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Keywords: Nonlinear dynamics, Chaotic attractor, Fixed point index, Euler characteristic.

Dedicated to Professor Tosio Kato on his 70^{th} birthday.

APPLICATION OF FIXED POINT THEORY TO CHAOTIC ATTRACTORS OF FORCED OSCILLATORS

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Abstract

A review of the structure of chaotic attractors of periodically forced second-order nonlinear oscillators suggests that the theory of fixed points of transformations gives information about the fundamental topological structure of attractors. First a simple extension of the Levinson index formula is proved. Then numerical evidence is used to formulate plausible conjectures about absorbing regions containing chaotic attractors in forced oscillators. Applying the Levinson formula suggests a fundamental relation between the number of fixed points or periodic points in a section of the chaotic attractor on the one hand, and a topological invariant of an absorbing region on the other hand.

1. Introduction

The topological theory of fixed points of transformations has a long and fruitful history of application to the study of nonlinear oscillations.

Consider a dynamical system defined by the following differential equation in a Banach space E with solution $\mathbf{x}(\mathbf{x_0}, t) \in E$:

$$\dot{\mathbf{x}} = f(\mathbf{x}, t) \qquad \text{for } t > 0 \tag{1}$$

with initial condition

$$\mathbf{x}(0) = \mathbf{x_0} \in E$$

and suppose that $f(\mathbf{x}, t)$ is periodic in t with period L. Then the transformation T defined by

$$T(\mathbf{x}(\mathbf{x_0}, t)) = \mathbf{x}(\mathbf{x_0}, t + L)$$

is the Poincaré map of the differential equation (1); this transformation completely characterizes the behavior of solutions of (1).

Levinson [**18**] introduced a broadly useful category of systems, those which are dissipative at large displacements. A system defined by (1) is said to be of class D if there exists an R such that for all $\mathbf{x_0}$

$$\limsup_{t\to\infty} \|\mathbf{x}(\mathbf{x_0}, t)\| < R.$$

Levinson showed in the case $E = \mathbf{R}^2$ that for a suitable integer $k > 0$ the set

$$\mathcal{J} = \cap_{n=1}^{\infty} T^{nk} B_R,$$

that is, the infimum of images of the ball B_R of radius R, is the maximal bounded invariant set of the differential equation (1); see Krasnoselskii and Zabrieko [**16**,§39] for a proof in the case of a completely continuous evolution operator in a Banach space. The goal of applied dynamical systems theory is to obtain, for a given equation(1), a decomposition of $\mathcal{J}$ into a finite number of disjoint transitive invariant subsets, called basic sets, among which are the attractors. Roughly speaking, attractors are the basic sets which absorb initial sets of positive measure; see Milnor [**20**] for a discussion of the definition of attractor.

The fixed points of the transformation T^n are periodic points of T, and correspond to periodic solutions of period nL of the system (1). Levinson also proved a useful theorem concerning the total number of fixed points of T^n for $E = \mathbf{R}^2$, based on the theory of index of fixed points. Fixed points of a transformation T of $\mathbf{R}^2$ may be divided into four types, according the characteristic roots of the linear part of T near the fixed point. Since T is orientation-preserving, the characteristic roots λ_1, λ_2 near any fixed point are constrained by $\lambda_1\lambda_2 > 0$. It is usual to assume that $|\lambda_i| \neq 1$ so that multiple fixed points are excluded; typically in applications if one or both roots have magnitude 1, then a small change in the parameters of equation (1) will separate the fixed points. Hence the four possible types of fixed points are:

completely stable,	$\lvert\lambda_1\rvert < 1, \lvert\lambda_2\rvert < 1,$
completely unstable,	$\lvert\lambda_1\rvert > 1, \lvert\lambda_2\rvert > 1,$
directly unstable,	$\lambda_1 > 1 > \lambda_2 > 0,$
inversely unstable,	$\lambda_2 < -1 < \lambda_1 < 0.$

Now suppose that T^n has only a finite number of isolated simple fixed points, and let the number of fixed points in each of the four categories be denoted by $C(n)$, $U(n)$, $D(n)$ and $I(n)$ respectively. Levinson proves that

$$C(n) + U(n) + I(n) = 1 + D(n).$$

This is sometimes called the Levinson index formula. The proof rests on the notion of rotation of the field of vectors connecting $\mathbf{x} \in E$ to $T^n\mathbf{x}$. Birkhoff, who used this notion to prove a famous geometric theorem of Poincaré [**3**], called these point-image vectors. As the point $\mathbf{x}$ makes one circuit around a closed curve in the plane, the point-image vector makes a net rotation $2\pi k$ which is an integer multiple of 2π. The sign of k indicates whether the point-image vector has rotated in the same (+) sense or the opposite (−) sense as the traversal of the closed curve by $\mathbf{x}$. The integer k is called the index of T on the closed curve. Because the point-image vector is directed inward on the circle of the radius R, the index there is 1. On a small circle surrounding a fixed point C, U, I, or D the rotation does not depend on the circle, so the point has index

$$\text{ind } C = \text{ind } U = \text{ind } I = 1$$
$$\text{ind } D = -1$$

Note that an inversely unstable point of T^n is a directly unstable point of T^{2n}.

The Levinson index formula is useful in applications, for example in verifying whether all fixed points of a given transformation have been identified. In particular, the existence of at least one periodic point of each period is guaranteed, although it may not be stable.

Levinson's proof relies on the ball B_R to construct a bounded open set Ω such that $T(\partial\Omega) \subset \Omega$. Recall that a trapping region or absorbing set Ω is a bounded open set such that $T(\Omega \cup \partial\Omega) \subset \Omega$. By analogy, if $T(\partial\Omega) \subset \Omega$ we may refer to $\partial\Omega$ as an absorbing boundary. For flows, $\partial\Omega$ is an absorbing boundary if and only if Ω is an absorbing set; but this equivalence does not hold for a general discrete time transformation T. To prove the index formula, only an absorbing boundary is required, but in applications the set Ω is usually absorbing as well.

In the applications discussed below, we require a simple generalization of Levinson's theorem involving bounded open domains $\Omega \subset \mathbf{R}^2$ with a finite number of holes. We suppose that each component of $\partial\Omega$ is a simple closed curve. Following Levinson we assume that $f(\mathbf{x}, t)$ is analytic in $\mathbf{x}$, which is true for the applications considered here.

THEOREM 1. *Suppose T^n has a finite number of fixed points, none of which lies in $\partial\Omega$, and suppose $T(\partial\Omega) \subset \Omega$. Then the rotation of T on $\partial\Omega$ (that is, the rotation on the outer boundary plus the rotations around all hole boundaries) equals the Euler characteristic χ_Ω of the domain Ω, and*

$$C_\Omega(n) + U_\Omega(n) + I_\Omega(n) = D_\Omega(n) + \chi_\Omega$$

where in each case only the points in Ω are counted.

