ISSN 0915-633X

NATIONAL INSTITUTE FOR FUSION SCIENCE

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(Received – Sep. 3, 1990)

NIFS-47 Sep. 1990

RESEARCH REPORT
NIFS Series

NAGOYA, JAPAN

SOLITONS AND CHAOS IN PLASMA

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ABSTRACT

Plasma exhibits a full of variety of nonlinear phenomena. Active research in nonlinear plasma physics contributed to explore the concepts of soliton and chaos. Structure of soliton equations and dynamics of low dimensional Hamiltonian systems are discussed to emphasize the universality of these novel concepts in the wide branch of science and engineering.

Keywords, soliton, chaos, inverse scattering transformation, Alfven soliton, optical fiber, standard map, relativistic standard map

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A manuscript prepared for the plenary session of the Fourth Asia-Pacific Physics Conference, held during August 13 and 17th,1990 at Yonsey University, Sseoul, Korea

1. Introduction

Plasma is a treasure land of instabilities[1], which give rise to a full variety of nonlinear phenomena. It is formidable challenge for scientists and engineers to confine such plasma and to heat it up to high temperature over 10^8 K. Their sturdy effort over the four decades, however, has achieved the break through to produce high temperature plasma with properties specified by the Lawson criterion[2]. Endeavor of plasma research during these periods may be best characterized as studies of collisionless plasma, i. e. physics of conservative systems. When nuclear fusion reaction occurs in the high temperature plasma, species of the constituent particles change into other species and produced energy is carried away. Thus, we are confronted with dissipative systems. No one will deny that physics of fusion plasma poses unexplored problem and awaits new challenge of younger researchers for coming decades.

Resisting provocation of such challenge, however, I will speak on solitons and chaos in plasma as a summary report of what plasma physics contributed to advancement in the field of fundamental physics during the past half centuries. As a collection of charged particles under influence of electro-magnetic field, plasma exhibits very complex behavior, which could be described in terms of the interplay of the individual mode (particles) and the collective mode (waves). When the level of fluctuations exceeds thermal level, plasma is identified as in the turbulent state. The transport properties of the collisionless plasma are attributed to the turbulent fluctuation in plasma.[3] Yet, studies of plasma turbulence are suffering from critical limitations of lack of good knowledge of nonlinearity. Understanding of the strong plasma turbulence requires penetrating knowledge of genuine nonlinear entities such as solitons and chaos.

2. Rediscovery of K-dV Equation

Examining nonlinear propagation of collisionless hydromagnetic waves, Gardner and Morikawa[4] reduced a full set of nonlinear magnetohydrodynamic equation to a simple nonlinear evolution equation,

$$\frac{\partial}{\partial t} q + q \frac{\partial}{\partial x} q + \delta^2 \frac{\partial^3}{\partial x^3} q = 0 \qquad 1)$$

which is known as the Korteweg-de Vries (K-dV in short) equation[5) for a shallow water wave propagation.

Observing that Gardner and Morikawa's reduction is based on proper account of balancing effect of the nonlinear steepening effect and the wave dispersion, Taniuti et al.[6) have developed the reductive perturbation theory for weakly dispersive waves as well as for strongly dispersive waves. Analyzing contributions of the higher order terms in the reductive perturbation theory, Ichikawa et al.[7) have shown that the reductive perturbation theory isolates fully contributions of the nonlinear terms in the lowest order equation and the higher order equations become linear equations, thus the reductive perturbation theory provides the scheme to treat soliton as the normal modes of the given system.

3. Birth of Soliton and Quest for Chaos

Upon the introduction of high speed electronic computer in the early 1950's, as one of the research program in the Sherwoods Project, Fermi, Pasta and Ulam[8) undertook numerical experiment to observe the equipartition of energy among the normal modes of one dimensional coupled system of anharmonic oscillators. On the contrary to their expectation, energy is not distributed over the entire mode of the system, but is shared among the lowest modes and after a finite time of elapse the system returns to the original state. This observation of the Fermi-Pasta-Ulam recurrence acted as a holy spring of the novel concepts of soliton and chaos.

