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Calculation of Magnetic Field of Helical Coils

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Abstract

Formulae of magnetic field calculation of finite size helical coils with rectangular cross section [Kakuyugo Kenkyu 57 (1988) 318] are extended to the case that the size of the coil, or the relative position to the guiding curve, varies along the arc length of the guiding curve. The error caused by the inappropriate formulae or inappropriate current model is discussed.

KEYWORDS: helical coils, numerical calculation, magnetic field

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This is the English version of the paper submitted to Kakuyugo Kenkyu [in Japanese].

§1 Introduction

Finite size of helical coils should be taken into account when the magnetic field is calculated for the actual helical coils. The magnetic field of such finite size coils can be calculated by using the multiple filaments. If the electric current in the helical coils is approximated to be uniform through the cross section, simple formulae for the magnetic field can be used to calculate the magnetic field of the helical coils with rectangular cross section. The formulae used in that calculation are given in the previous paper.[1]

In that paper, the helical coils are described by determining the cross section perpendicular to the guiding curve passing through the center of the coils. It is convenient in some cases to use the other curve as the guiding curve, which does not pass through the center of the coil cross section. There is also the case that the coil consists of several elements, each with rectangular cross section, and the relative position to the guiding curve is a function of the arc length. It is also of theoretical interest to consider the helical coils such that the cross section and/or current density varies along the arc length.[2] In this paper, the formulae given in Ref.1 are extended to these cases that the size or the relative position from the guiding curve may vary along the guiding curve.

Even when the guiding curve describing the helical coil does not pass through the center of the coil, the formulae in Ref.1 are applicable with slightest change as long as the relative position is fixed. However, if the relative position to that curve is not constant, as is the case of heliotron-E, the new formulae given in this paper must be used.

In the next section the formulae for the magnetic field valid to the more general case than in Ref.1 are derived. In §3, the possible error caused by the inappropriate formulae, or improper model for the helical current, is briefly discussed.

§2 Formulae for the Magnetic Field

The helical coils are specified by determining a spatial curve called the guiding curve, $\boldsymbol{r}_G(l)$, l being the arc length along the curve, and the cross section in the plane perpendicular to the curve. The tangent vector $\boldsymbol{t}(l)$ of the guiding curve, and the other two unit vectors $\boldsymbol{u}(l)$ and $\boldsymbol{v}(l)$, in the plane perpendicular to $\boldsymbol{t}$ consists unit triad to construct local orthogonal coordinate axes. Then, the point on the coil can be expressed as

$$\boldsymbol{r}_C(\xi,\eta,l) = \boldsymbol{r}_G(l)+X(\xi,l)\boldsymbol{u}(l)+Y(\eta,l)\boldsymbol{v}(l). \tag{1}$$

We define

$$X(\xi,l) = X_L(l)+X_W(l)\xi, \quad X_H(l) \equiv X_L(l)+X_W(l),$$
$$Y(\eta,l) = Y_L(l)+Y_W(l)\eta, \quad Y_H(l) \equiv Y_L(l)+Y_W(l), \tag{2}$$

in order to be able to consider the varying cross section. The coil is assumed to occupy the volume

$$X_L(l) \leqq X \leqq X_H(l), \quad Y_L(l) \leqq Y \leqq Y_H(l),$$

or

$$0 \leqq \xi \leqq 1, \quad 0 \leqq \eta \leqq 1.$$

We assume that the coil current flows along the lines (ξ, η) = const. with uniform current density. We introduce the quantities

$$\kappa_u = \boldsymbol{u}\cdot\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}l}, \quad \kappa_v = \boldsymbol{v}\cdot\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}l}, \quad \tau = \boldsymbol{v}\cdot\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}l}. \tag{3}$$

κ_u and κ_v being the curvature, and τ being the torsion, so that

$$\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}l} = \kappa_u(l)\boldsymbol{u}(l)+\kappa_v(l)\boldsymbol{v}(l),$$

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}l} =-\kappa_u(l)\boldsymbol{t}(l)+\tau(l)\boldsymbol{v}(l),$$

