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Internal Structures of Self-Organized Relaxed States and Self-Similar Decay Phase

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A thought analysis on relaxation due to nonlinear processes is presented to lead to a set of general thoughts applicable to general nonlinear dynamical systems for finding out internal structures of the self-organized relaxed state without using "invariant". Three applications of the set of general thoughts to energy relaxations in resistive MHD plasmas, incompressible viscous fluids, and incompressible viscous MHD fluids are shown to lead to the internal structures of the self-organized relaxed states. It is shown that all of the relaxed states in these three dynamical systems are followed by self-similar decay phase without significant change of the spatial structure. The well known relaxed state of $\nabla \times \mathbf{B} = \pm\lambda\mathbf{B}$ is shown to be derived generally in the low β plasma limit.

Keywords : internal structure, self-organized relaxed state, self-similar decay phase, nonlinear dissipative dynamical system, thought analysis, resistive MHD plasma, incompressible viscous fluid, RFP, spheromak, FRC.

§ 1. Introduction

Much attention has been paid to the relaxation phenomena of magnetically confined plasmas in toroidal devices such as for the reversed field pinch (RFP) experiment[1-6] and for the spheromak experiment.[7-9] J. B. Taylor has given the remarkable fundamental explanation for the relaxation mechanism in the RFP discharge.[10] He showed from his idealized theory that the equation of the force-free field, $\nabla \times \mathbf{B} = \lambda \mathbf{B}$ with a constant profile of λ, represents "the minimum-energy state" which is called "the fully relaxed state", by introducing the conjecture on "the time invariant" of "the total helicity". For a cylindrical plasma he derived the well-known $\beta = 0$ Bessel function model (BFM) configuration from the equation.[10,11] The gross features of the relaxed plasmas in the experiments of the RFP and the spheromak are well described by the force-free field equation, $\nabla \times \mathbf{B} = \lambda \mathbf{B}$ with the constant profile of λ.[1-14] The detailed experimental measurements show, however, that the relaxed states of plasmas deviate somewhat from the fully relaxed state of $\nabla \times \mathbf{B} = \lambda \mathbf{B}$, and have finite pressure gradient and nonuniform profile of λ.[1-9] This deviation is considered to result from the high resistive boundary plasmas. Taking account of the experimental RFP plasma which has the finite pressure gradient and satisfies the boundary condition that the current density $\mathbf{j} = \mathbf{0}$ at the wall, the present author had introduced the partially relaxed state model (PRSM)[12-14] and developed numerical codes for the RFP equilibria and for the mode transition point of the relaxed states by introducing the energy principle with partial loss of helicity in the boundary region.[15-18] It has been shown that the experimental data of the RFP plasma in the TPE-1RM15 device [5,6] are well fitted by the numerical results of the PRSM.[14,18]

Other energy relaxation theories have been also reported, as the modifications of the theory by Taylor[10,11] for the explanation of experimental plasmas, by using infinit set of global invariant concerning with helicity[19] or by using the minimun dissipation rate or the minimum entropy production rate under the constraint of the constant time-averaged rate of supply of helicity[20] or the assumption of the total helicity invariant[21]. All of these theories mention above are based essentially on the concept of "helicity".

On the other hand, recent experimental data have clarified that in the ZP-2 device,[22] which is a simple toroidal Z pinch without toroidal coiles for the toroidal flux and therefore has no initial total helicity, there still appears the relaxation of the field configuration to lead to the spontaneous generation of the toroidal field within a few tens of μs in the produced toroidal plasma.[22–24] The relaxed state of the plasma becomes to have finite total helicity and to be close to the state of $\nabla \times \mathbf{B} = \lambda \mathbf{B}$ that cannot be determined by the initial total helicity,[22–24] contrary to the theory by Taylor.[10,11] On the other hand, three-dimensional MHD simulations have also clarified that the relaxation takes place to lead to the state given nearly by $\nabla \times \mathbf{B} = \lambda \mathbf{B}$.[25] An important point to consider is that in the MHD simulations in ref.25 they do not solve any equations for hilicity but they do only solve equations of mass, momentum, and energy (or equivalently the entropy equation) together with Maxwell's equations and Ohm's law, where $\mu_o \mathbf{j} = \nabla \times \mathbf{B}$ is used by neglecting the displacement current. This fact indicates that the quantity of helicity does not dominate the process of relaxation but is used for a kind of classification or labeling to describe some part of the process. The both results by the experiments and the MHD simulations mentioned above suggest that we need a new theory for obtaining the relaxed state without using "helicity invariant for time".

