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Theory of Longitudinal Adiabatic Invariant in the Helical Torus

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Abstract

The theory on the longitudinal adiabatic invariant valid for the helical magnetic field is developed on the basis of the variable transformation from the guiding center variables to the new adiabatic variables. The theory uses the Hamiltonian formalism with non canonical variables. Under the assumption that the rotational transform per period is small, the adiabatic invariant is defined by integral along the toroidal direction, not along the field line. The transition between the passing and the ripple trapped states is investigated; the jump of adiabatic invariant and other variables in the transition process is evaluated. The change of distribution function in the variable transformation is also discussed.

KEYWORD: adiabatic invariant, distribution function, Hamiltonian formalism, helical torus, transition

§1. Introduction

Understanding of particle orbit is important for containing high temperature plasma in the asymmetric magnetic confinement systems. The particle orbit in the toroidal helical devices is very complicated, and their primary knowledge are obtained by integrating drift equations of motion in a given magnetic field for various initial conditions. A lot of works are devoted to such vast calculations. The other technique to study particle orbit in the magnetic fields is that based on the (longitudinal) adiabatic invariant.[1-5]

The usual concept of the adiabatic invariant is

$$J_{/\!/} = \oint m v_{/\!/} \mathrm{d}l, \tag{1.1}$$

where $mv_{/\!/}$ is the momentum parallel to the magnetic line of force and the integration is along the magnetic line of force. Unfortunately, in the helical torus with non-zero rotational transform, the label for the magnetic lines of force can not be used as coordinates. The particles circulating along the toroidal direction may not be considered as doing periodic motion along the magnetic lines of force.

The purpose of this paper is to formulate the theory of the adiabatic invariant in the helical torus. The Hamiltonian formalism in terms of non-canonical variables[6,7] is used to perform the systematic perturbation expansion. We first introduce the Hamiltonian system to the guiding center variables, and then define the transformation to the new set of adiabatic variables, which correspond to the periodic motion along the toroidal direction.

The helical torus with our interest has many periods in the toroidal direction, N, and the rotational transform per period is considered as a small parameter, $\iota/N \ll 1$. The particles are considered doing fast periodic motion in the toroidal direction, periodic motion between magnetic hill, or circulating motion in the toroidal direction, depending on the pitch angle between their velocity and the magnetic field. In this description the particles are divided into only three types: ripple trapped particles, passing particles in the positive direction, and passing particles in the negative direction. This is very much simplified picture, compared to the theory based on the average along the magnetic lines of force, where there are many kinds of trapped particles: trapped in single well, trapped in double well, and so on [see Fig.1].

In the theory of the adiabatic invariant, the transition between the trapped and untrapped state is important.[8] The adiabatic invariant may change its value in the course of transition. The collisionless diffusion as well as the chaos of particle motion can be caused by the change of adiabatic invariant.

The adiabatic invariant is closely related to the description of plasma on the basis of the bounce averaged particle distribution function. The relation of the bounce averaged distribution function to the distribution function of the guiding center is also important to establish the entire understanding of the particle behaviors in the helical torus.

In Section 2, the drift equation of motion is written in Hamiltonian form with non-canonical variables. The adiabatic variables describing fast bounce motion in the toroidal direction and slower motion across the poloidal direction are introduced; In Section 3, the perturbation expansion with respect to the small parameter is carried out to obtain the concrete form of the transformation. The transition between the different states is discussed in Section 4. Section 5 is devoted to the discussion of the transformation of the distribution function associated with the transformation of variables introduced in Section 3. Summary and conclusions of this paper are stated in Section 6.

The scope of this paper is restricted to the formal development of the theoretical method. Application of the theory to the problems of interest such as the evaluation of particle loss or evaluation of transport coefficients in the helical magnetic field configurations is left to the other paper.

§2. Basic Equations

The drift equation of motion in the static magnetic and electric field can be written as [6]

$$\dot{\boldsymbol{r}} = \frac{v_{/\!/}}{B_*}\{\boldsymbol{B}+ \mathrm{curl}\ (\rho_{/\!/}\boldsymbol{B})\}, \tag{2.1}$$

where dot denotes time derivative and

$$\rho_{/\!/} = \frac{mv_{/\!/}}{eB}, \qquad v_{/\!/} = \left(\frac{2}{m}\right)^{1/2}(E-\mu B-e\Phi_{E})^{1/2}, \tag{2.2}$$

and

$$B_* \equiv \boldsymbol{b}\cdot\boldsymbol{B}_*, \quad \boldsymbol{b} \equiv \boldsymbol{B}/B, \quad \boldsymbol{B}_* \equiv \boldsymbol{B}+\rho_{/\!/}\,\mathrm{curl}\,\boldsymbol{B}. \tag{2.3}$$

Here E is the energy, μ is the magnetic moment, m and e are mass and charge of the particle, and Φ_E is the static electric potential. We also use the notation

$$\ell_{\circ} = \boldsymbol{b}\cdot \mathrm{curl}\,\boldsymbol{b} = \frac{\boldsymbol{B}\cdot \mathrm{curl}\,\boldsymbol{B}}{B^2}, \tag{2.4}$$

so that $B_*=B(1+\rho_{/\!/}\ell_0)$.

We shall write eq.(2.1) in the Hamiltonian form.

We introduce the curvilinear coordinate (ψ, θ, φ), in which the magnetic field can be expressed as

$$B = \nabla\psi\times\nabla\theta-\nabla\Psi_p(\psi)\times\nabla\varphi,$$

$$= B_\varphi\nabla\varphi+B_\theta\nabla\theta+B_\psi\nabla\psi. \tag{2.5}$$

The rotational transform is

$$\iota(\psi) = \frac{d\Psi_p}{d\psi} = \frac{B^\theta}{B^\varphi}. \tag{2.6}$$

The Jacobian is

$$\sqrt{g} = \left(\frac{\partial r}{\partial\psi}\times\frac{\partial r}{\partial\theta}\right)\cdot\frac{\partial r}{\partial\varphi} = (\nabla\psi\times\nabla\theta\cdot\nabla\varphi)^{-1}. \tag{2.7}$$

The covariant and contravariant component of any vector are expressed by the subscript and superscript, respectively. Assuming the magnetostatic equilibrium with scalar pressure

$$j\times B = \nabla p(\psi),\ \ j = \mathrm{curl}\, B, \tag{2.8}$$

we adopt the coordinates such that B_φ and B_θ are constant on each magnetic surface (Boozer coordinate).[9] Since B_θ is proportional to the toroidal current within the magnetic surface, $B_\theta=0$ in the vacuum field, or in the currentless equilibrium. Also $B_\psi=0$ in the vacuum field. In this coordinate

$$\sqrt{g}B^\varphi = 1,\quad \sqrt{g}B^\theta = \iota,\quad B^\psi = 0,$$

$$\sqrt{g}\,j^\psi = 0,\quad \sqrt{g}\,j^\theta = \frac{\partial B_\psi}{\partial\varphi}-\frac{dB_\varphi}{d\psi},\quad \sqrt{g}\,j^\varphi = \frac{dB_\theta}{d\psi}-\frac{\partial B_\psi}{\partial\theta}, \tag{2.9}$$

and

$$B^2 = B^\varphi B_\varphi+B^\theta B_\theta = \frac{1}{\sqrt{g}}[B_\varphi+\iota B_\theta],\quad j\cdot B = j^\theta B_\theta+j^\varphi B_\varphi. \tag{2.10}$$

We denote the canonical coordinates and momenta q and p, respectively, and put $z=(z^1,z^2,z^3,z^4)=(\psi,\theta,\varphi,\rho_\parallel)$. Then $z^i = z^i(q,p)$. The time derivative of the variable z^i is given by

$$\dot{z}^i = [z^i, H], \tag{2.11}$$

where

$$[F,G] = \frac{\partial F}{\partial \boldsymbol{q}}\cdot\frac{\partial G}{\partial \boldsymbol{p}} - \frac{\partial F}{\partial \boldsymbol{p}}\cdot\frac{\partial G}{\partial \boldsymbol{q}}, \tag{2.12}$$

is the Poisson bracket and $H(\boldsymbol{q},\boldsymbol{p})$ is the Hamiltonian. Since Hamiltonian H is considered as $H(\boldsymbol{z})$, eq.(2.11) is rewritten to the form

$$\dot{z}^i = \sum_j \sigma^{ij}\frac{\partial H}{\partial z^j}, \tag{2.13}$$

where

$$\sigma^{ij} = [z^i, z^j] \tag{2.14}$$

Because of the invariance of the Poisson brackets with respect to the canonical transformation, the matrix σ^{ij} does not depend on the choice of canonical variables $(\boldsymbol{q},\boldsymbol{p})$. With use of Hamiltonian form of equation we can treat to non-canonical variables which have no explicit relation to the canonical variables, and to perform the systematic calculation to find the constant of motion.