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}l} =-\kappa_v(l)\boldsymbol{t}(l)-\tau(l)\boldsymbol{u}(l). \tag{4}$$

Since the Jacobian of the coordinates (ξ,η,l) is

$$\sqrt{g} = X_\xi Y_\eta\{1-\kappa_u X-\kappa_v Y\}, \tag{5}$$

the current density j is expressed as

$$\sqrt{g}\,\boldsymbol{j} = I_c\frac{\partial \boldsymbol{r}_c}{\partial l} = I_c\left\{(1-\kappa_u X-\kappa_v Y)\boldsymbol{t}+(-\tau Y+\frac{\partial X}{\partial l})\boldsymbol{u}+(\tau X+\frac{\partial Y}{\partial l})\boldsymbol{v}\right\}, \tag{6}$$

I_c being the total current flowing the coil.

The guiding curve $\boldsymbol{r}_G$ can be described by the rotating surface $(r_G(\theta),z_G(\theta))$ and the relation between θ and φ, φ being the toroidal angle. As the rotating surface, for instance, we may put

$$r_G(\theta) = R_0+a_c\cos\theta, \quad z_G(\theta) = a_c\sin\theta. \tag{7}$$

The relation between θ and φ, called winding law, can be most generally expressed as

$$\theta = \theta(p), \quad \varphi = \varphi(p),$$

in terms of a parameter p.

In order to specify the coil cross section, the direction of $\boldsymbol{u}$ must be determined; then the vector $\boldsymbol{v}$ is determined as $\boldsymbol{v}=\boldsymbol{t}\times\boldsymbol{u}$. For the sake of simplicity, we often choose its direction normal to the surface (7).

The magnetic field at the point $\boldsymbol{r}_p$ is given by Bio-Savart's law

$$\boldsymbol{B}(\boldsymbol{r}_p) = \frac{\mu_0}{4\pi}\oint \mathrm{d}l\int_0^1 \mathrm{d}\xi\int_0^1 \mathrm{d}\eta\frac{\sqrt{g}\,\boldsymbol{j}\times\boldsymbol{R}}{R^3}, \quad \boldsymbol{R} = \boldsymbol{r}_p-\boldsymbol{r}_c. \tag{8}$$

where μ_0 is the magnetic permiability in the vacuum. The point $\boldsymbol{r}_p$ is represented in the local coordinate system around the guiding curve at l,

$$\boldsymbol{r}_\mathrm{p}(x,y,z) = \boldsymbol{r}_\mathrm{G}(l)+x\boldsymbol{u}(l)+y\boldsymbol{v}(l)+z\boldsymbol{t}(l). \tag{9}$$

Then the integral across the cross section can be carried out analytically, and we obtain

$$\boldsymbol{B} = \oint \delta\boldsymbol{B}dl, \qquad \delta\boldsymbol{B} = \delta B_x\boldsymbol{u}(l)+\delta B_y\boldsymbol{v}(l)+\delta B_z\boldsymbol{t}(l), \tag{10}$$

where

$$\delta B_\alpha = \frac{\mu_0}{4\pi}\sum_{i=1}^{2}\sum_{k=1}^{2}(-1)^{i+k}F_\alpha(x_i,y_k,z) \qquad (\alpha=x,y,z), \tag{11}$$

and

$$\begin{aligned} F_x(x_i,y_k,z) = {} & (c+b_yz)\ \ln(R_{ik}+x_i)+(\tau z-\kappa_v x_i)\ \ln(R_{ik}+y_k) \\ & +\kappa_u R_{ik}+(a_y+\tau x+\kappa_v z)\ \arctan\left(\frac{x_i y_k}{zR_{ik}}\right), \end{aligned} \tag{12}$$

$$\begin{aligned} F_y(x_i,y_k,z) = {} & -(c+b_xz)\ \ln(R_{ik}+y_k)+(\tau z+\kappa_u y_k)\ \ln(R_{ik}+x_i) \\ & -\kappa_v R_{ik}-(a_x-\tau y+\kappa_u z)\ \arctan\left(\frac{x_i y_k}{zR_{ik}}\right), \end{aligned} \tag{13}$$

$$\begin{aligned} F_z(x_i,y_k,z) = {} & \{a_y+\tau(x+x_i)\}\ \ln(R_{ik}+y_k)+\{-a_x+\tau(y+y_k)\}\ \ln(R_{ik}+x_i) \\ & +(b_x-b_y)R_{ik}-2\tau z\ \arctan\left(\frac{x_i y_k}{zR_{ik}}\right). \end{aligned} \tag{14}$$