The set of general thoughts to find internal structures of the self-organized relaxed states without using any time invariant has been reported by the present author, using a thought analysis on relaxation due to nonlinear processes with dissipation itself.[26] Here, the word "thought analysis" means that we investigate logical structures, ideas or thoughts used in the objects being studied, and try to find some key elements for improvement and/or some other new thoughts which involve generality.[26-28] An example of the application of the set of general thoughts to the energy relaxation of the MHD plasma has been shown in ref.26 to lead to the equation of $\nabla \times \nabla \times \mathbf{B} = \lambda^2 \mathbf{B}$ for the internal structures of the relaxed states, which has the force-free field branch $\nabla \times \mathbf{B} = \pm\lambda\mathbf{B}$, and also lead to the mode transition condition of the relaxed state, which are derived by Taylor.[10,11] Remarkable points of the applied theory in ref.26 are the followings: (a) The relaxed state of the force-free field branch $\nabla \times \mathbf{B} = \lambda\mathbf{B}$ and the mode transition condition can be derived from the set of general thoughts without using "helicity" and "invariant", whose concepts are essential in the theory by Taylor.[10,11] (b) The applied theory permits the quasi-steady energy flow through the boundary surface, as is indeed the case in most experiments, and leads to a more general relaxed state of $2\eta\mathbf{j} = \alpha\mathbf{A}$ for plasmas having spatially dependent resistivity η. This result leads directly to the experimental fact of $\mathbf{j} = 0$ near the wall, as is indeed the case in all experiments where η goes up to infinity near the boundary wall.

Other two examples of the application of the set of general thoughts for finding out the internal structures of the self-organized relaxed states to incompressible viscous fluids and to incompressible viscous MHD fluids are reported in ref.29 to lead to equations which must be satisfied by the relaxed fluid flow patterns and the relaxed field profiles. The obtained equations for the incompressible MHD fluids would yield the field profiles and the flow patterns realized in the magmas as the result of the earth dynamo.[30]

It is interesting to investigate and clarify the property of the internal structures of the self-organized relaxed states realized in nonlinear dissipative dynamical systems such as the resistive MHD plasma, the incompressible viscous fluid, and the incompressible viscous MHD fluid. In this paper, a detailed description of the thought analysis on the relaxation due to nonlinear processes with dissipation is presented to lead to the set of general thoughts for finding out the internal structures of the self-organized relaxed states, whose internal structures are hardest to change themselves in their time evolutions and therefore followed by the self-similar decay phase without significant change of their internal distributions. Three applications of the set of general thoughts to resistive MHD plasmas, incompressible viscous fluids, and incompressible viscous MHD fluids are shown in detail to lead to the internal spatial structures of the self-organized relaxed states and their self-similar decay phases. In Sec.2, the thought analysis on self-organized relaxed states is presented in detail to lead to the set of general thoughts to find out the internal structures of the relaxed states. The first application of the set of general thoughts to resistive MHD plasmas is shown in Sec.3 in detail, where some examples of axisymmetric plasmas such as the diffused Z pinch plasma, the screw pinch plasma, the RFP plasma in the cylindrical approximation, and the field reversal configuration (FRC) plasma are presented together with a proof of the existence of the self-similar decay phase after the realization of the self-organized relaxed state. Other two applications of the set of general thoughts to incompressible viscous fluids and incompressible viscous MHD fluids are presented in Sec.4 and Sec.5, respectively, where the internal structures of the self-organized relaxed states and the self-similar decay phases are also described. Conclusing remarks are presented in Sec.6.