If we introduce the quantities

$$P_i = \boldsymbol{p}\cdot\frac{\partial \boldsymbol{q}}{\partial z^i}, \tag{2.15}$$

and the Lagrange brackets

$$\omega_{ij} = \frac{\partial P_j}{\partial z^i} - \frac{\partial P_i}{\partial z^j} = \frac{\partial \boldsymbol{q}}{\partial z^i}\cdot\frac{\partial \boldsymbol{p}}{\partial z^j} - \frac{\partial \boldsymbol{q}}{\partial z^j}\cdot\frac{\partial \boldsymbol{p}}{\partial z^i}, \tag{2.16}$$

the matrix (ω_{ij}) is inverse of the matrix (σ^{ij}). Since

$$P_1 = P_\psi = e\rho_\parallel B_\psi, \quad P_2 = P_\theta = e(\rho_\parallel B_\theta + \psi),$$

$$P_3 \equiv P_\varphi = e\rho_{/\!/}B_\varphi - \Psi_p, \quad P_4 \equiv P_{\rho_{/\!/}} = 0, \tag{2.17}$$

we have

$$(\omega_{ij}) = \begin{pmatrix} 0, & e\sqrt{g}B_*^\varphi, & -e\sqrt{g}B_*^\theta & -eB_\psi \\ -e\sqrt{g}B_*^\varphi, & 0, & 0, & -eB_\theta \\ e\sqrt{g}B_*^\theta, & 0, & 0, & -eB_\varphi \\ eB_\psi, & eB_\theta, & eB_\varphi, & 0 \end{pmatrix}. \tag{2.18}$$

This matrix can be easily inverted to yield the matrix (σ^{ij}).

Thus, eq.(2.1) can be expressed in the form

$$\dot\psi = [\psi, H] = [\psi,\rho_{/\!/}]\frac{\partial H}{\partial \rho_{/\!/}} + [\psi,\varphi]\frac{\partial H}{\partial \varphi} - [\theta,\psi]\frac{\partial H}{\partial \theta},$$

$$\dot\theta = [\theta, H] = [\theta,\rho_{/\!/}]\frac{\partial H}{\partial \rho_{/\!/}} + [\theta,\psi]\frac{\partial H}{\partial \psi} - [\varphi,\theta]\frac{\partial H}{\partial \varphi},$$

$$\dot\varphi = [\varphi, H] = [\varphi,\rho_{/\!/}]\frac{\partial H}{\partial \rho_{/\!/}} + [\varphi,\theta]\frac{\partial H}{\partial \theta} - [\psi,\varphi]\frac{\partial H}{\partial \psi},$$

$$\dot\rho_{/\!/} = [\rho_{/\!/}, H] = -[\psi,\rho_{/\!/}]\frac{\partial H}{\partial \psi} - [\theta,\rho_{/\!/}]\frac{\partial H}{\partial \theta} - [\varphi,\rho_{/\!/}]\frac{\partial H}{\partial \varphi} \tag{2.19}$$

with the Hamiltonian

$$H(\rho_{/\!/},\psi,\theta,\varphi) = \frac{1}{2}\frac{e^2}{m}\rho_{/\!/}^2 B^2 + \mu B + e\Phi_E. \tag{2.20}$$

The values of Poisson bracket are given as

$$[\psi,\rho_{/\!/}] = 0, \quad [\theta,\rho_{/\!/}] = \frac{B_*^\theta}{eBB_*}, \quad [\varphi,\rho_{/\!/}] = \frac{B_*^\varphi}{eBB_*},$$

$$[\theta,\psi] = \frac{B_\varphi}{e\sqrt{g}BB_*}, \quad [\varphi,\theta] = \frac{B_\psi}{e\sqrt{g}BB_*}, \quad [\psi,\varphi] = \frac{B_\theta}{e\sqrt{g}BB_*}. \tag{2.21}$$

These Hamiltonian equations of motion is different from those given by Boozer.[10-12] If one assume that B_ψ=0, eqs.(2.17) give relations between the set of variables $(\psi,\rho_{/\!/})$ and the set of momenta (P_θ,P_φ)

$$P_\theta = P_\theta(\psi,\rho_{/\!/}), \quad P_\varphi = P_\varphi(\psi,\rho_{/\!/}), \tag{2.22}$$

that is, the variables ψ and $\rho_{/\!/}$ are related to the momenta conjugate to θ and φ.

$$\psi = \psi(P_\theta,P_\varphi), \quad \rho_{/\!/} = \rho_{/\!/}(P_\theta,P_\varphi). \tag{2.23}$$

Since we do not assume B_ϕ=0, no explicit relation between physical variables and canonical variables exists. This is the reason for using the rather complicated form of equations in con-canonical variables.

In the next section, we will consider the transformation to the new variables $Z \equiv (Z^1,Z^2,Z^3,Z^4)$ such that $Z^4 \equiv J_{/\!/}$ is the adiabatic invariant associating with the fast periodic motion along the toroidal direction. Before doing the concrete calculations, we will discuss some features of our problem due to the smallness of dimension.

We shall consider the transformation to the new adiabatic variables $Z = (Z^i) = (\alpha,\beta,\omega,J_{/\!/})$, anticipating that $J_{/\!/}$ is the adiabatic invariant relating fast periodic motion between magnetic well, and ω is the angle variable. The standard method of transformation is to construct the set of variables satisfying the relations

$$\{J_{/\!/},\alpha\} = \{J_{/\!/},\beta\} = \{\alpha,\omega\} = \{\beta,\omega\} = 0, \tag{2.24}$$

and

$$H = H(\alpha,\beta,J_{/\!/}). \tag{2.25}$$

If the transformation satisfying eqs.(2.24) is found, the pair of variables ($J_{/\!/}$, ω) becomes canonical conjugate, and the time derivative of $J_{/\!/}$becomes

$$\dot{J}_{/\!/} = \{J_{/\!/},H\} = \{J_{/\!/},\omega\}\,\frac{\partial H}{\partial\omega}, \tag{2.26}$$

which vanishes because of eq.(2.25). Hence $J_{/\!/}$ becomes constant of motion, or adiabatic invariant.

However, the Jacobi's identity

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\} = 0, \tag{2.27}$$

incorporating with eqs.(2.24) poses restrictions to the other Poisson brackets $\{\omega,J_{/\!/}\}$ and $\{\alpha,\beta\}$. The Poisson bracket $\{J_{/\!/},\omega\}$ cannot depend on α

and β; and it is essentially constant. On the other hand $[\alpha,\beta]$ is only function of α and β. These restrictions are too strong, so that we cannot find such a transformation, as is shown later. To avoid this difficulty we will relax the constraint to the following

$$[J_{/\!/},\alpha] = [J_{/\!/},\beta] = 0, \tag{2.28}$$

with eq.(2.25). The time derivative of $J_{/\!/}$ also vanishes under these conditions.

In the following we shall use the notations

$$\gamma \equiv [\omega, J_{/\!/}], \tag{2.29}$$

$$\sigma^{\alpha\beta} \equiv [\beta,\alpha], \tag{2.30}$$

$$\sigma^{\omega\alpha} \equiv [\alpha,\omega], \qquad \sigma^{\omega\beta} \equiv [\beta,\omega], \tag{2.31}$$

and

$$\{F,G\} \equiv \frac{\partial F}{\partial J_{/\!/}}\frac{\partial G}{\partial \omega}-\frac{\partial F}{\partial \omega}\frac{\partial G}{\partial J_{/\!/}}, \tag{2.32}$$

$$(F,G) \equiv \frac{\partial F}{\partial \alpha}\frac{\partial G}{\partial \beta}-\frac{\partial F}{\partial \beta}\frac{\partial G}{\partial \alpha}. \tag{2.33}$$

Then

$$[F,G] = -\frac{\gamma}{\varepsilon}\{F,G\}-\sigma^{\alpha\beta}(F,G)-\sigma^{\omega\alpha}\left\{\frac{\partial F}{\partial \omega}\frac{\partial G}{\partial \alpha}-\frac{\partial F}{\partial \alpha}\frac{\partial G}{\partial \omega}\right\}-\sigma^{\omega\beta}\left\{\frac{\partial F}{\partial \omega}\frac{\partial G}{\partial \beta}-\frac{\partial F}{\partial \beta}\frac{\partial G}{\partial \omega}\right\}. \tag{2.34}$$

The factor ε^{-1} in the first term of this equation expresses that the periodic motion related to the action variable $J_{/\!/}$ and the angle variable ω is the fast one.

The equation of motion in the adiabatic variables are written as

$$\dot{\alpha} = -\sigma^{\alpha\beta}\frac{\partial H}{\partial \beta}, \qquad \dot{\beta} = \sigma^{\alpha\beta}\frac{\partial H}{\partial \alpha}, \tag{2.35}$$

$$\dot{J}_{/\!/} = 0, \qquad \dot{\omega} = \frac{\gamma}{\varepsilon}\frac{\partial H}{\partial J_{/\!/}}-\sigma^{\omega\alpha}\frac{\partial H}{\partial \alpha}-\sigma^{\omega\beta}\frac{\partial H}{\partial \beta}. \tag{2.36}$$

§3. Adiabatic Variables

We now describe the physical meaning of the small parameter ε. Since the particle energy is considered to be small, it is natural to assume that $\rho_{/\!/}$ scales as ε. If we assume that the rotational transform ι is of order of unity, trapped particles carry out the periodic motion along the magnetic lines of force. For passing particles, however, the motion along the magnetic line of force is not periodic because of the rotational transform. Therefore, the rotational transform ι is assumed of order of ε, in order to make the motion periodic in the lowest order. Since the rotational transform appears in the combination ι/N, our ordering is expressed as

$$\frac{\rho_{/\!/}}{a_p} \sim \frac{\iota}{N} \sim \varepsilon \ll 1, \tag{3.1}$$

a_p being the characteristic length in the poloidal direction, say minor radius of the plasma.

