summation over $\vec{\rho}_*$ is limited inside the super-cell, S_0 the total area of super-cell, and the integral $\int \hat{B}^\sigma_\theta d\vec{\rho}$ is a total magnetic field belonging to a single particle located at $\vec{\rho}_*$. Contributions to the macrofield $\vec{B}$,(4.8), from inner particles in the supper-cell are cancelled out, because all the microscopic magnetic field lines assigned to a single particle are closed(See Fig.2). Only particles located at the boundary of the super-cell contribute to the magnetic field $\vec{B}$. The boundary of the super-cell is composed of six segments, $\beta = 1, \ldots, 6$, (see Fig.4). $\vec{B}$ defined by eq.(4.8) now becomes

$$\vec{B} = \sum_{\beta=1}^{6} \hat{e}_\beta \sum_{\vec{\rho}_*} \sum_{a,\sigma} n_{a,\sigma}(\vec{\rho}_*) \int \hat{e}_\beta \cdot \hat{B}^\sigma(\vec{\rho} - \vec{\rho}_*) d\vec{\rho}/S_0. \tag{4.9}$$

where the summation over $\vec{\rho}_*$ is on the β-th segment. The integral over $\vec{\rho}$ is performed over the inner-half space on the side of the super-cell to yield,

$$\int \hat{e}_\beta \cdot \hat{B}^\sigma(\vec{\rho} - \vec{\rho}_*) d\vec{\rho} = \int_0^\pi d\theta \sin\theta \int_{r_1}^{r_2} dr \sigma b = 2\sigma b(r_2 - r_1). \tag{4.10}$$

Definition of the current density in the z -direction is not simple, since any contributions from individual particles are canceled even from particles located at the boundary. To

obtain a finite value of macroscopic current J_z, we require the Ampere's law for $\vec{B}$ and J_z, i.e. $\nabla \times \vec{B} = J_z \hat{z}$, so that we have a formula, from the Stokes theorem,

$$\int\int dxdy J_z = \oint \vec{B} \cdot d\vec{l}, \tag{4.11}$$

where the integrations are over the whole space of the super-cell for the integral in the left-hand side and along the boundary for the right-hand side. Thus, from eqs. (4.9) and (4.10), the current J_z is

$$\begin{aligned} J_z &= \frac{1}{S_0} \oint \vec{B} \cdot d\vec{l} \\ &= \frac{1}{S_0^2} \sum_{\beta=1}^{6} \int \hat{e}_\beta \cdot d\vec{l} \sum_{\vec{\rho}_*} \sum_{a,\sigma} n_{a,\sigma}(\vec{\rho}_*)(2\sigma b(r_2 - r_1)) \\ &= \frac{1}{S_0^2} \sum_{\beta=1}^{6} L_\beta \sum_{\vec{\rho}_*} \sum_{a,\sigma} n_{a,\sigma}(\vec{\rho}_*) 2\sigma b(r_2 - r_1), \end{aligned} \tag{4.12}$$

where L_β is the length of the β -th segment. It is to be noted that both $\vec{B}$ and J_z can be determined just from the surface observation.

4.3 "Quantization" of Lorentz force

Updating rules to impart the momentum increase in the super-cell in proportion to the Lorentz force $\vec{J} \times \vec{B}$ is a central part of the MHD · CA. Expressions (4.9) and (4.12) lead to a formula for Lorentz force

$$\vec{J} \times \vec{B} = \frac{1}{S_0} f I \vec{b}, \tag{4.13}$$

with

$$\vec{b} = \sum_{\beta=1}^{6} (\hat{e}_{\beta+4} + \hat{e}_{\beta+5}) \sum_{\vec{\rho}_*} \sum_{a,\sigma} n_{a,\sigma}(\vec{\rho}_*)\sigma, \tag{4.14}$$

$$I = -\sum_{\vec{\rho}_*} \sum_{a,\sigma} n_{a,\sigma}(\vec{\rho}_*)\sigma, \tag{4.15}$$

where summation over $\vec{\rho}_*$ in (4.15) is throughout the boundary of the super-cell, and

$$f = \frac{4b^2(r_2 - r_1)^2 \sum_\beta L_\beta}{\sqrt{3} S_0^2}. \tag{4.16}$$

Notice that $\vec{b}$ is the sum of $\hat{e}_\beta$ multiplied by integer over $\beta = 1, 2, \cdots 6$, and that I is an integer. If we here impose a quantization condition for the Lorentz force,