§ 2. Thought Analysis on Self – Organized Relaxed States

First, we show a thought analysis on relaxation where we try to analyse the concept of "relaxation due to nonlinear processes with dissipation" in order to understand the basic structure of the thoughts included in it.[26] We now consider a general nonlinear dynamical system with dissipation that consists of quantities $\mathbf{q}(t, \mathbf{x})$. Here, t is time, $\mathbf{x}$ denotes m-dimensional space variables, and $\mathbf{q}$ represents a set of physical quantities having n elements, some of which are vectors such as $\mathbf{B}$ and $\mathbf{j}$, and others are scalors such as the mass density, the energy density, the specific entropy and so on. Time evolutions of $\mathbf{q}$ are given by definite equations such as the conservation laws of mass, momentum, and energy, and the Maxwell equations or the laws ruling $\mathbf{q}(t, \mathbf{x})$ of the nonlinear dynamical system in general sense. Integrating one element, w, such as the energy density in $\mathbf{q}$ over the space volume, we can define a global quantity of $W(t)$, such as the energy of the system, as $W(t) = \int w(\mathbf{q})\, d\mathbf{x}$. We can recognize from this definition of $W(t)$ that the values of both $W(t)$ and its time derivative $\mathrm{d}W/\mathrm{d}t$ depend essentially on the internal structure (i.e. the internal distributions of $\mathbf{q}$) of the dynamical system. The value of $\mathrm{d}W/\mathrm{d}t$ represents the loss rate or the dissipation rate with respect to W of the system. We can also understand that the relation between $\mathrm{d}W/\mathrm{d}t$ and $W(t)$ is embedded essentially in the laws ruling $\mathbf{q}$ of the nonlinear dynamical system with dissipation.

When we follow the time evolution of $W(t)$, we would observe usually "very rapid decay phases" and "quasi-steady slow decay phases" almost periodically. (If $W(t)$ is the total entropy, then we observe rapid increasing phases and slow ones). We would then consider that in the rapid decay phases some nonlinear processes must take place to change the internal structure (i.e. the internal distributions of $\mathbf{q}$) of the system so drastically that the value of $W(t)$ decreases very rapidly. In other words, the internal distributions of $\mathbf{q}$ in this rapid decay phases are such distributions that make the value of $|\ \mathrm{d}W/\mathrm{d}t\ |$ very large and lead themselves to the drastic change of their own internal structures. We call these rapid decay phases with drastic nonlinear change of internal structure as "the relaxation phase".

After each of these relaxation phases, we observe the quasi-steady slow decay phases. We would think and expect for the quasi-steady slow decay phases that the system must have relaxed and reached by itself to the state with its own peculiar internal spatial distribution such that makes the value of | dW/dt | minimum (i.e. the minimum loss rate or the minimum dissipation rate with respect to W) after each of those relaxation phases. We recognize and call this relaxed state with "the minimum dissipation rate of W" as "the self-organized quasi-steady relaxed state".

Since we can observe the value of W at the time when the self-organized quasi-steady relaxed state has realized just after the relaxation phase of interest, we express here the time and the observed value of W at the relaxed state, respectively as t_R and W_R. We then come to the following general description on the internal structure of the self-organized relaxed state:

"The relaxation phase of interest with the drastic change of the internal structure continues itself until and terminates itself at the time of t_R, at which time the internal spatial distribution has reached the peculiar spatial distribution such that yields the minimum dissipation rate of W and therefore is hardest to change its own internal spatial distribution for its own instantaneous amount of the containing quantity $W = W_R$ at t_R. The state with this peculiar spatial distribution at t_R is called the self-organized quasi-steady relaxed state."

This description is rewritten simply as follows:[26)]

"The internal structure of the self-organized quasi-steady relaxed state" must satisfy "$W = W_R$" and must have also "the minimum value of $|dW/dt|$".

It should be noted clearly here that the sentence of "must satisfy $W = W_R$" shown above does not mean that W is an invariant for time, but indicates that "$W = W_R$ is the necessary condition as a global constraint" for "finding out the internal structure (which is the function of the spatial variables $\mathbf{x}$) at the time of the relaxed state from various distributions", and W is never invariant for time.