Examining the long wave length behavior of the discrete coupled oscillator equation, Zabusky and Kruskal[9) have reduced it to the K-dV equation, and numerically examined the collision processes of solitary wave solutions. In spite of its nonlinearity, two solitary waves retains their original form after the collision. Thus, they were led to propose to call the K-dV solitary wave as "soliton".

At the same time, the observation of Fermi-Pasta-Ulam recurrence phenomena renewed active interest to investigate the ergodic behavior of dynamical systems, and led us to examine "chaos" in the low dimensional nonlinear dynamical systems. Here, it would be appropriate to mention that the classical problems of orbital stability in stellar dynamics[10] shares the common problems with the long time behavior of plasma particles confined in the fusion devices[11] or particle beams in the high energy accelerators[12].

Yet the most striking achievement of the studies on the stochasticity is the discovery of the period doubling root to the chaos[13], which provides the new way of looking for the onset of turbulence[14] .

4. Inverse Scattering Transformation

The mysterious secret of K-dV soliton has been uncovered by the genius discovery of the inverse scattering transformation for the K-dV equation[15]. Subsequent extension of the method to the cubic nonlinear Schrodinger equation [16] inspired Ablowitz et al.[17] to formulate the 2 2 matrix representation of the inverse scattering transformation, which succeeded to unify the K-dV equation, the modified K-dV equation, the cubic nonlinear Schrodinger equation and the sine-Gordon equation as the completely integrable soliton equations.

We can generalize the A-K-N-S scheme to the following set of Lax-pair operator equations,

$$\frac{\partial}{\partial x} u_1 = -F(\lambda; q, r) u_1 + G(\lambda) q u_2$$
$$\frac{\partial}{\partial x} u_2 = G(\lambda) r u_1 + F(\lambda; q, r) u_2 \qquad 2,a)$$

and

$$\frac{\partial}{\partial t} u_1 = A(\lambda; q, r) u_1 + B(\lambda; q, r) u_2$$
$$\frac{\partial}{\partial t} u_2 = C(\lambda; q, r) u_1 - A(\lambda; q, r) u_2 \qquad 2,b)$$

Considering the compatibility conditions $(u_i)_{xt} = (u_i)_{tx}$ under the isospectral requirement $\lambda_t = 0$, we obtain relationship among the quantities A, B and C. Determining A, B and C, we can reduce the nonlinear evolution equations for q and r.

We list several cases here;

1) $F = i\lambda$, $G = 1$;

This is the scheme proposed by Ablowitz et al.[17]. The K-dV, modified K-dV, cubic nonlinear Schrodinger and sine-Gordon equations belong to this scheme.

2) $F = i\lambda^2$, $G = \lambda$;

Kaup and Newell[18] determined the functions A, B and C, and had shown that the derivative nonlinear Schrodinger equation is integrable by the inverse scattering transformation.

3) $F = i\alpha\lambda^2 - 2\beta\lambda$, $G = \alpha\lambda - \beta/2$;

Wadati et al.[19] have shown that this scheme confirms that the superposition of A-K-N-S and K-N scheme valid for the generalized nonlinear Schrodinger equation

$$i\frac{\partial}{\partial t}q + \frac{\partial^2}{\partial x^2}q - i\alpha\frac{\partial}{\partial x}\left(|q|^2 q\right) + \beta |q|^2 q = 0 \qquad 3)$$

4) $F = i\lambda$, $G = \lambda$;

Under this scheme, Wadati et al.[20] derived the new types of soliton equation such as

$$i\frac{\partial}{\partial t}q + \frac{\partial^2}{\partial x^2}\left\{\frac{q}{(1+|q|^2)^{1/2}}\right\} = 0 \qquad 4)$$

and

$$\frac{\partial}{\partial t}q + \frac{\partial^2}{\partial x^2}\left\{\frac{1}{(1+q^2)^{3/2}}\frac{\partial q}{\partial x}\right\} = 0 \qquad 5)$$

Extending Eq.5, we[21] have studied propagation of a loop soliton along a string. El Naschie[22], referring to the Euler elastica, noticed the close similarity of this soliton looping and buckling of compressed strut and emphasized that the study of elastic models will provide useful informations on the interaction between the integrable soliton and the non-integrable chaos.