PROOF: We follow Levinson, who uses an idea from Birkhoff [**3**]: the net rotation of any continuous field of point-image vectors along a smooth curve Γ varies continuously as the curve is displaced to a different curve Γ' with the same endpoints, provided that neither Γ nor Γ' nor the region between them contains a fixed point of the transformation T which defines the point-image vector field. Let us agree to measure the rotation in units of 2π, so the rotation on a closed curve equals the index. If Γ is a closed curve, the rotation of the vector field in one trip around Γ is always an integer; so as Γ is deformed to another closed curve Γ' the rotation varies continuously yet is always an integer, and hence must remain constant.

Now consider the domain Ω containing h holes, and consider T from inside one such hole Ω_1 with boundary $\partial\Omega_1$ which is a subset of $\partial\Omega$. For any compact simply connected Ω_1, it is possible to extend the definition of T from $\partial\Omega_1$ continuously to Ω_1 in such a way that there is at most one fixed point in Ω_1; since T maps each point in $\partial\Omega_1$ to a point outside Ω_1, it is clear that exactly one fixed point, a completely unstable one, is needed. Thus we alter the problem by removing h holes and adding h completely unstable fixed points. After this modification, the new domain has a simple closed curve Γ as its boundary.

To this modified problem we apply Levinson's argument directly. Enclose each fixed point within Γ by a small circle, and connect the circles by cuts which form a tree, that is, without forming any closed circuit of cuts. The union of these circles and cuts can be regarded as a simple closed curve Γ', to which Γ can be deformed continuously without crossing any fixed points. Since T maps Γ to the interior of Ω,

$$\text{ind }(T,\Gamma) = 1 = \text{ind }(T,\Gamma').$$

Now the cuts in Γ' make no net contribution to the rotation of T on Γ', since each cut is traversed once in each direction, and the contributions of the two traversals cancel each other. Therefore the rotation of T on Γ' is determined by the rotation on each small circle, that is, the index of each fixed point:

$$\text{ind }(T,\Gamma') = C_\Omega(1) + U_\Omega(1) + h + I_\Omega(1) - D_\Omega(1)$$

where the fixed points in the original domain Ω are counted, and the additional points for the h holes are indicated explicitly. Recalling that the Euler characteristic $\chi_\Omega = 1-h$ proves the theorem for $n = 1$. The same arguments apply to T^n in place of T, proving the general case.

The Levinson index formula has been generalized to the case of finite-dimensional Euclidean space $E = \mathbf{R}^d$ by Shiraiwa [**26**]; thus it may even be applicable to infinite-dimensional systems for which the notion of rotation can be defined, such as completely continuous evolution operators. Examples would include parabolic initial-boundary problems, and delay-differential equations.

In what follows we consider examples of nonlinear oscillators in $\mathbf{R}^2$ for which chaotic attractors are at least well-documented numerically. There are as yet no rigorous proofs that chaotic attractors exist for such systems, since the theoretical possibility of confusing them with stable periodic solutions of period nL with n very large is quite difficult to exclude. Nevertheless, the numerical evidence for chaotic attractors is very convincing, and by concentrating on periodic solutions for small n, the structure of these apparently chaotic attractors can be to some extent understood.

2. Uniformly dissipative forced oscillators

Among the most extensively simulated of nonlinear oscillators has been the forced oscillator of Duffing type

$$\ddot{x} + k\dot{x} + \alpha x + x^3 = A_0 + A\sin\omega t. \tag{2}$$

In the case $A_0 = 0$ and $\alpha = 0$ (corresponding for example to a critically loaded Euler support column), an essentially complete survey of the two parameters k and A has been reported by Ueda[**31**]. For $\alpha < 0$ (Euler column loaded past the buckling point, twin-well potential), three independent parameters need to be surveyed; important results have been obtained by Ueda et al. [**33**], Holmes [**13**], Moon and Holmes [**21**], Holmes and Whitley [**14**], Ueda et al. [**34,35**]and Stewart [**27**], although a complete survey is still lacking. In this twin-well potential problem, a fundamental question is what conditions are required for a solution to escape from confinement in one well; this issue has been addressed by Thompson[**29**] for the asymmetric single-well potential oscillator

$$\ddot{x} + \beta\dot{x} + x - x^2 = F\sin\omega t; \tag{3}$$

See also refs. [**35**] and [**28**].

Numerical studies of equations (2) and (3) show robust long-term behavior governed by persistent homoclinic intersections of invariant manifolds of unstable periodic points of the Poincaré map, defined by sampling stroboscopically at $t = 2\pi n/\omega, n = 1, 2, \ldots$ The unstable manifold or outset consist of solutions asymptotic as $t \to -\infty$ to a periodic point, while the stable manifold or inset consists of solutions asymptotic as $t \to +\infty$ to a periodic point. Chaotic attractors in oscillators of Duffing type always appear to coincide with the closure of the outset of some unstable periodic motion of either the directly or indirectly unstable type.

Sometimes the existence of a homoclinic intersection can be proved. For example, in the twin-well Duffing oscillator it can be proved by a Melnikov analysis that there are persistent homoclinic intersections [**10**;pp.191ff.]; but while these guarantee horseshoe-like dynamics, it may still happen that the attractors are regular and periodic, with horseshoes governing only the transient behavior. This can happen for example when a basin boundary becomes tangled; see [**11**,**6**] for examples. It may also happen in the Duffing equation that chaotic attractors are observed numerically in parameter regimes where the invariant manifolds accessible to Melnikov analysis have no intersection; in such cases, other homoclinic intersections occur, belonging to invariant manifolds for which Melnikov analysis is not practical.

Thus at present, in ascribing long-term aperiodic behavior to the existence of a chaotic attractor in Duffing's equation, one relies on careful interpretation of the evidence of numerical simulations. In practice this requires exploratory numerical study of bifurcations of chaotic attractors, particularly the bifurcations which create or destroy a chaotic attractor — the boundary crises [**7**] or blue sky catastrophes [**30**;Ch.13]. By determining the numerical requirements for confirming such bifurcations, one learns by implication the requirements for identifying a chaotic attractor (as distinct from chaotic transient behavior).

In what follows, we propose to consider the insets and outsets of the unstable periodic motions with the lowest subharmonic number, with the goal of understanding under what conditions a tangled outset is contained in a chaotic attractor.

In the simplest case, it can happen that a chaotic attractor contains exactly one unstable orbit of lowest subharmonic number. In the Poincaré map, this orbit is always an unstable periodic point of indirectly unstable type, corresponding to a half-twist over the period of the orbit. An example is shown in Figure 1, taken from ref.[**33**], showing a Poincaré section in the $\mathbf{x} = (x, y = \dot{x})$ plane of equation (2) with $\alpha = 0$, $A_0 = 0.08$, $A = 0.3$, $k = 0.2$, and $\omega = 1$. There are two attractors, a fixed point S^1 and a chaotic attractor containing the fixed point I^1 (standing for inversely unstable). There is also an unstable saddle fixed point D^1 (directly unstable) with two positive multipliers; its inset, or stable manifold (the thick curve through D^1), separates the basin of S^1 from the basin of the chaotic attractor containing I^1. Note that neither branch of the outset of D^1 (thinner curve) is homoclinic (i.e. intersects the inset of D^1); a positive distance separates D^1 from each attractor.