As the explicit form of the transformation, we express the old variables in terms of the new variables, in the form of expansion

$$\psi = \alpha+\varepsilon\Psi^{(1)}(J_{/\!/},\omega,\alpha,\beta)+ \cdots,$$

$$\theta = \beta+\varepsilon\Theta^{(1)}(J_{/\!/},\omega,\alpha,\beta)+ \cdots,$$

$$\varphi = \Phi_0(J_{/\!/},\omega,\alpha,\beta)+\varepsilon\Phi^{(1)}(J_{/\!/},\omega,\alpha,\beta)+ \cdots,$$

$$\rho_{/\!/} = \varepsilon\{P_0(J_{/\!/},\omega,\alpha,\beta)+\varepsilon P^{(1)}(J_{/\!/},\omega,\alpha,\beta)+ \cdots \}. \tag{3.2}$$

The coefficients in eq.(2.34) as well as the Hamiltonian are also expanded as

$$\gamma = 1+\varepsilon\gamma^{(1)}(\alpha,\beta,J_{/\!/},\omega)+\cdots,$$

$$\sigma^{\alpha\beta} = \frac{1}{e}\left\{1+\varepsilon\sigma^{\alpha\beta(1)}(\alpha,\beta)+\varepsilon^2\sigma^{\alpha\beta(2)}(\alpha,\beta,J_{/\!/},\omega)+\cdots\right\},$$

$$\sigma^{\omega\alpha} = \varepsilon\sigma^{\omega\alpha(1)}(\alpha,\beta,J_{/\!/},\omega)+\cdots,$$

$$\sigma^{\omega\beta} = \varepsilon\sigma^{\omega\beta(1)}(\alpha,\beta,J_{/\!/},\omega)+\cdots,$$

$$H = \varepsilon^2\{H_0(J_{/\!/},\alpha,\beta)+\varepsilon H^{(1)}(J_{/\!/},\alpha,\beta)+\cdots\}. \tag{3.4}$$

The Jacobi's identity yields

$$\frac{\partial\sigma^{\omega\alpha(1)}}{\partial\alpha}+\frac{\partial\sigma^{\omega\beta(1)}}{\partial\beta}+\frac{1}{e}\frac{\partial\sigma^{\alpha\beta(2)}}{\partial J_{/\!/}} = 0, \tag{3.5}$$

$$\frac{\partial\sigma^{\omega\alpha(1)}}{\partial\omega} = \frac{\partial\gamma^{(1)}}{\partial\beta}, \qquad \frac{\partial\sigma^{\omega\beta(1)}}{\partial\omega} = -\frac{\partial\gamma^{(1)}}{\partial\alpha}. \tag{3.6}$$

The transformation required can be obtained by substituting these expressions (3.2),(3.3) into eqs.(2.19), and using relations (2.34). Then, the lowest order equations are

$$\{P_0,\Phi_0\} = \frac{1}{eB_\varphi}, \tag{3.7}$$

$$H_0(P_0,\Phi_0,\alpha,\beta) = \frac{1}{2}\frac{e^2}{m}P_0^2B^2+\mu B+e\Phi_s \equiv E(J_{/\!/},\alpha,\beta), \tag{3.8}$$

which can be solved in terms of the generating function

$$S(J_{/\!/},\Phi_0,\alpha,\beta) = e\int^{\Phi_0}(P_0B_\varphi-\Psi_p)\mathrm{d}\varphi, \tag{3.9}$$

where the integration is carried out to the direction of the particle velocity, for fixed ψ and θ. The longitudinal invariant $J_{/\!/}$ and phase variable ω are defined by the following integrals

$$J_{/\!/}(E,\alpha,\beta) = \frac{e}{2\pi}\oint(P_0B_\varphi-\Psi_p)\mathrm{d}\varphi, \tag{3.10}$$

$$\omega = \frac{\partial S}{\partial J_{/\!/}} \equiv \omega_0(J_{/\!/},\alpha,\beta,\Phi_0) = \left(\frac{\partial J_{/\!/}}{\partial E}\right)^{-1}\int^{\Phi_0}eB_\varphi\frac{\partial P_0}{\partial E}\mathrm{d}\varphi. \tag{3.11}$$

The lower bound of the integration in eqs.(3.9) and (3.11) is not explicitly written, which may depend on α and β.

We consider the case that the potential

$$U \equiv \mu B + e\Phi_E, \tag{3.12}$$

has only one maximum and minimum per period in the toroidal direction. The maximum of U is denoted as $U_{\max}(\psi,\theta)$. If electric potential Φ_E does not depend on φ, $U_{\max}=\mu B_{\max}+e\Phi_E$, where $B_{\max}$ is the maximum of the magnetic field strength with respect φ. Then, there are three types of particles: particles with $E<U_{\max}$ trapped between helical ripple (r), and the particles circulating around the torus with $E>U_{\max}$, (+) or (−) depending on the direction of its movement. For the ripple trapped particles (r), eq.(3.10) becomes to

$$J_{/\!/}(E,\alpha,\beta) \equiv J_r(E,\alpha,\beta) = \frac{e}{2\pi}\oint P_0 B_\varphi d\varphi. \tag{3.13}$$

For passing particles (+) and (−),

$$J_{/\!/}(E,\alpha,\beta) \equiv J_\pm(E,\alpha,\beta) = \frac{e}{2\pi}\int_0^{2\pi/N} |P_0| B_\varphi d\varphi \mp \frac{e\Psi_p}{N}. \tag{3.14}$$

The double sign in eq.(3.14) correspond to the direction of particle movement.

We now consider the next order equations:

$$\{P_0,\Psi^{(1)}\}+\frac{1}{e}[P_0,\alpha] = 0, \tag{3.15}$$

$$\{\Phi_0,\Psi^{(1)}\}+\frac{1}{e}[\Phi_0,\alpha] = \frac{B_\theta}{eB_\varphi}, \tag{3.16}$$

$$\{P_0,\Theta^{(1)}\}+\frac{1}{e}[P_0,\beta] = \frac{P_0}{eB_\varphi}\left(\frac{\partial B_\psi}{\partial\varphi}-\frac{dB_\varphi}{d\alpha}\right)+\frac{\iota}{eB_\varphi}, \tag{3.17}$$

$$\{\Theta^{(1)},\Phi_0\}+\frac{1}{e}[\beta,\Phi_0] = \frac{B_\psi}{eB_\varphi}. \tag{3.18}$$

From these equations, we can obtain

$$\Psi^{(1)} = \frac{1}{e}\frac{\partial S}{\partial \beta} - B_\theta P_0 + \Psi_0^{(1)}(\alpha,\beta), \tag{3.19}$$

$$\Theta^{(1)} = -\frac{1}{e}\frac{\partial S}{\partial \alpha} + B_\psi P_0 + \Theta_0^{(1)}(\alpha,\beta), \tag{3.20}$$

where, $\Psi_0^{(1)}$ and $\Theta_0^{(1)}$ include constant of integration, which will be chosen appropriately.

We note that the term in S in the first term in RHS of eq.(3.20) gives an interesting contribution. We shall put the toroidal angle of the two turning points of the ripple trapped particles as φ_{t1} and φ_{t2}. Because of

$$S(\varphi_{t1}) - S(\varphi_{t2}) = \pi J_{/\!/} + e\Psi_p(\varphi_{t1} - \varphi_{t2}), \tag{3.21}$$

the difference of $\Theta^{(1)}$ between these two points becomes

$$\Theta^{(1)}(\varphi_{t1}) - \Theta^{(1)}(\varphi_{t2}) = \iota(\varphi_{t1} - \varphi_{t2}). \tag{3.22}$$

This means that the actual orbit of the ripple trapped particles is along the magnetic line of force, even if the integrals for $J_{/\!/}$ is carried out along the φ direction for fixed θ. For passing particles, on the other hand, the movement in θ direction along the magnetic line of force is not included in $\Theta^{(1)}$, but in $\dot{\beta}$.

The first order Hamiltonian is

$$H^{(1)} = \Psi^{(1)}\frac{\partial H_0}{\partial \alpha} + \Theta^{(1)}\frac{\partial H_0}{\partial \beta} + \Phi^{(1)}\frac{\partial H_0}{\partial \Phi_0} + P^{(1)}\frac{\partial H_0}{\partial P_0} = 0. \tag{3.23}$$

The equation (3.23) yields

$$P^{(1)} = \Psi^{(1)}\frac{\partial P_0}{\partial \alpha}\bigg|_{E,\varphi} + \Theta^{(1)}\frac{\partial P_0}{\partial \beta}\bigg|_{E,\varphi} + \Phi^{(1)}\frac{\partial P_0}{\partial \varphi}\bigg|_{E}, \tag{3.24}$$

We now consider the equations derived from the second order contribution of $\{\theta,\psi\}$ and $\{\varphi,\rho_{/\!/}\}$

$$\frac{1}{e}[\Psi^{(1)},\beta] + \frac{1}{e}[\alpha,\Theta^{(1)}] + \sigma^{\alpha\beta(1)} + \{\Psi^{(1)},\Theta^{(1)}\}$$

$$= -\frac{1}{eB_\varphi}\left\{ \iota B_\theta + \sqrt{g}(B\cdot \mathrm{curl}\, B)P_0 \right\}, \tag{3.25}$$

$$\{P_0, \Phi^{(1)}\} + \{P^{(1)}, \Phi_0\} + \frac{1}{e}[P_0, \Phi_0] + \frac{\gamma^{(1)}}{eB_\varphi}$$

$$= -\frac{\Psi^{(1)}}{eB_\varphi^2}\frac{\mathrm{d}B_\varphi}{\mathrm{d}\alpha} - \frac{B_\theta}{eB_\varphi^2}\left\{ \iota + P_0\left(\frac{\partial B_\psi}{\partial \varphi} - \frac{\mathrm{d}B_\varphi}{\mathrm{d}\alpha}\right) \right\}. \tag{3.26}$$

From eq.(3.25), using eqs.(3.19) and (3.20), we obtain

$$\sigma^{\alpha\beta(1)} + \frac{\partial \Psi_0^{(1)}}{\partial \alpha} + \frac{\partial \Theta_0^{(1)}}{\partial \beta} = 0. \tag{3.27}$$