To conclude the present section, it would be important to mention about multi-dimensional behavior of solitons. For the ion acoustic wave in plasma, Kadomtsev and Petviashvili[23] derived an equation now bearing their names as

$$\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial t}q + \frac{\partial^3}{\partial x^3}q + 6q\frac{\partial}{\partial x}q\right\} = \alpha\frac{\partial^2}{\partial y^2}q \qquad 6)$$

for which Zakharov et al [24] presented a two dimensional inverse scattering transformation. The K-P equation ,Eq.6 accounts for the phenomena of soliton resonance, photographed off the coast of Oregon[25]. Experimental studies of oblique interaction of two ion acoustic solitons were made by many authors[26-28], which were in accordance with the result of theoretical prediction given by Yajima et al.[29].

5. Alfven Solitons and Solitons in Optical Fibers

Investigation of the Alfven wave propagation in a gaseous plasma attracts special interests in connection with plasma heating in the fusion devices. In the field of space plasma physics, large amplitude incompressible magnetic field variation observed in the solar wind has been attributed to nonlinear propagation of the Alfven wave, for which the derivative nonlinear Schrodinger equation

$$i\frac{\partial}{\partial t}b + \mu\frac{\partial^2}{\partial \xi^2}b + i\nu\frac{\partial}{\partial \xi}\left(|b|^2 b\right) = 0 \qquad 7)$$

plays the canonical role[30)-32]. b is defined as $b = (B_y + i B_z)/2B_0$ with the uniform magnetic field B_0 . $\xi = x - V_A t$ stands for the coordinate moving with the Alfven velocity V_A . The coefficients are given as $\mu = (V_A^2/2\Omega_i)$ and $\nu = (V_A^2/4B_0)$. For the plane wave boundary condition, setting $b = q(\xi,t)\exp\{i(k_0\xi - \omega_0 t)\}$, we[33] reduced Eq.7 to the generalized nonlinear Schrodinger equation, Eq.3. Suggesting that the spiky soliton could explain the interstellar magnetic shock [34], Kennel et al. have been carrying out detailed analysis[35].

The most active interest for the generalized nonlinear Schrodinger equation, however, arose in the field of optical fiber technology. We notice Eq.3 had been derived by Tzoar et al.[36] in connection with the nonlinear signal propagation in the optical fiber, but the fact that Eq.3 is one of the soliton equations was not recognized for a certain time[37]. Improving the accuracy of numerical analysis, Ohkuma et al.[38] have examined the soliton behavior of pulses propagating along the optical fibers. It is particularly interesting to observe that not only the K-dV equation, but also many other equations such as the generalized nonlinear Schrodinger equation posses their universal nature to describe nonlinear behavior of the systems in the full varieties of different branches of science and engineering.

6. Discretized Soliton Equations and Integrable Mapping

Although we have emphasized success of the analytical approach in the soliton theory, advancement of computational physics plays the key role in development of nonlinear science. In this regard, there have been extensive studies on the discretized soliton equations and the nonlinear differential-difference equations. For example, the system of pendulums coupled with linear spring is described by

$$\frac{\partial^2}{\partial t^2}\theta_n = \theta_{n+1} - 2\theta_n + \theta_{n-1} - K\sin\theta_n \qquad 8)$$

of which continuum limit is the sine-Gordon equation. The cubic nonlinear Schrodinger equation, Eq. 3 with $\alpha=0$, is reduced to

$$i\frac{\partial}{\partial t}q_n = q_{n+1} - 2q_n + q_{n-1} + \frac{1}{2}q_n q_n^*(q_{n+1} + q_{n-1}) \qquad 9)$$

for which Ablowitz and Ladik[39] presented the exact theory to construct soliton solutions. Observing there are other choices of the discretization of the cubic nonlinear term, Ross and Thompson[40] discussed the static solution with real amplitude of Eq. 9 in the form of symmetric difference equations

$$\phi_{n+1} + \phi_{n-1} = f(\phi_n) \qquad 10)$$

obtaining an expression of the invariant curves of

$$(\phi_{n+1}-\phi_n)^2+\frac{1}{2}\phi_{n+1}^2\phi_n^2=const \qquad 11)$$

Extending this approach, Quispel et al.[41] have been discussing relationships between the soliton equations and the integrable mapping.