$$f = 2, \tag{4.17}$$

then it becomes convenient to construct a scenario to apply the Lorentz force to particles in a super-cell 1) Observe the integer b_β defined by

$$b_\beta = \sum_{\vec{\rho}_*} \sum_{a,\sigma} n_{a,\sigma}(\vec{\rho}_*)\sigma, \tag{4.18}$$

for all segments, $\beta = 1, \cdots, 6$. 2) Observe the integer current I at the boundary,

$$I = -\sum_{\beta=1}^{6} b_\beta. \tag{4.19}$$

Next task is to change velocities of some particles inside the super-cell. The Lorentz force (4.13) is, after rearranging,

$$\vec{J} \times \vec{B} = \frac{1}{S_0} 2I[(b_2 + b_3 - b_5 - b_6)\hat{e}_1 + (b_3 + b_4 - b_6 - b_1)\hat{e}_2 + (b_4 + b_5 - b_1 - b_2)\hat{e}_3]. \tag{4.20}$$

3) If $I(b_2 + b_3 - b_5 - b_6) > 0 (< 0)$, search for $|I(b_2 + b_3 - b_5 - b_6)|$ particles with velocity $\hat{e}_4(\hat{e}_1)$ and change their velocities to $\hat{e}_1(\hat{e}_4)$. 4) If $I(b_3 + b_4 - b_6 - b_1) > 0 (< 0)$, search for $|I(b_3 + b_4 - b_6 - b_1)|$ particles with velocity $\hat{e}_5(\hat{e}_2)$ and change their velocities to $\hat{e}_2(\hat{e}_5)$. If $I(b_4 + b_5 - b_1 - b_2) > 0 (< 0)$, search for $|I(b_4 + b_5 - b_1 - b_2)|$ particles with velocity $\hat{e}_6(\hat{e}_3)$ and change their velocities to $\hat{e}_3(\hat{e}_6)$. Thus, the new scenario for the Lorentz force is fully deterministic.

4.4 Restriction from the equipartition of energies

It is widely believed that there exists an Alfven mode dominant inertia region in the fully developed MHD turbulence. It is characterized by the inequality

$$|v_A k| > \varepsilon^{1/3} k^{2/3} \equiv \tau(k)^{-1}, \tag{4.21}$$

where ε is a rate of energy supplied to eddies of size L per unit mass and unit time, $\tau(k)$ the characteristic life time of an eddy of size $k^{-1} (\ll L)$, and v_A the Alfven velocity. In the high wave-number region satisfying (4.21), the equipartition of the kinetic and magnetic energy holds,

$$\frac{1}{2} n < |\vec{\hat{u}}(\vec{k})| > = \frac{1}{2} < |\vec{\hat{b}}(\vec{k})|^2 > . \tag{4.22}$$

Here the Fourier components $\vec{\hat{u}}(\vec{k})$ and $\vec{\hat{b}}(\vec{k})$ are

$$\vec{\hat{u}}(\vec{k}) = \frac{1}{V} \int d\vec{\rho}_* \vec{U}(\vec{\rho}_*) \exp(-i\vec{k} \cdot \vec{\rho}_*), \tag{4.23}$$

$$\vec{\hat{b}}(\vec{k}) = \frac{1}{V} \int d\vec{\rho}_* \vec{B}(\vec{\rho}_*) \exp(-i\vec{k} \cdot \vec{\rho}_*), \tag{4.24}$$

where $\vec{\tilde{U}}(\vec{\rho}_*)$ and $\vec{\tilde{B}}(\vec{\rho}_*)$ are microscopic flow velocity and magnetic field, respectively.

The generalized MHD particle introduced in §4.1 has four parameters. We have imposed a constraint, quantization condition for the Lorentz force, but three free parameters are left. We now choose a set of parameters such that the equipartition low holds in the region with high k of the order of $2\pi/l_0$, where l_0 is a characteristic size of the single lattice.

4.5 Strauss equation

The Strauss equation is a reduced set of full MHD system, which is valid in low-β plasma of tokamak ordering. It is written as

$$(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla)\Omega_z = (\vec{B}\cdot\nabla)J_z + n\nu\Delta\Omega_z, \tag{4.25}$$

$$(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla)A_z = \frac{\partial\phi}{\partial z} + n\eta\Delta A_z, \tag{4.26}$$

$$u_z = 0, \tag{4.27}$$

$$\vec{u}_\perp = \nabla\phi\times\hat{z}, \tag{4.28}$$

$$\vec{B} = \nabla A_z\times\hat{z} + \hat{z}, \tag{4.29}$$

$$\Delta A_z = -J_z, \tag{4.30}$$

$$\Delta\phi = -\Omega_z. \tag{4.31}$$

If the z -dependence is totally neglected in eqs.(4.25) through (4.31), they are reduced to 2D · MHD.