Also shown in Fig. 1 is an enlarged view of the invariant manifolds of I^1. These are clearly homoclinic; since I^1 has negative multipliers, each branch of the outset (denoted by α in the Figure) of I^1 is the image under the Poincaré map of the other branch; so if one branch is homoclinic, the other must be also. (This is of course not true for D-type, i.e. positive multiplier periodic points.) The outset of I^1 is folded during each forcing

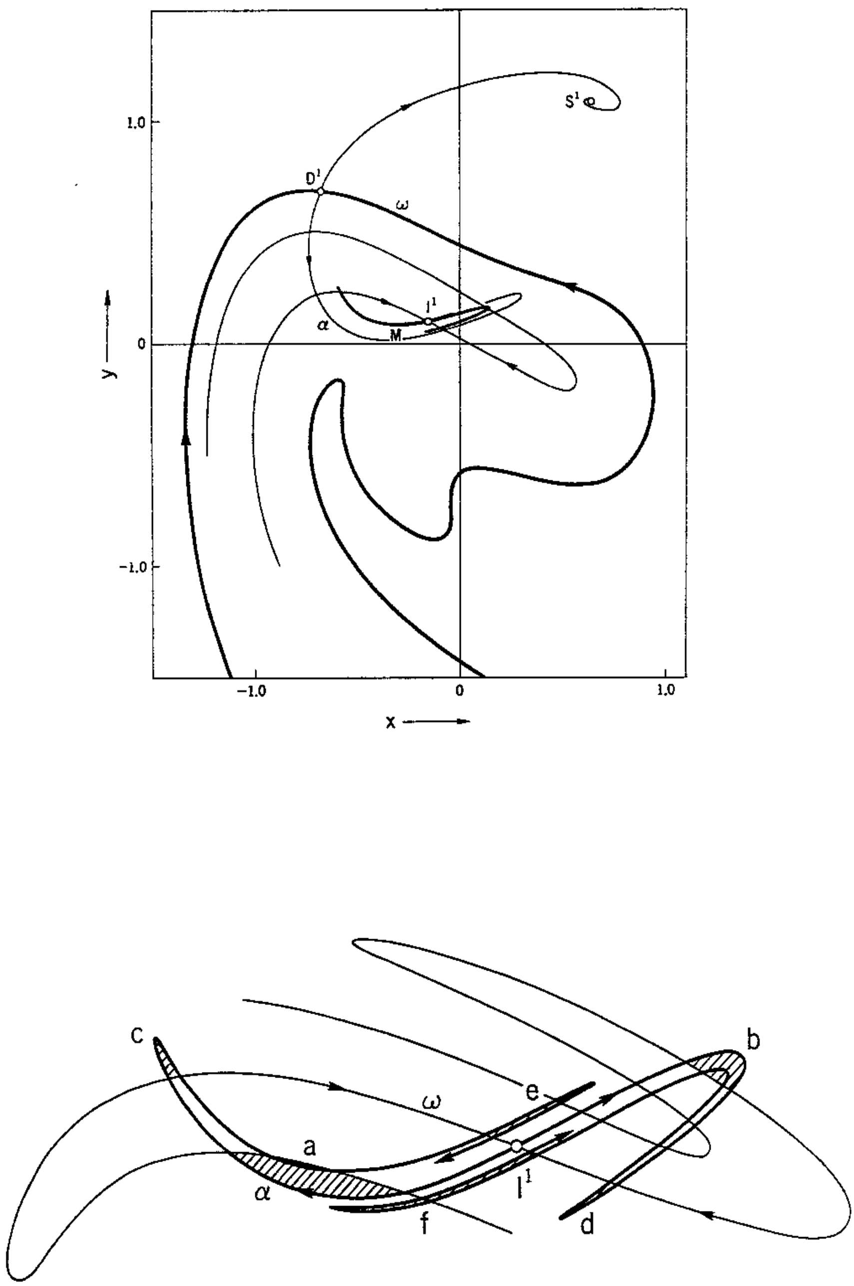

Figure 1. Simply folded band chaotic attractor in the Poincaré map of the forced Duffing equation, with part of the invariant manifolds

cycle by a simple folding action like that described by Rössler **[22]** in his synthesis of the folded band attractor; Rössler also identified the dollar sign formed by the S-shaped outset and the nearly straight portion of the intersecting inset as characteristic of the simplest chaotic attractor. The Hénon map attractor **[12]** in the orientation-preserving case also has this structure.; see also **[23]**, **[30]**, **[27]**, and **[8]**.

Variants of this band attractor, in which the unstable periodic orbit of lowest subharmonic number is a unique point I^n of period n, also occur commonly with n greater that one; the corresponding chaotic attractor locus contains n disconnected pieces. Each piece is similar in structure to Fig. 1, with the inset and outset of I^n forming a dollar sign, and one folding of the band is completed after n forcing cycles. An example with $n = 2$ can be found in ref. **[29]**.

A slightly more complicated attractor structure, occurring commonly in systems with symmetry, contains three unstable orbits of lowest period; two have negative multipliers while the third has positive multipliers. This chaotic attractor structure is predominant in the survey **[31]**, whose results are summarized in the parameter space chart of Figure 2; here eq. (2) is considered with $\alpha = 0$, $A_0 = 0$, $\omega = 1$, and the forcing amplitude A is denoted B in Fig. 2. (Note that the case $\alpha = 0$, $A_0 = 0$, $\omega \neq 1$ can be transformed to $\omega = 1$ by appropriate change of variables.)

A typical chaotic attractor is illustrated in Fig. 3, together with the locations of the three fixed points ${}^1D^1$, ${}^1I^1$ and ${}^2I^1$, and the invariant manifolds of ${}^1D^1$. The parameters here are $k = 0.1$, $A = 12$. Also depicted in Fig. 3 are small portions of the insets and outsets of all three fixed points. The structure shows a dollar sign based at ${}^1I^1$ and at ${}^2I^1$, both within a dollar sign based at ${}^1D^1$. There are heteroclinic Smale cycles, with the outset of ${}^1D^1$ intersecting the inset of each I, and the outset of each I in turn intersecting the inset of ${}^1D^1$. We note that in attractor structures such as Fig. 1 above, it is always possible to find a period two (or $2n$) point I^2 whose two images have dollar signs which are subordinated to the primary dollar sign in exactly the same configuration as ${}^1I^1$ and ${}^2I^1$ are subordinated to ${}^1D^1$ in Fig. 3.