The equation (3.26), which determines $\Phi^{(1)}$ and $\gamma^{(1)}$, is more complicated. Eliminating $P^{(1)}$ by using eq.(3.24), multiplying eq.(3.26) by B_φ and adding eq.(3.25), after some algebra we obtain

$$\frac{\partial}{\partial \varphi}\left(eB_\varphi \frac{\partial P_0}{\partial E}\Phi^{(1)} \right)_E + \frac{\partial}{\partial \alpha}\left(eB_\varphi \frac{\partial P_0}{\partial E}\Psi^{(1)} \right)_{E,\varphi} + \frac{\partial}{\partial \beta}\left(eB_\varphi \frac{\partial P_0}{\partial E}\Theta^{(1)} \right)_{E,\varphi}$$

$$+ \left(\gamma^{(1)} + \sigma^{\alpha\beta(1)} + \frac{\iota B_\theta}{B_\varphi}\right) eB_\varphi \frac{\partial P_0}{\partial E} + \frac{m}{e}\sqrt{g}\ell_0 = 0. \tag{3.28}$$

The quantity $\gamma^{(1)}$ is determined so that eq.(3.28) yields periodic solution $\Phi^{(1)}$. If we choose the additive constant for $\Psi^{(1)}$ and $\Theta^{(1)}$ such that

$$\oint \Psi^{(1)} \mathrm{d}\omega = \oint \Theta^{(1)} \mathrm{d}\omega = 0, \tag{3.29}$$

we have

$$\gamma^{(1)} + \sigma^{\alpha\beta(1)} = -\frac{\iota B_\theta}{B_\varphi} - \frac{m}{2\pi e}\frac{\partial E}{\partial J_{/\!/}} \oint \sqrt{g}\ell_0 \mathrm{d}\varphi, \tag{3.30}$$

which implies the relation

$$\gamma^{(1)} = -\sigma^{\alpha\beta(1)} - \frac{\iota B_\theta}{B_\varphi}, \tag{3.31}$$

for trapped particles, because the second term in RHS of eq.(3.30) vanishes. Note that eq.(3.31) is valid for both trapped and passing particles in case of vacuum magnetic field.

If we retain only terms up to first order, the equations of motion in the adiabatic variables become to

$$\dot{\alpha} = \frac{1}{e}\frac{\partial J_{/\!/}}{\partial \beta} \bigg/ \frac{\partial J_{/\!/}}{\partial E}, \tag{3.32}$$

$$\dot{\beta} = -\frac{1}{e}\frac{\partial J_{/\!/}}{\partial \alpha} \bigg/ \frac{\partial J_{/\!/}}{\partial E}, \tag{3.33}$$

and

$$\dot{\omega} = (1+\gamma^{(1)}) \bigg/ \frac{\partial J_{/\!/}}{\partial E}. \tag{3.34}$$

§4. Transition Between Ripple Trapped and Passing State

The particles are in either state: positive passing (+), negative passing (-), or ripple trapped (r). Within each state, the motion of particles are well described in terms of adiabatic variables, α, β, $J_{/\!/}$. When particles moves following eq.(3.32)-(3.33), they often reaches to the point

$$U_{\max}(\alpha_t, \beta_t) = E, \tag{4.1}$$

and the transition between different states occurs. In the lowest order description without considering the displacement of the particle orbit from adiabatic coordinates, getting to the transition point (α_t, β_t) in the state (I), the particle start the motion in the new state (F) with the new value of invariant $J_{/\!/}=J_{\text{F}}(\alpha_t, \beta_t)$.

In the vicinity of the transition point, the distance from the transition point d is defined as

$$d_+, d_- = \frac{E - U_{max}}{|\nabla U_{max}|}, \quad d_r = \frac{U_{max} - E}{|\nabla U_{max}|}, \tag{4.2}$$

where the gradient in the denominator is evaluated at the transition point. Since $J_+ + J_- = J_r$, on the curve $U_{max} = E$, the change of the distance from the transition point should satisfy the the relation

$$\dot{d}_+ + \dot{d}_- + 2\dot{d}_r = 0. \tag{4.3}$$

This means that three terms can not have the same sign. If the sign of $\dot{d}$ in one state is different from other two state, the state is called as the majority state,[5] and the other two state is called as minority states. When the initial state is the minority state, the final state after transition is the majority state. On the contrary, when the initial state is the majority state, the state after transition is either of the minority state; the final state is determined by the phase of the particles at the transition.

If we assume that the phase of particles approaching to the transition point distributes uniformly, the probability of transition to each minority state can be obtained by simple argument. We introduce the distribution of particles in each state as f_σ, σ being (+), (−) or (r); then the particle flux crossing the transition point satisfies the relation

$$\dot{d}_+ f_+ + \dot{d}_- f_- + 2\dot{d}_r f_r = 0. \tag{4.4}$$

Here, factor 2 in the third term means that ripple trapped particle have positive and negative velocity. If (+), say, is the majority state, and the probability from (+) to (−), and from (+) to (−) are denoted by $w_{(+\to-)}$, and $w_{(+\to r)}$, respectively, we can write

$$-\dot{d}_- f_- = w_{(+\to-)}\dot{d}_+ f_+, \quad -2\dot{d}_r f_r = w_{(+\to r)}\dot{d}_+ f_+. \tag{4.5}$$

Since the transition follows the mechanical process, and the distribution function conserves in that process,

$$f_- = f_- = f_r. \tag{4.6}$$

Thus we can conclude

$$w_{(+\to-)} = -\frac{\dot{d}_-}{\dot{d}_+}, \quad w_{(+\to r)} = -\frac{2\dot{d}_r}{\dot{d}_+}. \tag{4.7}$$

These results agree with the transition probability given in Ref.5.

Let us study the mechanism of the transition in detail. If we take into account the first order quantities, the jump of adiabatic variables is possible in the course of the transition. The jump in adiabatic invariant is especially important in the particle confinement in the helical torus.

In some transition process such as the orbit approaching to the X-point, the motion along φ direction slows down to the same order of magnitude of the motion across the magnetic field lines, and the treatment is only possible by integrating the original equations of motion eqs.(2.1) or (2.19). However, when the separatrix crossing occurs apart from the X-point, we can construct the theory on the basis of the adiabatic equations of motion.

At first we consider the case that (+) is the majority state.

The schematic situation of transition from (+) state is shown in Fig.2. The Fig.2(a) shows the orbit in (θ,φ) plane. The orbit *(1)* initially in (+) state is reflected by the magnetic hill, and the final state is (−). The orbit *(2)* is the example of the transition resulting to the (r) state. The same situation is shown in $(\rho_{/\!/},\varphi)$ planes in

Fig.2(b). The orbit (*1*) cross the separatrix at the points shown SX1 and SX2; while the orbit (*2*) cross the separatrix only once at the point SX1'.

We assume that the transition from (+) to (-) or (r) occurs at (α_{1-},β_{1-}). This means that $U_{max}(\alpha_{1+},\beta_{1+}) = E$. The transition from (+) to (-) is considered as composed in the following two steps: i) transition from (+) to (r) at $(\psi_1,\theta_1,\varphi_1)$, ii) transition from (r) to (-) at $(\psi_2,\theta_2,\varphi_2)$. At the first step, the orbit crosses the separatrix from the positive passing to the ripple trapped state (SX1). The toroidal angle of the X-point are denoted as φ_L, φ_R, such that $\varphi_R-\varphi_L=2\pi/N$. Although the real orbit has no jump, adiabatic variables may have jump from $(\alpha_{1+}, \beta_{1+})$ to (α_{1r},β_{1r}) at the separatrix crossing.

Jump of the adiabatic variables at SX1 can be calculated from the relations

$$\psi_1 = \alpha_{1+}+\Psi_+^{(1)}(\alpha_{1+},\beta_{1+},\varphi_1) = \alpha_{1r}+\Psi_r^{(1)}(\alpha_{1r},\beta_{1r},\varphi_1),$$
$$\theta_1 = \beta_{1+}+\Theta_+^{(1)}(\alpha_{1+},\beta_{1+},\varphi_1) = \beta_{1r}+\Theta_r^{(1)}(\alpha_{1r},\beta_{1r},\varphi_1), \tag{4.8}$$

with use of eqs.(2.19) and (2.20) for $\Psi^{(1)}$ and $\Theta^{(1)}$, and put $\alpha_{1+}=\alpha_{1r}=\alpha_1$, $\beta_{1+}=\beta_{1r}=\beta_1$, in the first order terms. For instance,

$$\begin{aligned}\beta_{1+}-\beta_{1r}-\Theta_r^{(1)}(\varphi_L) &= \Theta_r^{(1)}(\alpha_1,\beta_1,\varphi_1)-\Theta_r^{(1)}(\varphi_L)-\Theta_+^{(1)}(\alpha_1,\beta_1,\varphi_1),\\ &= -\int_{\varphi_L}^{\varphi_1}\frac{\partial}{\partial\alpha}(P_0B_\varphi-\Psi_p)\Big|_{J_r}d\varphi+\int_{\varphi_L}^{\varphi_1}\frac{\partial}{\partial\alpha}(P_0B_\varphi-\Psi_p)\Big|_{J_+}d\varphi,\\ &= -\int_{\varphi_L}^{\varphi_1}\Big\{\frac{\partial}{\partial\alpha}(P_0B_\varphi-\Psi_p)\Big|_E+\frac{\partial E}{\partial\alpha}\Big|_{J_r}\frac{\partial P_0}{\partial E}B_\varphi\Big\}d\varphi\\ &\quad+\int_{\varphi_L}^{\varphi_1}\Big\{\frac{\partial}{\partial\alpha}(P_0B_\varphi-\Psi_p)\Big|_E+\frac{\partial E}{\partial\alpha}\Big|_{J_+}\frac{\partial P_0}{\partial E}B_\varphi\Big\}d\varphi,\\ &= \left\{\frac{\partial J_r/\partial\alpha}{\partial J_r/\partial E}-\frac{\partial J_+/\partial\alpha}{\partial J_+/\partial E}\right\}\int_{\varphi_L}^{\varphi_1}\frac{\partial P_0}{\partial E}B_\varphi d\varphi.\end{aligned} \tag{4.9}$$

This calculation is only in formal meaning if the integration is carried out along the separatrix, because integral on the separatrix logarithmically diverges at the X-point. Hence the integration should be carried out along the pass distant of ϵ from separatrix inside for (r) and outside for (+). This causes error of the order of $\epsilon \log(1/\epsilon)$ for the result.