The lattice system for Strauss-CA is hexagonal in (x, y) space. The length l of the MHD particle is finite. Each particle is free to move in (x, y) space, but is inhibited to move in z -direction because of the restriction, $u_z = 0$. Total system is composed of a set of layers perpendicular to the z-axis. Particular particles are confined to a particular layer, but they undergo interactions among themselves. Only interactions between adjacent layers are due to terms, $\partial J_z/\partial z$ in (4.25) and $\partial\phi/\partial z$ in (4.26). They result from shears in the z -direction of the perpendicular Lorentz force and flow velocity. Scenario to incorporate such shear effects can be constructed as a local interaction between adjacent layers.

5 Some remarks

Recent development of the lattice gas method has given a novel tool for numerical solution of incompressible Navier-Stokes equation or MHD. Cellular automata have

been paid a renewed interest, not only because they are simple dynamical systems and are tractable theoretical models, but also because they are quite suitable for implementation on computers. Construction of good CA can simulate a natural system in the scaling limit. The time evolution of CA can be traced rigorously by a computer, while the numerical solution of partial differential equation cannot proceed without inevitable roundoff errors or numerical instabilities. The lattice gas method is thought to have an advantage over the ordinary finite element or difference methods, in the tact that the numerical solution of CA has no difficulty in the problem of complicated boundary conditions. S. A. Orszag and V.Yakhot [13] however argued that the computational requirement for present lattice gas method is much more severe, if applied to turbulence, than for conventional solution of differential equation. In spite of this criticism, efforts in this direction is continued for instance by S. Chen et al.[14]

A different MHD · CA from those given in §§ 2 and 4 has been developed by H. Chen and W. H. Matthaeus[15]. Montgomery-Doolen model[6] incorporated a nonlocal computation of Lorentz force by spatially differencing the coarse-grained magnetic potential A_z, via $-\nabla A_z \Delta A_z$. The present new model described in § 4 however needs no spatial gradient of potential A_z. It can directly observe the macroscopic current density and the magnetic field from the diagnosis of particles located only at the boundary of super-cell. H. Chen and W. Matthaeus[15] presented alternative MHD · CA by completely local computation of magnetic Lorentz force. Their MHD · CA are stochastic in contrast with the resent MHD · CA which are characterized by fully deterministic algorithm.

As for CA machine, one should refer to papers in Ref.2.

The lattice gas model has been successful in the parabolic differential equations, such as hydrodynamics and magnetohydrodynamics. H. Chen et al[16] have shown that it is possible to construct CA for solving the linear wave equation. Application of CA also to other natural system has recently been performed. M. Gerhardt and H. Schuster[17] introduced CA describing a chemical system, a certain heterogeneous catalytic reaction. The numerical results demonstrate a self-sustained organization of circular and spiral wave-like structures, which are very similar to those observed in the Belousov-Zhabotinskii reaction. Quantum CA were first investigated by G. Grossing and A. Zeilinger[18]. As the speed of computer operation increases, they have to get smaller in size. Thus the understanding of quantum effects on computer or CA is becoming of significance. The quantum CA are not described by Boolian logic, since it is not certain whether the value at a given cite is 0 or 1.

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Figure captions

Fig. 1. Hexagonal Lattice system.

Fig. 2. The structure of the generalized MHD particle. The particle is an annulus with inner and outer radii r_1, r_2 and length l. It contains the closed magnetic field $\hat{B}_\theta$ whose magnitude and direction are characterized by b and σ.

Fig. 3. Radial profile of some quantities inside the MHD particle; a) azimuthal magnetic field $\hat{B}^\sigma_\theta(r)$, b) axial current density $\hat{J}_z(r)$, and z -component of magnetic vector potential $\hat{A}^\sigma_z(r)$, where σ indicates the orientation of magnetic field. The parameter b is defined by a formula, $\hat{B}^\sigma_\theta(r) = \sigma b/r$.