This attractor was presented as the "Japanese attractor" by Ruelle in **[24]**. The symmetric double-band structure of Fig. 3 is also predominant in eq. (2) for $\alpha < 0$ (twin-well potential) whenever the forcing amplitude is large enough that a chaotic attractor visits both potential wells. The same structure can also occur in single-well motions provided the potential well is symmetric, as in the damped forced pendulum; see for example ref. **[23]**, where invariant manifold structures are described.

We note that the structure shown in Figure 3 will usually survive a small perturbation of the symmetry, so that it is not a degenerate structure.

As with the simple band, the double band attractor structure can also occur as a subharmonic, with one point D^n and two points ${}^1I^n$ and ${}^2I^n$, generating an n-piece attractor.

In addition to the fixed points shown in Figure 3, there are only two additional fixed points: a completely stable sink lying outside the left edge of the rectangular region in Figure 3; and a directly unstable fixed point ${}^2D^1$ whose inset forms the boundary between the basin of attraction of the chaotic attractor and the basin of the sink. Since this inset is not tangled, the basin boundary has a regular global structure and remains remote from the chaotic attractor. Thus it is not difficult to draw a closed curve enclosing the chaotic attractor, and excluding the sink and all of the basin boundary, in

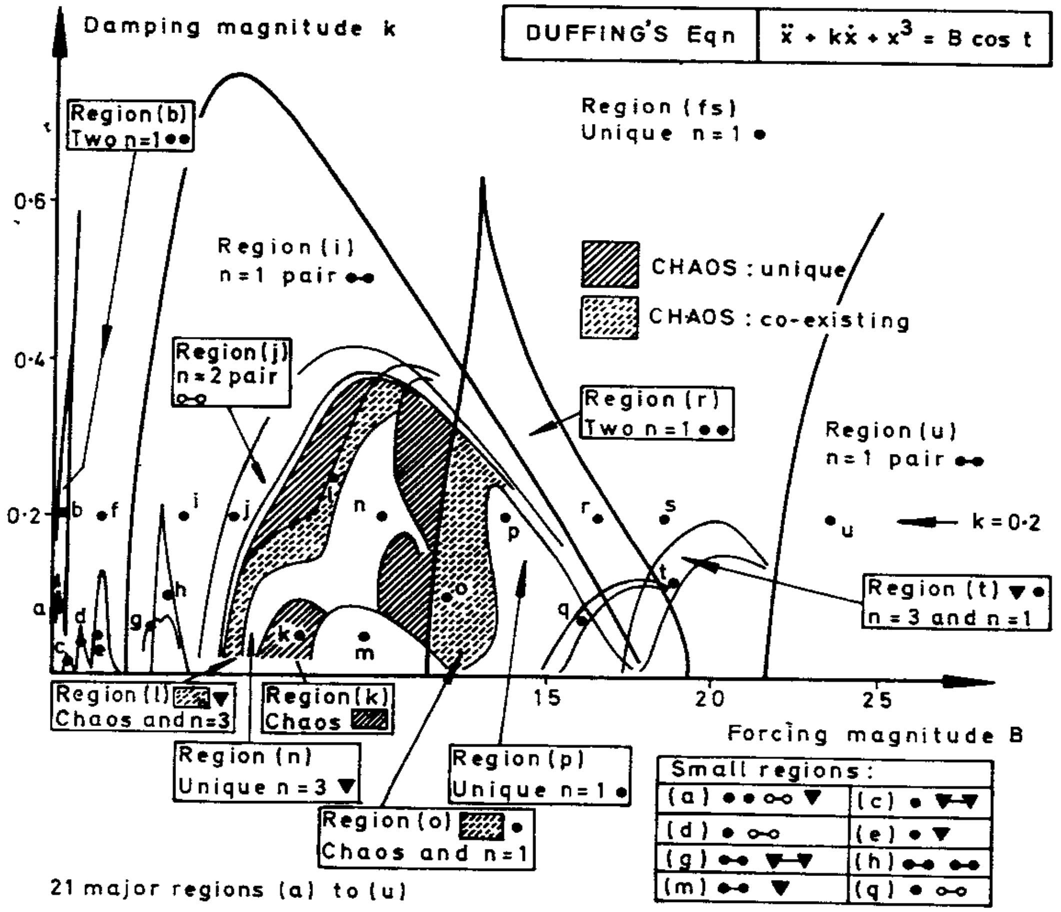

Figure 2. Results of Ueda's survey of attractors in Duffing's equation.