If we introduce the quantities obtained by integrating along the pass distant ϵ from the separatrix

$$Y_\epsilon(\varphi) = \left(\frac{\partial J_+}{\partial E}\right)^{-1}\int_{\varphi}^{\varphi_H} eB_\varphi\left|\frac{\partial P_0}{\partial E}\right| d\varphi, \tag{4.10}$$

$$\overline{Y}_\epsilon(\varphi) = \left(\frac{\partial J_+}{\partial E}\right)^{-1}\int_{\varphi_L}^{\varphi} eB_\varphi\left|\frac{\partial P_0}{\partial E}\right| d\varphi \equiv 2\pi - Y_\epsilon(\varphi). \tag{4.11}$$

then we obtain

$$\begin{aligned} \alpha_{1+}-\alpha_{1r} &= \Psi_r^{(1)}(\varphi_L) = \Psi_r^{(1)}(\varphi_H), \\ \beta_{1+}-\beta_{1r} &= \frac{\iota}{N}\overline{Y}_\epsilon(\varphi_1)+\Theta_r^{(1)}(\varphi_L) = -\frac{\iota}{N}Y_\epsilon(\varphi_1)+\Theta_r^{(1)}(\varphi_H). \end{aligned} \tag{4.12}$$

Here $\Psi_r^{(1)}(\varphi_H)$, $\Psi_r^{(1)}(\varphi_L)$, $\Theta_r^{(1)}(\varphi_H)$ and $\Theta_r^{(1)}(\varphi_L)$ are displacement of the ripple trapped orbit at the reflecting point. (For passing particles, the constants in eqs.(2.19) and (2.20) are chosen such that $\Psi^{(1)}=\Theta^{(1)}=0$ at $\varphi=\varphi_L$.)

$$\Psi_r^{(1)}(\varphi_H) = \Psi_r^{(1)}(\varphi_L), \qquad \Theta_r^{(1)}(\varphi_H)-\frac{\pi\iota}{N} = \Theta_r^{(1)}(\varphi_L)+\frac{\pi\iota}{N}. \tag{4.13}$$

The quantities defined in eqs.(4.10) and (4.11) are related to the phase of the particles at the transition

$$\overline{Y}_\epsilon(\varphi) = \wp(\omega_+) = 2\pi-\wp(\omega_-) = 2\wp(\omega_r), \tag{4.14}$$

$$Y_\epsilon(\varphi) = 2\pi-\wp(\omega_+) = \wp(\omega_-) = 2(\wp(\omega_r)-\pi), \tag{4.15}$$

where $0\leqq\wp(\omega)<2\pi$ stands for the principal value of ω, defined by

$$\wp(\omega) \equiv \omega(\varphi)-\omega(\varphi_L). \tag{4.16}$$

Thus, once the quantities Y_ϵ or $\overline{Y}_\epsilon$ are related with the phase ω, we can tend ϵ to zero without losing its meaning.

The conditions for continuity of orbit at the second separatrix crossing SX2 are

$$\psi_2 = \alpha_{2-}+\Psi_-^{(1)}(\alpha_{2-},\beta_{2-},\varphi_2) = \alpha_{1r}+\Psi_r^{(1)}(\alpha_{2r},\beta_{2r},\varphi_2),$$
$$\theta_2 = \beta_{2-}+\Theta_-^{(1)}(\alpha_{2-},\beta_{2-},\varphi_2) = \beta_{1r}+\Theta_r^{(1)}(\alpha_{2r},\beta_{2r},\varphi_2). \tag{4.17}$$

Noting that the sign of P_0 is negative, from these relations we obtain

$$\alpha_{2-}-\alpha_{2r} = \Psi_r^{(1)}(\varphi_H) = \Psi_r^{(1)}(\varphi_L),$$
$$\beta_{2-}-\beta_{2r} = -\frac{\iota}{N}Y_\epsilon(\varphi_2)+\Theta_r^{(1)}(\varphi_H) = \frac{\iota}{N}\overline{Y}_\epsilon(\varphi_2)+\Theta_r^{(1)}(\varphi_L). \tag{4.18}$$

The change of adiabatic variables between SX1 and SX2 is calculated by using eqs.(3.32)-(3.33) as

$$\alpha_{2r}-\alpha_{1r} = \frac{1}{2e}\frac{\partial J_r}{\partial\beta}\{Y_\epsilon(\varphi_1)+Y_\epsilon(\varphi_2)\},$$
$$\beta_{2r}-\beta_{1r} = -\frac{1}{2e}\frac{\partial J_r}{\partial\alpha}\{Y_\epsilon(\varphi_1)+Y_\epsilon(\varphi_2)\}. \tag{4.19}$$

Summing up eqs.(4.9),(4.17) and (4.18), we obtain

$$\alpha_{2-}-\alpha_{1+} = \frac{1}{2e}\frac{\partial J_r}{\partial\beta}\{Y_\epsilon(\varphi_1)+Y_\epsilon(\varphi_2)\} = \frac{1}{e}\left\{Y_\epsilon(\varphi_1)\frac{\partial J_+}{\partial\beta} + Y_\epsilon(\varphi_2)\frac{\partial J_-}{\partial\beta}\right\}.$$

$$\beta_{2-}-\beta_{1+} = \frac{\iota}{N}\{Y_\epsilon(\varphi_1)-Y_\epsilon(\varphi_2)\}-\frac{1}{2e}\frac{\partial J_r}{\partial\alpha}\{Y_\epsilon(\varphi_1)+Y_\epsilon(\varphi_2)\}$$
$$= -\frac{1}{e}\left\{Y_\epsilon(\varphi_1)\frac{\partial J_+}{\partial\alpha} + Y_\epsilon(\varphi_2)\frac{\partial J_-}{\partial\alpha}\right\}. \tag{4.20}$$

Since $U_{\max}(\alpha_{2-},\beta_{2-})=E$, we have

$$(\alpha_{2-}-\alpha_{1+})\frac{\partial U_{\max}}{\partial\alpha}+(\beta_{2-}-\beta_{1+})\frac{\partial U_{\max}}{\partial\beta} = 0, \tag{4.21}$$

or

$$Y_\epsilon(\varphi_1)[U_{\max},J_-] = Y_\epsilon(\varphi_2)[U_{\max},-J_-]. \tag{4.22}$$

Hence, when

$$\frac{1}{2\pi}Y_\epsilon(\varphi_1) < \frac{[U_{max}, -J_-]}{[U_{max}, J_+]}, \tag{4.23}$$

since the second separatrix crossing can occur, the final state is (−); on the other hand when

$$\frac{1}{2\pi}Y_\epsilon(\varphi_1) < \frac{[U_{max}, J_r]}{[U_{max}, J_+]}, \tag{4.24}$$

the final state is (r).

If the phase of particles reached to the transition point $(\alpha_{1+}, \beta_{1+})$ are uniformly distributed, the probability of the transition from (+) to (−) state is

$$w_{(+\to-)} = \frac{[U_{max}, -J_-]}{[U_{max}, J_+]}, \tag{4.25}$$

while probability from (+) to (r) state is

$$w_{(+\to r)} = \frac{[U_{max}, J_r]}{[U_{max}, J_+]}. \tag{4.26}$$

This is just the relation given in eq.(4.7).

At the lowest order, we can take

$$J_-(\alpha_{2-}, \beta_{2-}) = J_-(\alpha_{1+}, \beta_{1+}) = J_+(\alpha_{1+}, \beta_{1+}) + \frac{2e}{N}\Psi_p(\alpha_{1+}). \tag{4.27}$$

To the first order we have

$$\Delta J_- = J_-(\alpha_{2-}, \beta_{2-}) - J_-(\alpha_{1+}, \beta_{1+}) = -\frac{2\iota}{N}Y_\epsilon(\varphi_1)\frac{\partial J_-}{\partial \beta}. \tag{4.28}$$

Now we consider the case that (r) is the majority state, and the transition from (r) to (+) and (−). The schematic situation is shown in Fig.3. The particle making transition to (+) state crosses separatrix in the phase range $\wp\omega_r < \pi$, and one making transition to (−) state crosses separatrix in the phase range $\pi < \wp\omega_r < 2\pi$. The probability of transition is determined the ratio of the velocity of separatrix crossing. To be noted is that the transition points from (r) state is not determined by the condition $U_{\max}(\alpha_r, \beta_r)=E$, but

$$U_{\max}\left(\alpha_r+\Psi_r^{(1)}(\varphi_L),\ \beta_r+\Theta_r^{(1)}(\varphi_R)-\frac{\iota}{N}Y_\epsilon(\varphi)\right) = E. \tag{4.29}$$

In the case of $\sqrt{mE}/eB_0a \ll \iota/N$, we can put $\Psi_r^{(1)}(\varphi_{rL})=0$. As for the $\Theta_r^{(1)}$, eq.(4.29) shows that

$$\left\{\Theta_r^{(1)}(\varphi_R)-\frac{\iota}{N}Y_\epsilon\right\}\frac{\partial U_{\max}}{\partial\beta} \leq 0, \tag{4.30}$$

in order that $U_{\max}(\alpha_r, \beta_r) \geqq E$. This condition puts restriction to the choice of $\Theta_0^{(1)}$ of the additive constant to eqs.(3.19)-(3.20), depending to the nature of the transition and the direction of the motion.