We may understand that this nonlinear dynamical system just experiences almost periodically the relaxation phases and the quasi-steady slow decay phases accompanied by the self-organized quasi-steady relaxed state on the way of the time evolution through essentially the dissipation processes. In other words, the realization of the self-organization depends essentially on the fact that the present nonlinear dynamical system is the open system with respect to W and that the dissipation rate of W, i.e. $\mid \mathrm{d}W/\mathrm{d}t \mid$, is determined by the internal structures (i.e. the internal spatial distributions of $\mathbf{q}(t, \mathbf{x})$). We may also understand that the mechanism of the periodical realization of the self-organized quasi-steady relaxed state itself is embedded essentially in the laws ruling $\mathbf{q}(t, \mathbf{x})$ of the nonlinear dynamical system with dissipation.

The thought analysis on the concept "relaxation due to nonlinear processes with dissipation" itself of the general nonlinear dynamical system mentioned above leads us to the following set of general thoughts, { [I] and [II] }, to find internal structures of the self-organized quasi-steady relaxed state with respect to W in the system, where the thought [I] is on the relaxation phase and the thought [II] is on the internal structure of the relaxed state: [26,29]

[I] In the relaxation phases, some nonlinear processes with dissipation must take place to change the internal spatial structure of $\mathbf{q}(t, \mathbf{x})$ so drastically that the value of $W(t)$ decreases (or increases) very rapidly. The relaxation phase continues itself until and terminates itself at the time when the internal spatial structure has reached the peculiar spatial structure such that yields the minimum dissipation rate of W and therefore is hardest to change its own internal spatial distribution for its own instantaneous amount of the containing quantity W.

[II] The peculiar internal spatial structure of $\mathbf{q}(t_R, \mathbf{x})$ of the self-organized quasi-steady relaxed state just after each relaxation phase must have

$$\textit{the minimum value of } \ |\, \mathrm{d}W/\mathrm{d}t \,| \ \textit{ with } W \ = \ W_R \, , \tag{1}$$

where t_R denotes the time when the self-organized relaxed state has realized just after the relaxation phase of interest, and W_R is the value of W measured at the time of t_R. Here, "$W = W_R$" is the necessary condition that must be satisfied by the internal strauctures of the relaxed state because of the measured value and becomes "the global constraint" for "finding out the internal spatial structure of the relaxed state from various distributions", and W is, of course, not the invariant for time.

Using the variational technique with respect to the spatial variables $\mathbf{x}$ for $\mathbf{q}(t_R, \mathbf{x})$, i.e. using the variations of $\delta\mathbf{q}(\mathbf{x})$, in order to find out the internal spatial structure and the minimum value of $|\mathrm{d}W/\mathrm{d}t|$ at the time of the quasi-steady relaxed state, which is given by the two thoughts { [I], [II] } with eq.(1), we obtain the following mathematical expressions:[26,29]

$$\delta F \ = \ 0, \tag{2}$$

$$\delta^2 F \ > \ 0, \tag{3}$$

where F is the functional defined by $F = |\mathrm{d}W/\mathrm{d}t|$ - αW; δF and $\delta^2 F$ are the first and second variations of F; and α is the Lagrange multiplier. When the boundary values of x_k component of some elements q_j in $\mathbf{q}$ are given such as by the property of the given boundary materials and/or measurements at the relaxed state, then the boundary conditions of the variations $\delta\mathbf{q}(\mathbf{x})$ are written as

$$\delta q_{jk} \ = \ 0 \quad \text{at the boundary} \, . \tag{4}$$

We should notice here that the present theory shown above is neither "the energy principle" nor "the variational principle" based on some "invariant for time".

The set of general thoughts { [I], [II] } with eqs.(1)-(4) to find internal structures of the self-organized relaxed states would be common for all dynamical systems including physical systems, chemical systems, biological systems, and/or economical systems, in general.

§ 3. Application to Resistive MHD Plasmas

We now apply the set of general thoughts { [I], [II] } with eqs.(1) - (4) to the resistive MHD plasma which is described by the following simplified equations with Ohm's law of $\eta\mathbf{j} = \mathbf{E} + \mathbf{u} \times \mathbf{B}$,

$$\rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{j} \times \mathbf{B} - \nabla p, \tag{5}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times (\eta\mathbf{j}), \tag{6}$$

where the viscosity is assumed to be negligibly small. Using Maxwell's equations and Ohm's law, we obtain the time derivative of the magnetic energy $W_m = \int (B^2/2\mu_o)\mathrm{d}v$ as follows,