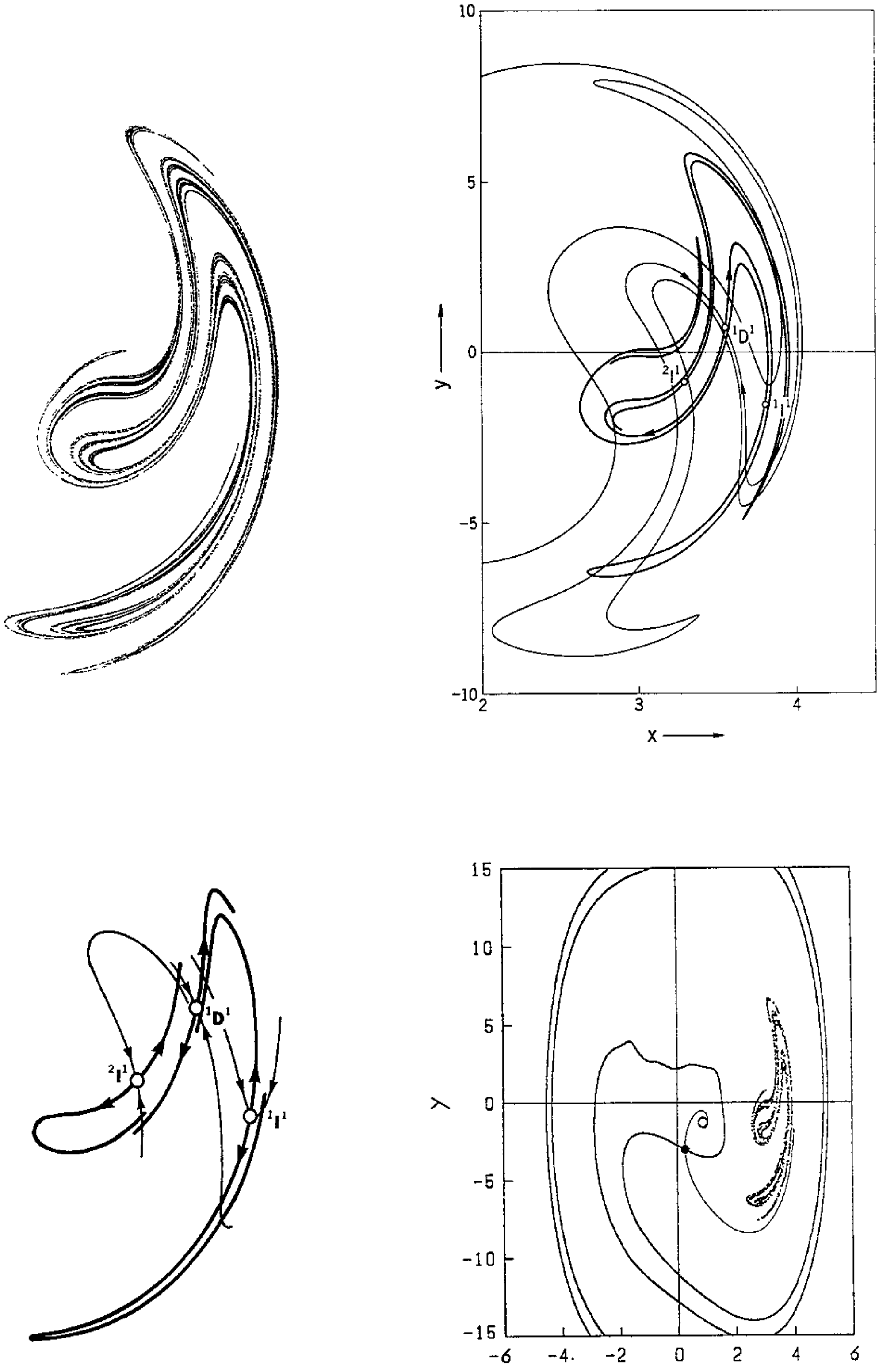

Figure 3. The Japanese attractor in Duffing's equation, and its invariant manifold structure

such a way that the closed curve may be taken as $\partial\Omega$ in Theorem 1. This implies that in Ω

$$I_\Omega(1) = 2 = D_\Omega(1) + \chi_\Omega$$

as required. Note that the rotation of T on $\partial\Omega$ equals $\chi_\Omega = 1$ for any simply connected domain satisfying $T(\partial\Omega) \subset \Omega$ containing the chaotic attractor and lying within its basin; if Ω can be chosen to contain no basic set other than the chaotic attractor, this suggests that χ_Ω may be an invariant of the chaotic attractor.

More generally, numerical evidence suggests the following conjecture for chaotic attractors of uniformly dissipative systems in $\mathbf{R}^2$, that is, systems for which $\operatorname{div} f(\cdot, t)$ is bounded above by a constant less than zero uniformly in $\mathbf{x}$ and t.

CONJECTURE 1. *Suppose a uniformly dissipative system (1) has a chaotic attractor which is generic in the sense that it does not touch its basin boundary nor is it on the threshold of explosion in size, and let n be the least period of any periodic point in the attractor. Then the chaotic attractor can be enclosed by a domain Ω homotopic to n disjoint separated balls containing no points of period n other than those in the attractor, satisfying $T(\partial\Omega) \subset \Omega$, and the numbers of unstable periodic points in the attractor (counting each of the n images) satisfy*

$$I_\Omega(n) = D_\Omega(n) + n.$$

This would imply the following as a consequence.

COROLLARY. *Every chaotic attractor in a uniformly dissipative system (1) in $\mathbf{R}^2$ contains among the periodic points of least period at least one indirectly unstable point.*

3. Forced self-oscillators

A third attractor structure occurs in forced second-order oscillators, which are self-oscillatory. To observe this structure we consider an oscillator of mixed Duffing-van der Pol type

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x^3 = A\cos\omega t \tag{4}$$

as studied by Ueda and Akamatsu [**33,32**]. Figure 4 shows a Poincaré section of the attractor at parameter values $\mu = 0.2$, $A = 17$, and $\omega = 4$, and parts of the invariant manifold structure. Here the only period one fixed point is the completely unstable point U^1 in the central region, at a distance from the attractor locus. The multipliers of this fixed point are a complex conjugate pair in magnitude; the area expansion near U^1 makes self-oscillation with $A = 0$ possible.

The lowest period of any unstable periodic point lying within the attractor in this case is two, and there are four with positive multipliers ${}^1D_j^2$ and ${}^2D_j^2, j = 1, 2$, and four with negative multipliers ${}^1I_j^2$ and ${}^2I_j^2, j = 1, 2$. Here the subscript j identifies the two images of a period two orbit; the eight points correspond to two pairs of orbits. D-type points alternate with I-type points; each I point has a dollar sign homoclinic structure, and there are Smale cycles of heteroclinic connections between the D and I points. This is seen more clearly in Figure 5, where the cycles are considered in two groups; those involving ${}^1D_j^2$ and those involving ${}^2D_j^2$. Each group shows an essential structure similar