Turning to the sine-Gordon equation , I want to refer the recent paper by Goedde et al.[42] on the parametric instabilities of the discrete sine-Gordon equation. They examined the dynamical behavior of the system at coarse discretization (chaos in few degrees of freedom) and at fine discretization (integrability in the continuum limit) when the total system energy is held constant. At the same time, the discrete sine-Gordon equation attracts much interests of solid state physicists, since the equation is known to be the basic equation of the Frenkel-Kontrova dislocation model. Considering the static solution of Eq.8, Aubry and many others[43)-44)] have discussed the problem of transitions between the incommensurate and commensurate states by the standard map,

$$\begin{aligned} P_{n+1} &= P_n - (K/2\pi)\sin(2\pi X_n) \\ X_{n+1} &= X_n + P_{n+1} \end{aligned} \qquad 12)$$

which is reduced from Eq.8 by setting $\theta_n = 2\pi X_n$ and $\theta_{n+1} - \theta_n = 2\pi P_n$

7. Chaos in Low Dimensional Hamiltonian Systems

Having discussed a connection between the soliton equations and integrable mapping, we are now led to study non-integrable mapping in low dimensional Hamiltonian systems. Helleman[45] emphasized that mechanics is not in good shape,contrary to the preoccupation implanted through the present day physics course on classical mechanics. The most Hamiltonian systems are non-integrable, and many orbits exhibit sensitive dependence on the initial condition, (though their temporal evolution is deterministic). Hence, the chaotic behavior appears already in systems with only 2 or 3 degrees of freedom[46].

All of these aspects of low dimensional Hamiltonian systems are best illustrated by the two dimensional area-preserving standard map, Eq.12. Statistical properties of the standard map have been one of the central problems in plasma physics, since the magnetic surface of toroidal fusion devices, the motion of charged particles in the magnetic mirrors and many other problems are best studied by the simple map of Eq. 12. We[47] have shown that analysis of symmetry properties of the orbits provides useful information on the stochasticity of the Hamiltonian systems.

Recently, we have undertaken investigation of the relativistic standard map[48],

$$P_{n+1} = P_n - (K/2\pi)\sin(2\pi X_n)$$
$$X_{n+1} = X_n + P_{n+1}/(1+\beta^2 P_{n+1}^2)^{1/2}, \quad \beta = v_p/c \qquad 13)$$

which describes the relativistic dynamics of charged particles under the action of repetitive kicks of the electrostatic wave packet, with the phase velocity of the fundamental mode $v_p = \omega_0/k_0$ [49]. In spite of its naive appearance of the relativistic modification over the standard map, we have discovered extremely intricate interplay of the nonlinear effect and the relativistic effect in determining the chaotic behavior of the system.

8. Concluding Remark

In the present discussion on soliton and chaos in plasma, I have tried to picturize the universality of the novel concept of soliton and chaos in dealing with nonlinear phenomena in science and engineering. With regards to the proper problems in plasma physics, I will refer to several investigators on interplay of soliton and chaos, such as on the process of chaotic emission of solitons in nonlinear inhomogeneous media[50]. Though these works are based on computational analysis, their observation provides us clear insight to develop strong plasma turbulence theory. Furthermore, I should mention that a number of experimental studies on chaos in plasma[51)-52] awaits further investigation to explore the true understanding of the nonlinear phenomena in plasma. To conclude the present talk, once again let me emphasize that success of controlled thermonuclear fusion is going to open the new horizon of fundamental physics in the coming century.

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