$$\frac{\mathrm{d}W_m}{\mathrm{d}t} = -\int \{ \eta\,\mathbf{j}\cdot\mathbf{j} + (\mathbf{j} \times \mathbf{B}) \cdot \mathbf{u} \}\mathrm{d}v - \oint (\mathbf{E} \times \mathbf{H}) \cdot \mathrm{d}\mathbf{s}, \tag{7}$$

where $\oint$ denotes the surface integral over the boundary. For simplicity, it is assumed here, as is indeed the case in most experiments, the plasma internal energy and the mass flow energy are negligible compared to the magnetic energy W_m. We pick up the magnetic energy W_m of the system and look for the self-organized quasi-steady relaxed state with respect to W_m. In the quasi-steady relaxed state, we may assume $\mathbf{u} \cong 0$, and may obtain the following equilibrium equation from eq.(5),

$$\nabla p = \mathbf{j}_R \times \mathbf{B}_R\,, \tag{8}$$

where the subscript R denotes the quantities at the quasi-steady relaxed state. We assume here, for simplicity, that the resistivity η has a fixed spatial dependence like as $\eta(\mathbf{x})$ at the quasi-steady relaxed state, as is indeed the case in all experiments where η goes up to infinity near the boundary wall. Substituting W_m and $|\mathrm{d}W_m/\mathrm{d}t|$ with $\mathbf{u} = 0$ respectively to W and $|\mathrm{d}W/\mathrm{d}t|$ in the set of general thoughts { [I], [II] } with eqs.(1)-(4) to find the internal structures of the self-organized quasi-steady relaxed state, we obtain the followings,[26)]

$$\delta F \;=\; \int (2\eta\, \delta\mathbf{j}\cdot\mathbf{j} \;-\; \frac{\alpha}{\mu_o}\delta\mathbf{B}\cdot\mathbf{B})\mathrm{d}v \;=\; 0, \tag{9}$$

$$\delta^2 F \;=\; \int (2\eta\, \delta\mathbf{j}\cdot\delta\mathbf{j} \;-\; \frac{\alpha}{\mu_o}\delta\mathbf{B}\cdot\delta\mathbf{B})\mathrm{d}v \;>\; 0, \tag{10}$$

where the values of the Poynting vector $\mathbf{E}\times\mathbf{H}$ on the boundary surface in $\mathrm{d}W_m/\mathrm{d}t$ are assumed to be given so that the surface integral terms vanish in both δF and $\delta^2 F$ by the boundary conditions of eq.(4), for simplicity. Using $\mu_o\delta\mathbf{j} = \nabla\times\delta\mathbf{B}$, the vector formula of $\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot\nabla\times\mathbf{a} - \mathbf{a}\cdot\nabla\times\mathbf{b}$, and the Gauss theorem, we obtain the followings from eqs.(9) and (10),

$$\delta F \;=\; \frac{2}{\mu_o}\int \delta\mathbf{B}\cdot\{\,\nabla\times(\eta\mathbf{j}) \;-\; \frac{\alpha}{2}\mathbf{B}\}\mathrm{d}v \;-\frac{2}{\mu_o}\oint(\,\eta\mathbf{j}\times\delta\mathbf{B}\,)\cdot\mathrm{d}\mathbf{s} = \;0, \tag{11}$$

$$\delta^2 F \;=\; \frac{2}{\mu_o}\int \delta\mathbf{B}\cdot\{\,\nabla\times(\eta\delta\mathbf{j}) \;-\; \frac{\alpha}{2}\delta\mathbf{B}\}\mathrm{d}v \;-\frac{2}{\mu_o}\oint(\,\eta\delta\mathbf{j}\times\delta\mathbf{B}\,)\cdot\mathrm{d}\mathbf{s} \;>\; 0\,. \tag{12}$$