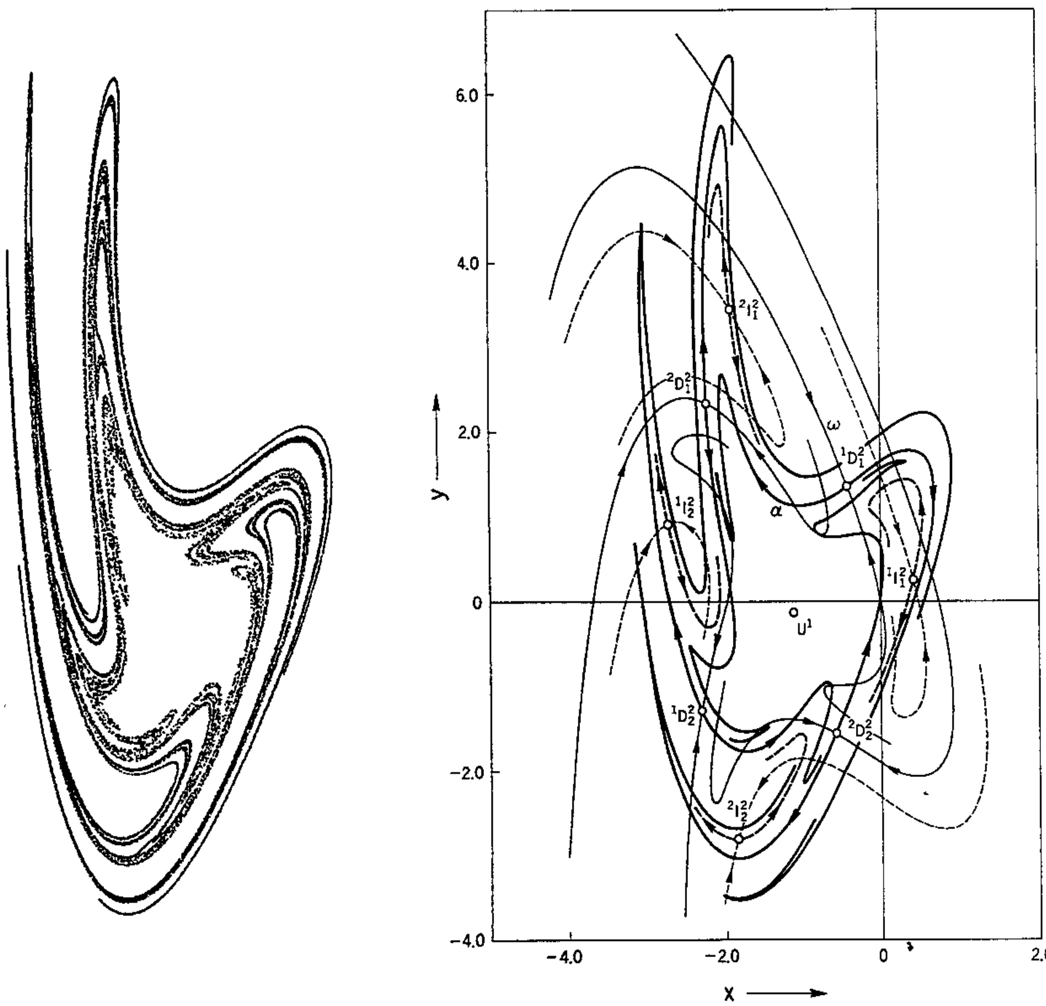

Figure 4. A Birkhoff attractor discovered by Ueda and Akamatsu in a forced oscillator of mixed Duffing-van der Pol type, with invariant manifolds.

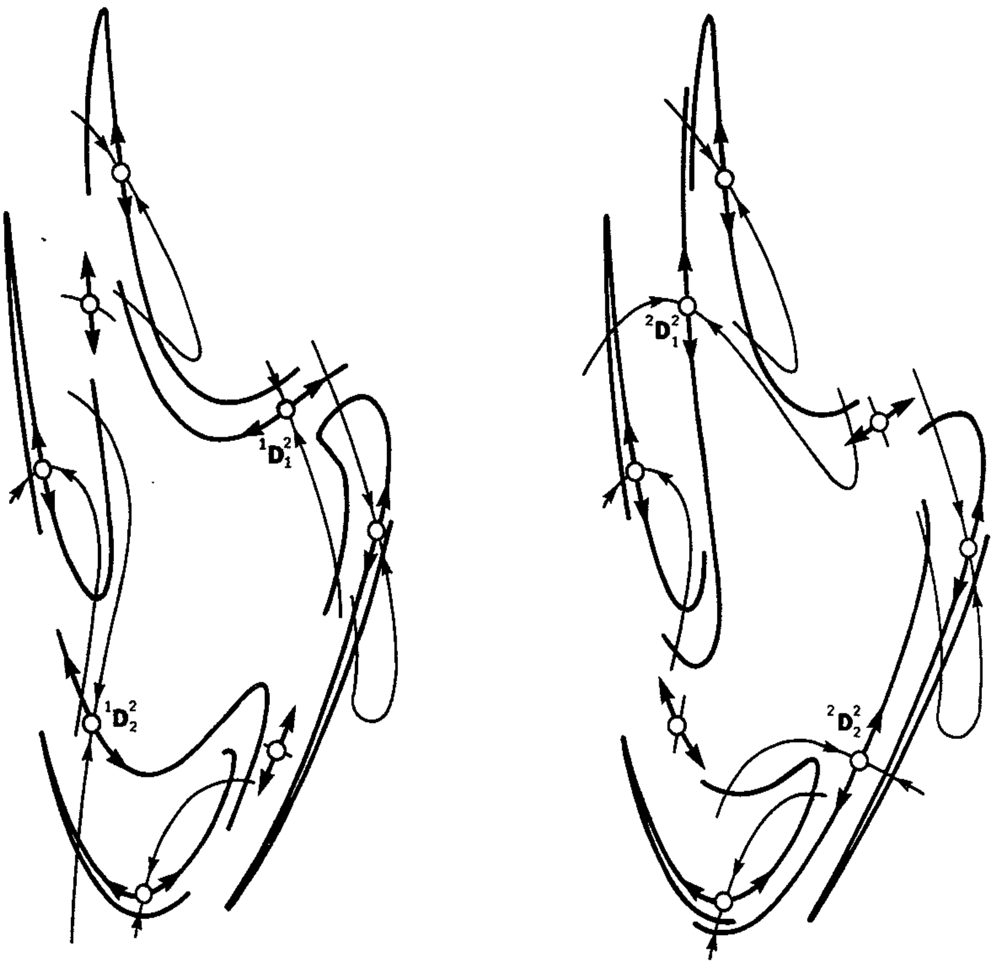

Figure 5. Smale cycles in the structure of the Birkhoff attractor.

to the double band of the Japanese attractor. The overall structure is a circular chain of dollar signs connected by D points.

In fact the same structure underlies a chaotic attractor discovered by Shaw [**25**] in a variant of the van der Pol equation of the form

$$\begin{aligned} \dot{x} &= y - A\sin\omega t \\ \dot{y} &= -x + (1-x^2)y. \end{aligned} \tag{5}$$

With $A = 0$ this is equivalent to the van der Pol equation without forcing. The usual forced van der Pol would have the periodic forcing in the second equation, acting on the acceleration, but here Shaw has moved the forcing to act on the velocity. (Although chaotic transients are common in the conventionally forced van der Pol equation, chaotic attractors are rare, and either velocity forcing or a cubic restoring force is necessary to make chaotic attractors appear robustly in simulations.) In the Shaw-van der Pol system, a decrease in forcing amplitude will cause the outsets of the I points to shrink until, at some critical value, they become just tangent to the insets of the D points; at lower forcing, the circular chain of cycles is broken, and the single attractor comes apart, leaving a pair of period two band attractors, i.e. a subharmonic of the structure of Fig. 1. This is illustrated in Fig. 13.9 of [**30**]; a similar bifurcation was reported in [**9**].

The circular chain of dollar signs occurs in variants with different subharmonic numbers. For example, it occurs in the forced pendulum whenever a chaotic attractor exists whose motions go through top dead center, i.e. are not confined to a single potential well. The dynamics in both the pendulum and the forced van der Pol systems correspond to a situation anticipated by Birkhoff [**4**] in 1932; even though Birkhoff discussed invariant sets without reference to attractors, the attractor of Figure 4 is referred to as a Birkhoff attractor. In the full three-dimensional flow, with time t replaced by $\theta = t/\omega(\text{mod}2\pi)$, this attractor is a folded (fractal) torus.

For the Birkhoff attractor in Figure 4, it is believed that all periodic points of period one or two are shown. Note that U^1 is also a period two point, and $U(2) = U(1) = 1$. Consider a domain Ω bounded by a circle of large radius R, and excluding a small circle of radius r containing U^1. Along the outer and the inner circle, points are mapped to the interior of Ω, so Theorem 1 applies; the rotation of T on $\partial\Omega$ is zero, and

$$D_\Omega(2) = 4 = I_\Omega(2)$$

as required.