The analysis of the other case is made completely in the similar manner. The properties on the transition are summarized in Table I.

§5. Distribution Function

In this section, we consider the transformation of the distribution function associated with the transformation of variables introduced in Section 2. The distribution function of drift particles $f_G(\rho_{/\!/}, \mu, \psi, \theta, \varphi)$ is transformed to that of the adiabatic variables $f_A(J_{/\!/}, \mu, \alpha, \beta, \omega)$. The average of an arbitrary function $w(\rho_{/\!/}, \mu, \psi, \theta, \varphi)$ is defined as

$$\langle w\rangle = \int w(\rho_{/\!/}, \mu, \psi, \theta, \varphi) f_G(\rho_{/\!/}, \mu, \psi, \theta, \varphi) e\sqrt{g}B_* B d\rho_{/\!/} d\mu d\psi d\theta d\varphi, \tag{5.1}$$

which can also be expressed in terms of f_A as

$$\langle w\rangle = \sum_s s\int w(\rho_{/\!/}, \mu, \psi, \theta, \varphi) f_A(J_{/\!/}, \mu, \alpha, \beta, \omega)\frac{1}{\gamma\sigma^{\alpha\beta}} dJ_{/\!/} d\mu d\alpha d\beta d\omega, \tag{5.2}$$

where $s=\pm 1$ is the sign of particle velocity. Here we have used the relation

$$e\sqrt{g}B_*B\frac{\partial(\rho_{/\!/},\psi,\theta,\varphi)}{\partial(J_{/\!/},\omega,\alpha,\beta)} = \frac{1}{\gamma\sigma^{\alpha\beta}}. \tag{5.3}$$

The quantity w is expanded with respect to ε as

$$w = w(P_0,\mu,\alpha,\beta,\Phi_0)+\varepsilon\left\{\Psi^{(1)}\frac{\partial w}{\partial\alpha}+\Theta^{(1)}\frac{\partial w}{\partial\beta}+\Phi^{(1)}\frac{\partial w}{\partial\varphi}+P^{(1)}\frac{\partial w}{\partial P_0}\right\} + \cdots,$$

$$= w(P_0,\mu,\alpha,\beta,\Phi_0)+\varepsilon\left\{\Psi^{(1)}\frac{\partial w}{\partial\alpha}\bigg|_E+\Theta^{(1)}\frac{\partial w}{\partial\beta}\bigg|_E+\Phi^{(1)}\frac{\partial w}{\partial\varphi}\bigg|_E\right\} + \cdots. \tag{5.4}$$

It is convenient to consider the function f_A as function of E,α,β, and φ, instead of $J_{/\!/}$, α, β, and ω.

$$\langle w\rangle = \sum_s s\int w(\rho_{/\!/},\mu,\psi,\theta,\varphi)f_A(E,\mu,\alpha,\beta,\Phi_0)\frac{1}{\gamma\sigma^{\alpha\beta}}eB_\varphi\frac{\partial P_0}{\partial E}\mathrm{d}E\mathrm{d}\mu\mathrm{d}\alpha\mathrm{d}\beta\mathrm{d}\Phi_0. \tag{5.5}$$

Then, from the condition that eq.(5.1) and eq.(5.2) give same results for arbitrary function w, we obtain the following relation

$$f_G = f_A -\varepsilon\left\{\Psi^{(1)}\frac{\partial f_A}{\partial\alpha}\bigg|_{E,\varphi}+\Theta^{(1)}\frac{\partial f_A}{\partial\beta}\bigg|_{E,\varphi}+\Phi^{(1)}\frac{\partial f_A}{\partial\varphi}\right\}+ \cdots. \tag{5.6}$$

In deriving eq.(5.6) we have used the approximation

$$(e\gamma\sigma^{\alpha\beta})^{-1} =1-\varepsilon(\gamma^{(1)}+\sigma^{\alpha\beta(1)})+\cdots, \tag{5.7}$$

and eq.(3.28).

Equation (5.6) can be also derived from the drift kinetic equation

$$\frac{\partial f_G}{\partial t}+\dot\psi\frac{\partial f_G}{\partial\psi}+\dot\theta\frac{\partial f_G}{\partial\theta}+\dot\varphi\frac{\partial f_G}{\partial\varphi} = C(f_G,f_G), \tag{5.8}$$

where C is the collision term. Expanding f_G into small parameter ε as

$$f_G = f_0+\varepsilon f_1+ \cdots, \tag{5.9}$$

with

$$\frac{\partial f_0}{\partial\varphi} = 0, \tag{5.10}$$

and assuming the time derivative as well as the collision term is of smallness of ε^2, we have

$$\frac{\partial f_0}{\partial t}\frac{\partial J_\parallel}{\partial E}+\frac{1}{e}\frac{\partial J_\parallel}{\partial \theta}\frac{\partial f_0}{\partial \psi}-\frac{1}{e}\frac{\partial J_\parallel}{\partial \psi}\frac{\partial f_0}{\partial \theta} = \oint \frac{\sqrt{g}B}{v_\parallel}C(f_0,f_0)\mathrm{d}\varphi, \tag{5.11}$$

and

$$f_1 = \Psi^{(1)}\frac{\partial f_0}{\partial \psi}+\Theta^{(1)}\frac{\partial f_0}{\partial \theta}, \tag{5.12}$$

with $\Psi^{(1)}$ and $\Theta^{(1)}$ given by eqs.(3.19)-(3.20).

As an application of eq.(5.6) we consider the number density and the parallel velocity of drift particles. The distribution function f_A is assumed independent to φ, or ω (bounce averaged). For the number density we put

$$\langle n_G\rangle = e\oint \mathrm{d}\varphi\sqrt{g}\int B_* B f_G \mathrm{d}\rho_\parallel \mathrm{d}\mu, \tag{5.13}$$

$$\langle n_A\rangle = \sum_S s\oint \mathrm{d}\omega\int \gamma f_A \mathrm{d}J_\parallel \mathrm{d}\mu$$

$$= \sum_S\oint \mathrm{d}\varphi\int \gamma e B_\varphi\left|\frac{\partial P_0}{\partial E}\right| f_A \mathrm{d}E\mathrm{d}\mu. \tag{5.14}$$

Then we have

$$\langle n_G\rangle = \langle n_A\rangle-\varepsilon\sum_S\oint \mathrm{d}\varphi\left\{\frac{\partial}{\partial\alpha}\int \sqrt{g}B^2\left|\frac{\partial P_0}{\partial E}\right|\Psi^{(1)}f_A \mathrm{d}E\mathrm{d}\mu\right.$$

$$\left.+\frac{\partial}{\partial\beta}\int \sqrt{g}B^2\left|\frac{\partial P_0}{\partial E}\right|\Theta^{(1)}f_A \mathrm{d}E\mathrm{d}\mu\right\}. \tag{5.15}$$

As is easily seen, the total number of particle remains constant.

For the parallel velocity, since $v_\parallel-\mu\ell_0/e$ is the real parallel velocity of the particle, we have

$$\int\left\{ev_\parallel-\mu\ell_0\right\}f_z B_* B \mathrm{d}\rho_\parallel \mathrm{d}\mu$$

$$= \sum_{s} s \int f_{A} \mathrm{d}E\mathrm{d}\mu$$

$$+\varepsilon \sum_{s} s \int \left\{ \frac{m\ell_0}{eB}\left[|v_{/\!/}| - \frac{\mu B}{|v_{/\!/}|} \right] f_{A} - \Psi^{(1)} \frac{\partial f_{A}}{\partial \alpha} - \Theta^{(1)} \frac{\partial f_{A}}{\partial \beta} \right\} \mathrm{d}E\mathrm{d}\mu . \quad (5.16)$$

The odd part of the distribution function with respect to $v_{/\!/}$ contributes to the first term while the even part contributes to the second term in eq.(5.11). This is just the analogy to the diamagnetic current in the non-uniform plasma in the drift particles.[13] For instance, if the distribution function f_{A} is nearly Maxwellian, independent to β, then f_{A} for passing particles can be approximated as

$$f_{A} = f_{A}^{(0)}(E,\alpha) + \frac{m}{e\iota} \frac{\partial f_{A}^{(0)}}{\partial \alpha} \oint \frac{v_{/\!/} B_{\varphi}}{B} \frac{\mathrm{d}\varphi}{2\pi} + \cdots . \quad (5.17)$$

Here we have assumed the inequality $\sqrt{mE}/eBa_{\mathrm{p}} \ll \iota/N$, and the effects of collisions are ignored. The parallel electric current can be written as

$$\frac{j_{/\!/}}{B} = \sum_{i,e} e^2 \int v_{/\!/} f_{G} B_{*} \mathrm{d}p_{/\!/} \mathrm{d}\mu - \frac{p}{B^2}\ell_0$$

$$= \frac{\mathrm{d}p}{\mathrm{d}\alpha} \left\{ -\frac{1}{\iota} \left\langle \frac{B_{\varphi}}{B^2} \right\rangle_{\varphi} - \frac{\partial}{\partial \beta} \int^{\varphi} \left[\frac{B_{\varphi}}{B^2} - \left\langle \frac{B_{\varphi}}{B^2} \right\rangle_{\varphi} \right] \mathrm{d}\varphi + \frac{B_{\theta}}{B^2} - \left\langle \frac{B_{\theta}}{B^2} \right\rangle_{\varphi} \right\} + \cdots , \quad (5.18)$$

where

$$\left\langle \cdots \right\rangle_{\varphi} = \oint \cdots \frac{\mathrm{d}\varphi}{2\pi} , \quad (5.19)$$

stands for the average with respect to φ. The second term in the right hand side of the first line of eq.(5.18) is the parallel component of the magnetization current. The second line corresponds to the Pfirsch-Schlüter current in the torus.