Here we have introduced the following notations:

$$c \equiv 1-\kappa_u x-\kappa_v y, \qquad R_{ik} \equiv (x_i^2+y_k^2+z^2)^{1/2}, \tag{15}$$

$$x_1 \equiv x-X_\mathrm{H},\ x_2 \equiv x-X_\mathrm{L}, \quad y_1 \equiv y-Y_\mathrm{H},\ y_2 \equiv y-Y_\mathrm{L}, \tag{16}$$

$$a_x \equiv \frac{dX_\mathrm{L}}{dl}+b_xX_\mathrm{L}-b_xx, \qquad a_y \equiv \frac{dY_\mathrm{L}}{dl}+b_yY_\mathrm{L}-b_yy,$$

$$b_x \equiv \frac{d\ln X_\mathrm{W}}{dl}, \qquad b_y \equiv \frac{d\ln Y_\mathrm{W}}{dl}. \tag{17}$$

If we put

$$\frac{dX_\mathrm{L}}{dl}=\frac{dY_\mathrm{L}}{dl}=\frac{dX_\mathrm{W}}{dl}=\frac{dY_\mathrm{W}}{dl}=0,$$

or $a_x = a_y = b_x = b_y = 0$, the results are reduced to the formulae given in Ref.1.

The integral with respect to l is carried out numerically.

§3 Error of Calculations

In calculating the magnetic field of the helical coils inappropriate models for the coil are often used. An example is the use of straight elements to approximate the helical coils. The essential difference of the helical coil and the other coils such as toroidal coils in tokamak or Ying-Yang coils is the presence of the torsion. If the straight elements without torsion are used in calculation some kind of systematic error arises. The other example is the application of the formulae in Ref.1 to the case of the varying cross section, or the varying position relative to the guiding curve.

We shall consider the error caused by the use of inappropriate formulae, assuming that the integration is carried out precisely. As the magnetic field is calculated by the Bio-Savart's law

$$B \propto \int \frac{j \times R}{R^3} \mathrm{d}V, \tag{18}$$

the calculated magnetic field satisfies the divergence-free condition $\mathrm{div}\, B = 0$. On the other hand , taking the curl of the magnetic field, we have

$$\mathrm{curl}\, B \propto \mathrm{curl} \int \mathrm{grad}\left(\frac{1}{R}\right) \times j \mathrm{d}V$$

$$\propto \mathrm{grad} \int \frac{\mathrm{div}\, j}{R} \mathrm{d}V = -\ \mathrm{grad} \int \frac{\dot{\sigma} \mathrm{d}V}{R}. \tag{19}$$

This means that the necessary condition for the curl-free field is $\operatorname{div} j=0$. The quantity $\dot{\sigma} \equiv -\operatorname{div} j$ means the charge density accumulating in unit time. Then the expression (19) corresponds to the electrostatic field produced by the accumulating charge density.