Thus chaotic attractors in simple self-oscillatory systems may have either the structure of a pleated full torus, whose Poincaré section is contained in an annular domain Ω, or that of a folded band, whose section is contained in a domain Ω equivalent to n balls. This motivates a conjecture corresponding to Conjecture 1 for systems which are dissipative except near a single completely unstable fixed point of T. Again we assume a chaotic attractor which is generic in the sense that it does not touch its basin boundary, nor is it on the threshold of an explosion in size.

CONJECTURE 2. *Let n be the least period of any periodic point in a generic chaotic attractor of a system (1) which is uniformly dissipative except in a cylinder enclosing the trajectory of a completely unstable fixed point U^1. Then the Poincaré section of the attractor can be enclosed in a domain Ω with $T(\partial\Omega) \subset \Omega$ which is either homotopic to n balls as in Conjecture 1, or is homotopic to an annulus surrounding U^1 and contains no other points of period n; in the second case,*

$$I_\Omega(n) = D_\Omega(n).$$

As a corollary, we could again conclude that $I_\Omega(n)$ is at least one. The same is also conjectured for systems such as the forced pendulum

$$\ddot{\theta} + k\dot{\theta} + \sin\theta = A_0 + A\sin\omega t \tag{6}$$

which are uniformly dissipative on$[0, 2\pi) \times (-R, R)$ with 0 and 2π identified and suitably large R.

In other words, in systems of the types considered, every chaotic attractor is composed of folded band dollar sign structures of homoclinic inversely unstable points of lowest period.

4. Relation to one-dimensional non-invertible maps

In a different but related context, the above-mentioned corollaries concerning the presence of an inversely unstable point can be proved for dynamical systems defined by non-invertible iterated maps $f(X)$ of an interval or of a circle

$$X_{n+1} = f(X_n) \tag{7}$$

Such maps approximate the dynamics of the chaotic attractors observed in the systems defined by differential equations (2) through (6) in regimes where areas contract rapidly, for example at large values of the damping coefficient k in the Duffing and pendulum equations. Unfortunately, a rigorous connection between one-dimensional non-invertible map dynamics and two-dimensional diffeomorphism dynamics can at present only be made under the assumption that the Jacobian determinant (area ratio) of the two-dimensional map may be chosen arbitrarily small, as in [**2**]. For linearly damped oscillators, the area ratio is

$$J = e^{-k2\pi/\omega}.$$

To make this small, either ω must be small or k large. Small ω means very slow forcing, a regime of minimal interest. On the other hand, large k means overdamping which makes chaos impossible, as proved by Levi [**17**]. Thus there is at present no guarantee that the following theorem about chaotic attractors of (7) can be applied to forced oscillators, even though there are large regimes of equations (2), (3), (5), and (6) where chaotic attractor dynamics are well approximated by one-dimensional maps.

THEOREM 2. *Let f be a continuously differentiable non-invertible map of an interval (or of a circle) having only generic fixed points. Suppose further that f is such that the attractors of (7) are a finite collection of periodic points and closed subintervals (cf. Theorem 2.4 of [15]). If the iterated map (7) has a chaotic attractor containing in its interior an unstable fixed point D with $f'(D) > 1$, then the attractor also has in its interior an unstable fixed point I with $f'(I) < -1$.*

PROOF: If D is in the interior of an attractor, then D must have at least one pre-image; in case there are several, choose the first one to the right of D, and call it A.

Since $f'(D) > 1$, f rises above the bisectrix immediately to the right of D. Therefore it must cross the bisectrix between D and A; if it crosses more than once, let B be the crossing closest to D. Then $f'(B) < 1$.

There must be a preimage of B between D and B; indeed successive preimages of B must be asymptotic to D from the right. If $f'(B)$ were greater than -1, B would be attracting, and the basin of attraction for B would accumulate at D. This contradicts the hypothesis that the attractor containing D is a finite collection of intervals with D in its interior. Therefore $f'(B) < -1$, that is, $B = I$.

A second theorem about one-dimensional maps shows that the behavior of one-dimensional maps does not completely explain the presence of inversely unstable points in forced oscillators. The following simple lemma is preparation for the statement of the theorem.

LEMMA. *Let f be as in Theorem 1, and suppose that an unstable fixed point D with $f'(D) > 1$ exists and is homoclinic, in the sense that there exists an orbit $P_{n+1} = f(P_n)$, $n = 0, -1, -2, -3, \ldots$ with $P_0 = D$, and $\lim P_n = D$ as $n \to -\infty$. (cf. [5]). Suppose further that f has only a single relative extremum (critical point) in the interval from D to P_{-1}. Then there is a fixed point I with $f'(I) < 0$ lying between the critical point and P_{-1}.*

THEOREM 3. *Let f be as in the previous lemma. Then if I lies in the interior of a chaotic attractor, D also lies in the same attractor, as does the interval from D to I.*

PROOF: Suppose without loss of generality that P_{-1} is to the right of D. Consider P_{-2} the pre-image of P_{-1}, that is, the second pre-image of D in the homoclinic orbit. Since there is no relative extremum of f between I and P_{-1}, the slope there is always negative, so P_{-2} must lie not between I and P_{-1}, but between D and I.

Now the critical point C also lies between D and I, so in addition to P_{-2} there is a second point Q_{-2} with $f(Q_{-2}) = P_{-1}$. Either P_{-2} or Q_{-2} lies between C and I. But since $f' < 0$ between C and I, any chaotic attractor including I must include the whole interval from C to I, and hence includes a neighborhood of either P_{-2} or Q_{-2}. By invariance the attractor includes a neighborhood of D, which proves the theorem.

The significance of Theorem 3 can be appreciated by referring to an example illustrated in Fig. 6, computed from the forced Duffing oscillator eq. (2) with $A_0 = 0$, $\alpha = -1$, $\omega = 1$, $k = 0.25$. For a range of A values near 0.19, there are within each potential well two attractors, due to nonlinear resonance. Each of these attractors is confined to a single potential well. We consider only the right hand well, with $x > 0$. Within this well there are two basins of attraction separated by a basin boundary which is the inset of the fixed point ${}^2D$ of the Poincaré map. (There is also a fixed point