§6. Summary and Discussion

In this paper the new theory on the longitudinal adiabatic invariant in the helical torus is presented. The introduced adiabatic variables describes not only the averaged motion with respect to the toroidal direction, but also the motion deviating from its averaged position.

The difference of the adiabatic invariant in this paper and that given in Ref.5 should be noted; the integral in eq.(3.10) is not along the pass of the actual orbit but a fictitious orbit described by the adiabatic equations of motion [eqs.(3.32)-(3.33)]. If eq.(3.10) is rewritten in terms of the actual coordinates of particles with the aids of eqs.(3.19) and (3.20), its constancy on time is only up to the first order. The invariance of $J_{/\!/}$ in our theory is correct up to any order asymptotically.

The transition between the passing and the trapped state is also discussed. At the transition the jump of adiabatic variables occurs, and as its result the value of adiabatic invariant changes. Such jump may make the particle motion chaotic, as the particles repeat the transitions between the positive and negative passing states.

The jump of adiabatic variables at the transition is expressed in terms of the phase ω, the time dependence of which is described by eq.(3.34). However, since the phase grows very fast any small error in eq.(3.34) may cause substantial difference after long time interval, and the phase of the particle at the transition point cannot predicted by eq.(3.34). In that sense the phase of the fast motion at the transition point can be determined only statistically.

The relation between distribution function for guiding center particles and that for bounce averaged distribution function is established. The expression for the parallel current is derived.

The use of bounce averaged distribution function is useful in the the study of the neoclassical transport in helical torus.[14] The average with respect to toroidal angle reduces dimensionality, and the relation between helical torus and axisymmetric tokamak becomes more clearly observed. Application of the theory to the neoclassical transport will be given in future.

The application to the particle orbit is discussed in the other paper.[15,16]

The formulation given in this paper is restricted to quasi-static magnetic configurations. The time derivative of the fields does not appear in the equation of motion. The allowance of the slow temporal variation of the magnetic field as well as the electric field is not difficult, but it is out of scope of this paper.

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Table I.

Change of variables at the transition.

initial state (+):

$$\to(-):\ \frac{[U_{\max},J_r]}{[U_{\max},J_-]}<\frac{\wp(\omega_-)}{2\pi}<1,\quad \Delta\alpha=-\frac{\iota}{N}\frac{\partial J_r}{\partial\beta}\frac{2\pi-\wp(\omega_+)}{[U_{\max},-J_-]}\frac{\partial U_{\max}}{\partial\beta},\quad \Delta\beta=\frac{\iota}{N}\frac{\partial J_r}{\partial\beta}\frac{2\pi-\wp(\omega_+)}{[U_{\max},-J_-]}\frac{\partial U_{\max}}{\partial\alpha},$$

$$\wp(\omega_-)=\frac{[U_{\max},J_+]}{[U_{\max},-J_-]}(2\pi-\wp(\omega_+)),\quad \Delta J_-=-\frac{\iota}{N}\frac{\partial J_r}{\partial\beta}(2\pi-\wp(\omega_-)).$$

$$\to(r),\ \frac{\wp(\omega_+)}{2\pi}<\frac{[U_{\max},J_r]}{[U_{\max},J_+]},\qquad \Delta\alpha=0,\qquad \Delta\beta=\frac{\iota}{N}(\pi-\wp(\omega_-)),$$

$$\wp(\omega_r)=\tfrac{1}{2}\wp(\omega_+),\qquad \Delta J_r=\frac{\iota}{N}\frac{\partial J_r}{\partial\beta}(\pi-\wp(\omega_+)).$$

initial state (−):

$$\to(+):\ \frac{[U_{\max},J_r]}{[U_{\max},J_-]}<\frac{\wp(\omega_-)}{2\pi}<1,\quad \Delta\alpha=-\frac{\iota}{N}\frac{\partial J_r}{\partial\beta}\frac{2\pi-\wp(\omega_-)}{[U_{\max},-J_+]}\frac{\partial U_{\max}}{\partial\beta},\quad \Delta\beta=+\frac{\iota}{N}\frac{\partial J_r}{\partial\beta}\frac{2\pi-\wp(\omega_-)}{[U_{\max},-J_+]}\frac{\partial U_{\max}}{\partial\alpha},$$

$$\wp(\omega_+)=\frac{[U_{\max},J_-]}{[U_{\max},-J_+]}(2\pi-\wp(\omega_-)),\quad \Delta J_+=-\frac{\iota}{N}\frac{\partial J_r}{\partial\beta}(2\pi-\wp(\omega_-)).$$

$$\to(r):\ \frac{\wp(\omega_-)}{2\pi}<\frac{[U_{\max},J_r]}{[U_{\max},J_-]},\qquad \Delta\alpha=0,\qquad \Delta\beta=\frac{\iota}{N}(\wp(\omega_-)-\pi),$$

$$\wp(\omega_r)=\tfrac{1}{2}\wp(\omega_-)+\pi,\qquad \Delta J_r=\frac{\iota}{N}\frac{\partial J_r}{\partial\beta}(\wp(\omega_-)-\pi).$$

initial state (r):

$$\to(+):\ \frac{[U_{\max},-J_-]}{[U_{\max},J_+]}<\frac{\wp(\omega_r)}{2\pi}<\frac{1}{2},\qquad \Delta\alpha=0,\quad \Delta\beta=\frac{\iota}{N}(2\wp(\omega_r)-\pi),$$

$$\wp(\omega_+)=2\wp(\omega_r),\qquad \Delta J_+=\frac{\iota}{N}\frac{\partial J_r}{\partial\beta}(2\wp(\omega_r)-\pi).$$

$$\to(-):\ \max\left(\frac{[U_{\max},-J_-]}{[U_{\max},J_-]},0\right)+\frac{1}{2}<\frac{\wp(\omega_r)}{2\pi}<1,\quad \Delta\alpha=0,\quad \Delta\beta=\frac{\iota}{N}(3\pi-2\wp(\omega_r)),$$

$$\wp(\omega_-)=2(\wp(\omega_r)-\pi),\qquad \Delta J_-=\frac{\iota}{N}\frac{\partial J_r}{\partial\beta}(3\pi-2\wp(\omega_r)).$$

Figure Captions

Fig.1 The change of magnetic field strength (a) along a field line, and (b) along the toroidal direction. In the case of (a) there are many trapped states, while in (b) only one trapped state.

Fig.2 The transition from (+) to (-) or (r). (a) The orbit in θ-φ plane. The bold line shows the locus of $E-U = 0$. (b) The orbit in $\rho_{/\!/}$ and φ plane. The separatrix is drawn by thin line. The particle (1) making transition from (+) to (-) crosses the separatrix at the two points, SX1 and SX2. The particle (2) making transition from (+) to (r) crosses the separatrix only at SX1'.

Fig.3 The transition from (r) to (+) or (1). (a) The orbit in θ-φ plane. The bold line shows the locus of $E-U = 0$. (b) The orbit in $\rho_{/\!/}$ and φ plane. The separatrix is drawn by thin line. The particle (1) making transition from (r) to (-) crosses the separatrix at the point SX1. The particle (2) making transition from (r) to (-) crosses the separatrix at SX2.