We shall now consider the case that the formulae given in Ref.1 are employed to the case of varying cross section. In that case the current in the coil is taken erroneously as

$$\sqrt{g}\,\boldsymbol{j}_* = I_c\left\{(1-\kappa_u X-\kappa_v Y)\,\boldsymbol{t}-\tau Y\boldsymbol{u}+\tau X\boldsymbol{v}\right\}, \tag{20}$$

instead of eq.(6). In order to take the divergence of this current we write the current in the form

$$\sqrt{g}\,\boldsymbol{j}_* = I_c\left\{(1-\kappa_u X-\kappa_v Y)\,\boldsymbol{t}-\tau Y\boldsymbol{u}+\tau X\boldsymbol{v}\right\}[\mathrm{H}(\xi)-\mathrm{H}(\xi-1)][\mathrm{H}(\eta)-\mathrm{H}(\eta-1)], \tag{21}$$

where

$$\mathrm{H}(x) = \begin{cases} 1 & (x>0), \\ 0 & (x<0), \end{cases}$$

is the Heaviside's step function. Since the contravariant component of the electric current in the directions of ξ and η are

$$\sqrt{g}\,j_*^{\xi} = -I_c Y_W\frac{\partial X}{\partial l}(1-\kappa_u X-\kappa_v Y)[\mathrm{H}(\xi)-\mathrm{H}(\xi-1)][\mathrm{H}(\eta)-\mathrm{H}(\eta-1)],$$

$$\sqrt{g}\,j_*^{\eta} = -I_c X_W\frac{\partial Y}{\partial l}(1-\kappa_u X-\kappa_v Y)[\mathrm{H}(\xi)-\mathrm{H}(\xi-1)][\mathrm{H}(\eta)-\mathrm{H}(\eta-1)],$$

we obtain

$$\dot{\sigma} = I_c\left\{\delta(\xi)\frac{\partial X_L}{\partial l}-\delta(\xi-1)\frac{\partial X_H}{\partial l}\right\}[\mathrm{H}(\eta)-\mathrm{H}(\eta-1)]$$

$$+I_c\left\{\delta(\eta)\frac{\partial Y_L}{\partial l}-\delta(\eta-1)\frac{\partial Y_H}{\partial l}\right\}[\mathrm{H}(\xi)-\mathrm{H}(\xi-1)]$$

$$-I_c\left\{\frac{\kappa_u\frac{\partial X}{\partial l}+\kappa_v\frac{\partial Y}{\partial l}}{1-\kappa_u X-\kappa_v Y}-\frac{\partial X_W}{\partial l}-\frac{\partial Y_W}{\partial l}\right\}[H(\xi)-H(\xi-1)][H(\eta)-H(\eta-1)]. \qquad (22)$$

Here $\delta(x)$ is the Dirac's delta function.

In the similar way, if the helical coil is approximated in terms of straight elements, $\dot{\sigma}$ proportional to the torsion appears

$$\dot{\sigma} = [H(\eta)-H(\eta-1)][\delta(\xi)-\delta(\xi-1)]\frac{I_c\tau Y}{X_W}$$

$$+[H(\xi)-H(\xi-1)][\delta(\eta)-\delta(\eta-1)]\frac{I_c\tau X}{Y_W}$$

$$+[H(\xi)-H(\xi-1)][H(\eta)-H(\eta-1)]\frac{I_c\tau(\kappa_u Y-\kappa_v X)}{1-\kappa_u X-\kappa_v Y}. \qquad (23)$$

Since these charges are multipoles, eq.(19) is expected to decrease rapidly as the point leaves from the coils. In the vicinity of the coils, or in the case of the coils of large size, however, these effects may be important. Making the spatial map of curl B seems to be useful to check the validity of the model and the calculation.

If the integration with respect to l in eq.(10) is approximated in the numerical calculation as the finite sum of the contribution from discrete points, the current is replaced by the sum of delta functions; $\dot{\sigma}$ appears as the results of such replacement. This gives a physical interpretation to the error cased by the numerical integration. However, such error can be decreased as small as possible by increasing the number of integration points. On the contrary, the divergence given in eqs.(22) or (23) can be resolved only by changing the model or the formulae employed.

§4 Summary

For the calculation of the magnetic field of the helical coils the the torsion of the helical coils must be taken into account to the model of the helical current and the formulae used in the calculation. If the cross section or the position of the coil relative to the guiding curve are varying in the direction of the coil current, the formulae given in this paper are useful.

For the case of the inappropriate formulae or the inappropriate model for the current is used in the calculation of the helical magnetic field, the check of the curl-free condition is necessary for the reliability of the calculation.

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