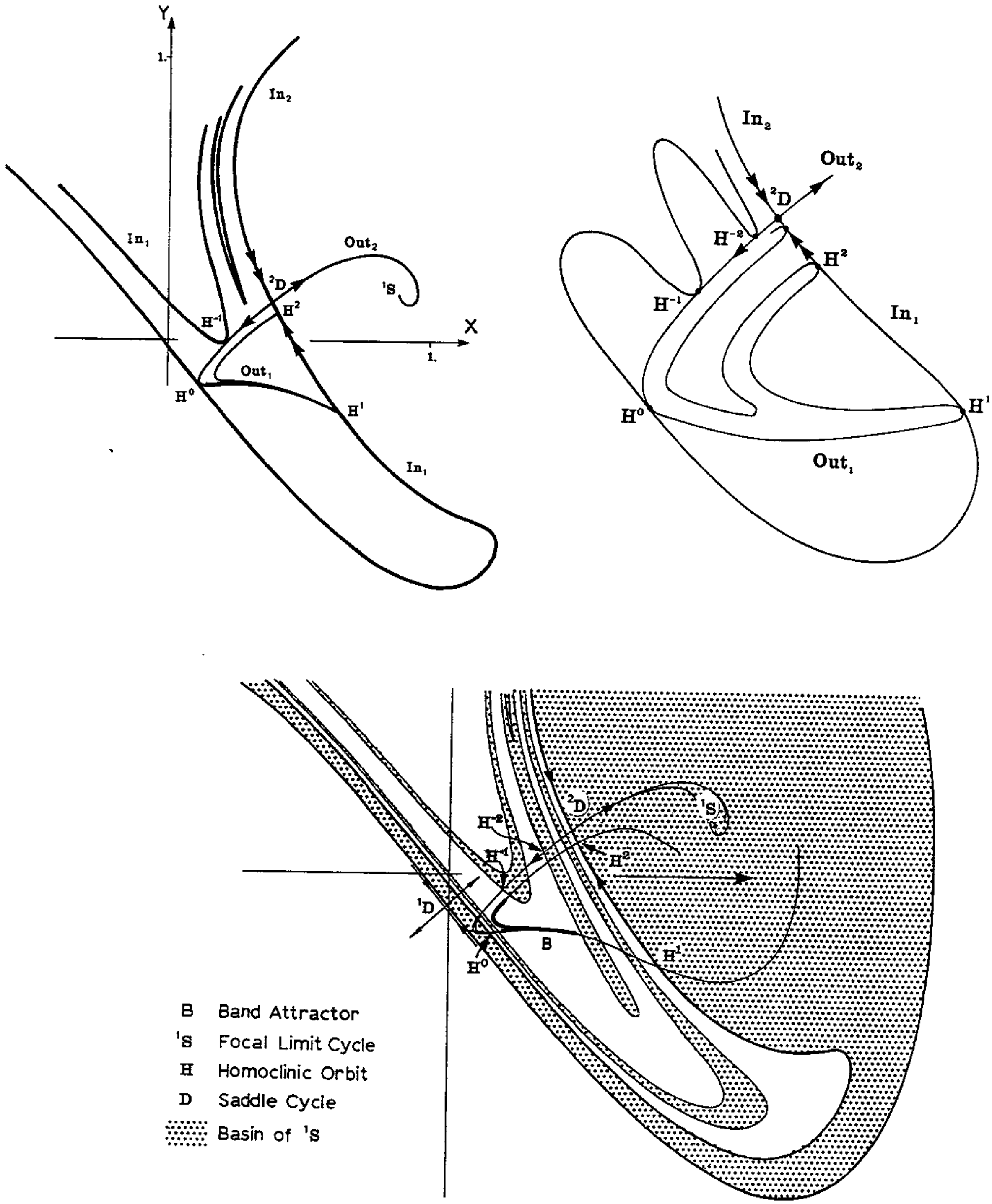

Figure 6. Homoclinic tangle in the twin-well potential Duffing oscillator creates a fractal basin boundary: tangency at $A = 0.1864$ (above); $A = 0.1923$ (below).

${}^1D$, near the origin, marking the dynamic barrier between the two potential wells.) At $A \simeq 0.1864$, the branch Out_1 of the outset of ${}^2D$ becomes tangent to the branch In_1 of the inset of ${}^2D$. Due to substantial damping, the outset is folded nearly onto itself, so for clarity a schematic drawing of the tangency is also shown. Increasing the forcing A further, persistent homoclinic intersections appear. The lower numerical portrait in Fig. 6 shows the situation for $A = 0.1923$, just before the chaotic attractor B experiences a loss of stability.

Thus in the range $0.1864 < A \leq 0.1923$ we find that one branch of the outset of ${}^2D$ is homoclinic, and the homoclinic points so formed surround one of the two attractors. The attractor within the tangle is periodic for $A = 0.1864$ and becomes chaotic (by period doubling) by the time A reaches 0.1923; the chaotic attractor contains exactly one fixed point I with negative multipliers and the dollar sign structure of Fig. 1. The effect of the tangle is a fractal structure of the basin boundary; but ${}^2D$ remains at a distance from both attractors. The horseshoe-like basic set containing ${}^2D$ has been called a chaotic saddle [**27**].

Furthermore, it turns out that the dynamic potential barrier ${}^1D$ is also homoclinic for a range of A values near $A = 0.1923$, and by the symmetry $x \to -x$, $\dot{x} \to -\dot{x}$, $t \to t + \pi/\omega$, both branches of ${}^1D$ are homoclinic if one is homoclinic. But ${}^1D$ is remote from the attractors; chaotic attractors containing ${}^1D$ only appear when the outset of I intersects the inset of ${}^1D$, for example at higher forcing; see [**34,35,28**]. In the regime of Fig. 6, the horseshoe-like basic set containing ${}^1D$ forms a double chaotic saddle in the shape of a figure eight.

The chaotic attractor labelled B in Figure 6 has highly compressed fractal layers (unlike the obviously fractal attractors of Figures 3 and 4), and it might appear that a one-dimensional approximation would adequately describe the dynamics. In fact it would be natural to obtain a one-dimensional parametrization of the dynamics in Figure 6 by projecting onto a line parallel to the outward eigenvector at ${}^2D$; the chaotic attractor, containing I, projects onto an interval which contains neither ${}^2D$ nor ${}^1D$. Such natural counterexamples to Theorem 3 are abundant in forced oscillators with low to moderate damping, but still Conjectures 1 and 2 are consistent with numerical evidence.

In order to construct an example in which a chaotic attractor might contain no inversely unstable point of lowest period, we consider a blue sky catastrophe documented in ref.[**1**]. There a Birkhoff attractor was surrounded by a basin boundary containing a period one fixed point in the Poincaré map. The chaotic attractor lies below this saddle D^1; the distance from attractor to D^1 goes to zero as a generic control is varied. After the bifurcation, there is a non-attracting tangle including the unstable periodic points of period two in the formerly attracting Birkhoff tangle, plus D^1. This could be made into an attractor by modifying the phase space to put another Birkhoff tangle above D^1, and also tangled with D^1. This can be achieved by making the system symmetric about D^1. The resulting attractor would contain a period one D point D^1 but no period one I point, only I points of period two or higher.

We note that this counter example (not yet verified numerically) would require a phase space in which any uniformly dissipative neighborhood of the attractor is topologically a disk with two holes, corresponding to the need for two completely unstable points in the center of the two Birkhoff tangles. Thus this example would not be inconsistent with Conjectures 1 and 2.

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