Fig. 4. Supercell is a large hexagon which is expected to contain a large number of MHD particles. The boundary of super-cell is composed of six segments, labeled as $\beta = 1, 2, \cdots 6$.

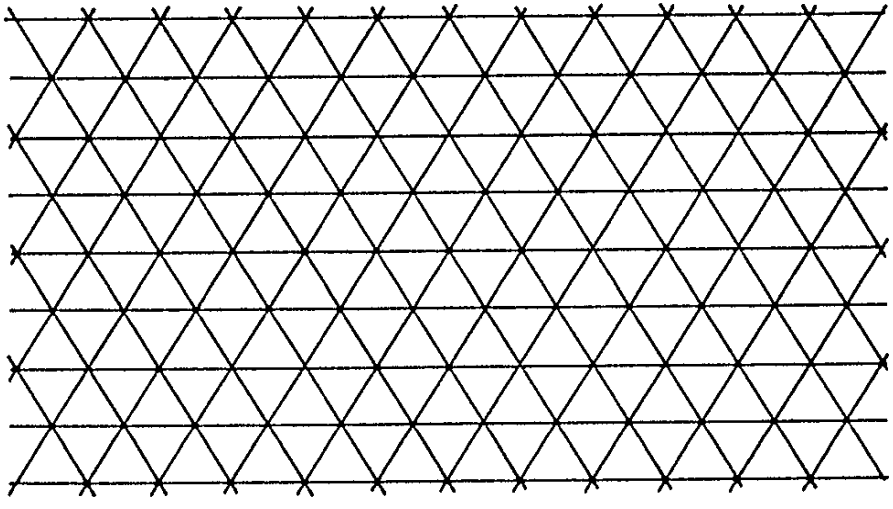


Fig. 1.

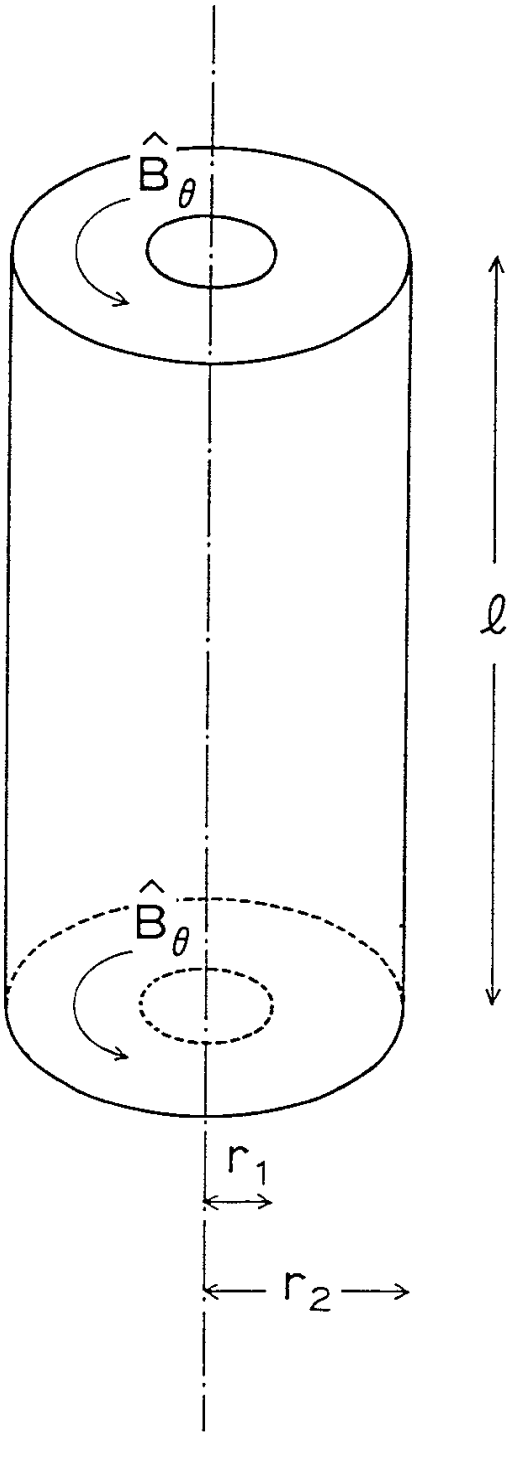
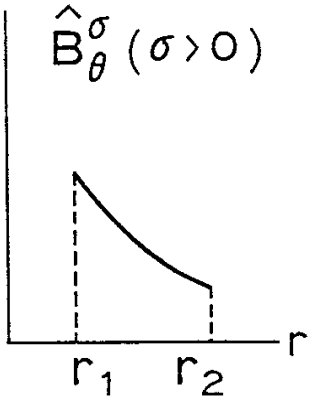
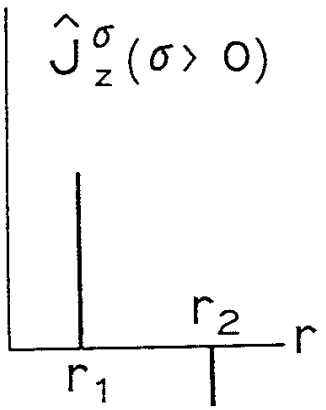