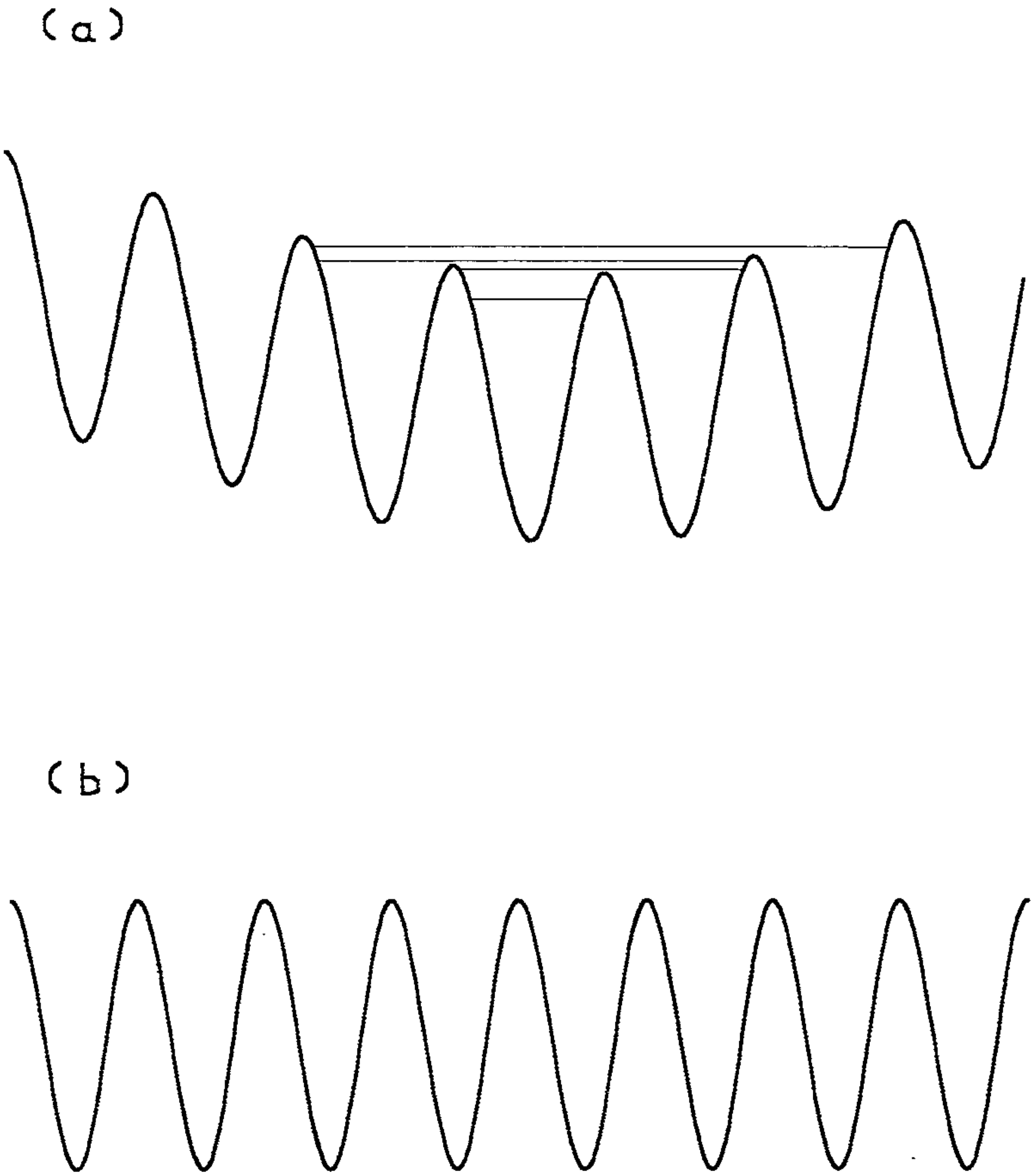

Fig.1

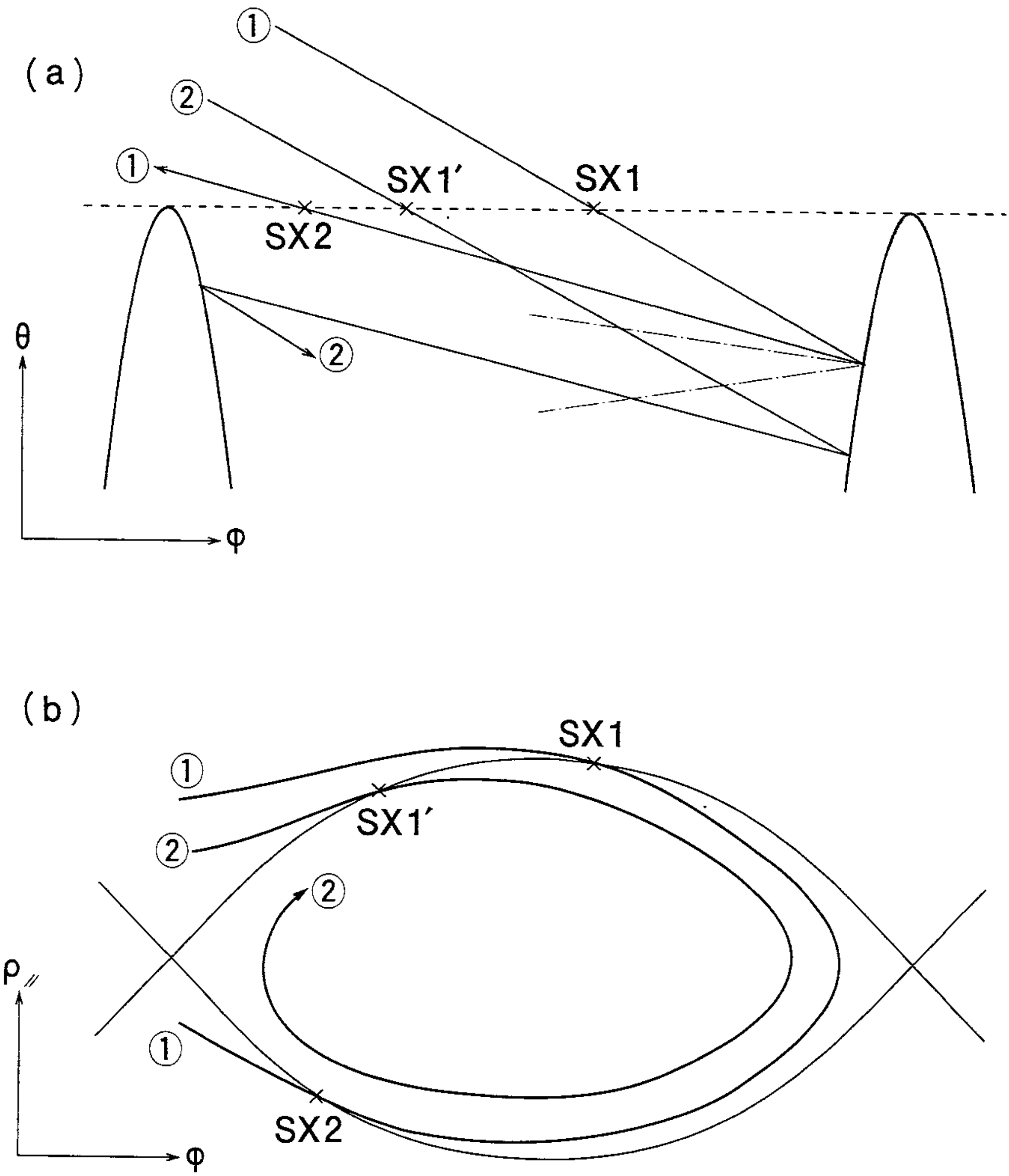

Fig.2

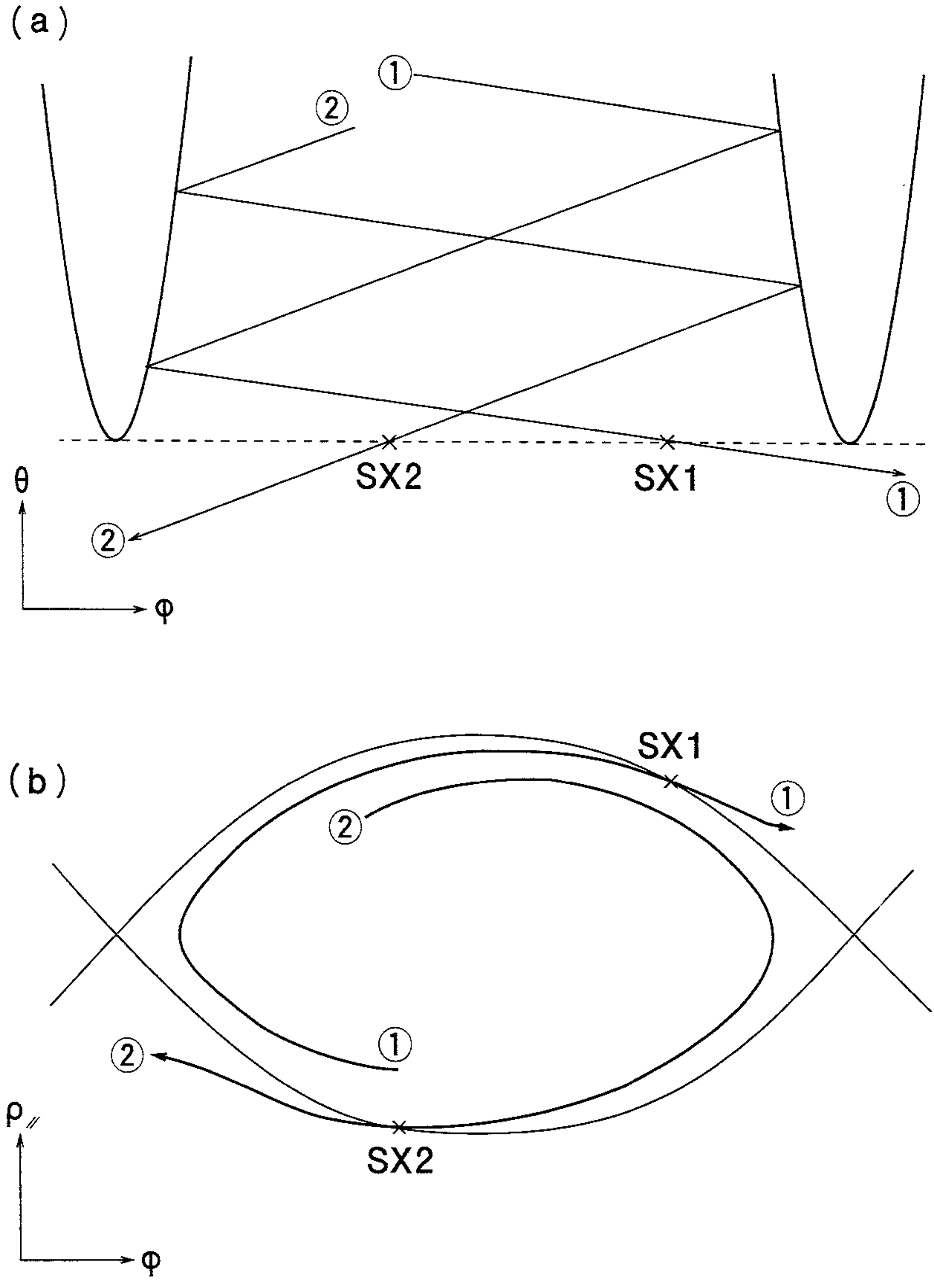

Fig.3