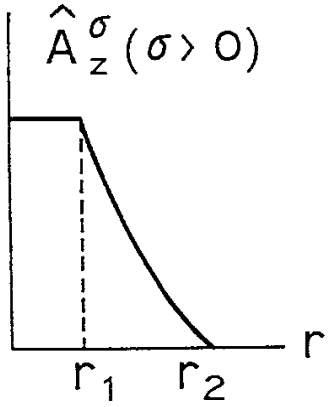
Fig. 2.



$$\hat{B}_\theta^\sigma = \begin{cases} 0 & , \ 0 < r < r_1 \\ \sigma b \dfrac{1}{r} & , \ r_1 < r < r_2 \\ 0 & , \ r_2 < r \end{cases}$$



$$\hat{J}_z^\sigma = \frac{\sigma b}{r_1}\,\delta(r - r_1) - \frac{\sigma b}{r_2}\,\delta(r - r_2)$$



$$\hat{A}_z^\sigma = \begin{cases} \sigma b\, \ell n \dfrac{r_2}{r_1} & , \ 0 < r < r_1 \\ \sigma b\, \ell n \dfrac{r_2}{r} & , \ r_1 < r < r_2 \\ 0 & , \ r_2 < r \end{cases}$$

Fig. 3.

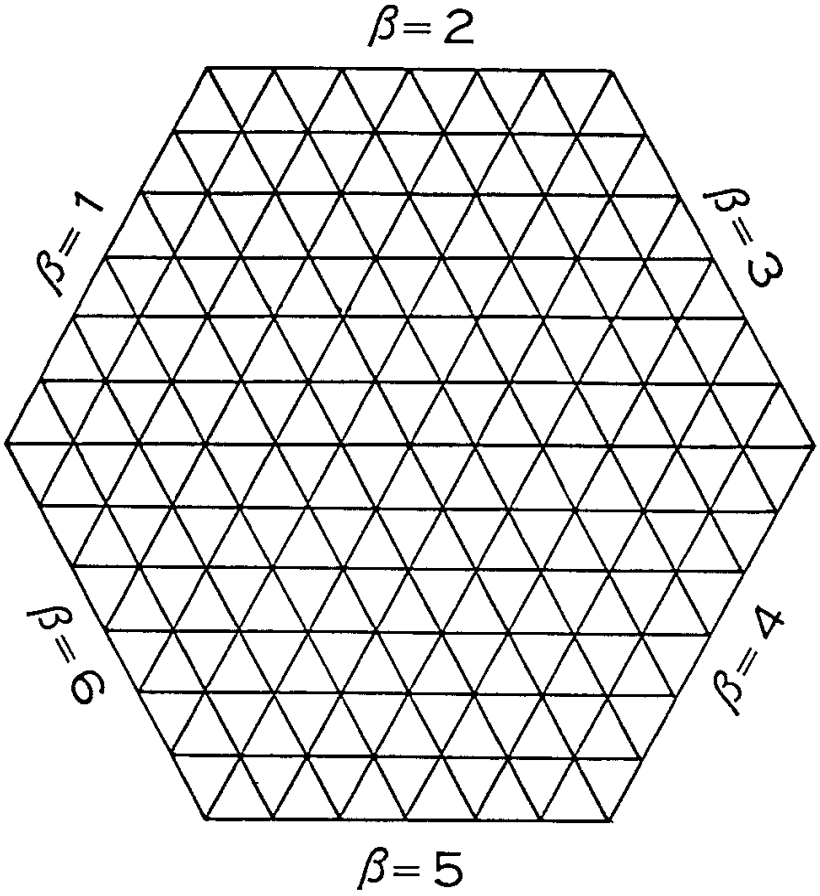


Fig. 4.