We then obtain the Euler-Lagrange equation from the volume integral term in eq.(11) for arbitrary variations of $\delta\mathbf{B}$ as follows,

$$\nabla\times(\eta\mathbf{j}) \;=\; \frac{\alpha}{2}\mathbf{B}\,. \tag{13}$$

When we use $\mu_o\mathbf{j} = \nabla\times\mathbf{B}$ instead of $\mu_o\delta\mathbf{j} = \nabla\times\delta\mathbf{B}$, we obtain the followings from eqs.(9) and (10), corresponding to eqs.(11) and (12),

$$\delta F = \int \delta \mathbf{j} \cdot (2\eta \mathbf{j} - \alpha \mathbf{A}) dv - \frac{\alpha}{\mu_o} \oint (\mathbf{A} \times \delta \mathbf{B}) \cdot d\mathbf{s} = 0, \tag{14}$$

$$\delta^2 F = \int \delta \mathbf{j} \cdot (2\eta \delta \mathbf{j} - \alpha \delta \mathbf{A}) dv - \frac{\alpha}{\mu_o} \oint (\delta \mathbf{A} \times \delta \mathbf{B}) \cdot d\mathbf{s} > 0 , \tag{15}$$

where $\mathbf{A}$ is the vector potential. We then obtain the Euler-Lagrange equation from the volume integral term in eq.(14) for arbitrary variations of $\delta \mathbf{j}$ as follows,[26)]

$$\eta \mathbf{j} = \frac{\alpha}{2} \mathbf{A}. \tag{16}$$

Taking rotation of eq.(16), we obtain eq.(13) again. Since $\mathbf{A}$ is finite near the boundary wall, the present result of eq.(16) leads directly to the experimental fact that the current density $\mathbf{j}$ goes to zero near the wall where η goes to infinity, as is indeed the case in all experiments.

We now have found that the self-organized quasi-steady relaxed state has the peculiar internal structure which satisfies eq.(13). Taking account of the assumption $\mathbf{u} \cong 0$ for the self-organized quasi-steady relaxed state, and substituting eq.(13) into eq.(6), we obtain the following,

$$\frac{\partial \mathbf{B}}{\partial t} \cong - \frac{\alpha}{2} \mathbf{B}. \tag{17}$$

Equation (17) gives us the following solution

$$\mathbf{B}(\mathbf{x}, t) \cong \mathbf{B}_R(\mathbf{x}) e^{-\frac{\alpha}{2} t}, \tag{18}$$

where $\mathbf{B}_R(\mathbf{x})$ is the solution of eq.(13) for the self-organized quasi-steady relaxed state. We see from eq.(18) that the field profile of $\mathbf{B}$ just after the realization of the self-organized relaxed state has the self-similar decay without significant change the spatial structure. The second term of eq.(6) and boundary conditions would lead to finite deviation from the self-similar decay gradually. We may recognize from eq.(13) for the self-organized quasi-steady relaxed state and eq.(18) for the time evolution of the relaxed **B** field that the present nonlinear dynamical system relaxes to the state that has attained such a peculiar internal spatial structure that yields the minimum dissipation rate of W_m and thereafter leads to the self-similar decay phase without significant change of the spatial structure.

We now assume the resistivity η to be constant, for simplicity. We then obtain the following from eq.(13),[26)]

$$\nabla \times \nabla \times \mathbf{B} \;=\; \lambda^2 \mathbf{B}\,, \tag{19}$$

$$|\,\lambda\,| = \sqrt{\frac{\alpha\mu_o}{2\eta}}\,, \tag{20}$$

where the Lagrange multiplier α is assumed to be positive. Equation (19) is the same with the equation used for the classical spheromak.[31,32)] According to ref.31, three independent solutions of eq.(19) with $\nabla \cdot \mathbf{B} = 0$ are given by

$$\mathbf{L}_m \;=\; \mathrm{grad}\psi_m, \;\; \mathbf{T}_m \;= \nabla \times (\mathbf{e}\psi_m), \;\; \text{and} \;\; \mathbf{S}_m \;=\; \frac{1}{\lambda}\nabla \times \mathbf{T}_m\,, \tag{21}$$

where $\mathbf{e}$ is a fixed unit vector, and ψ_m is a scalor function such that

$$\nabla^2 \psi_m \;+\; \lambda^2 \psi_m \;=\; 0\,. \tag{22}$$

Here, the solution of $\mathbf{L}_m$ may be excluded from the solutions for eq.(19), because $\nabla \times \mathrm{grad}\;\psi_m \equiv\; 0$. The general solution of eq.(19), $\mathbf{B}_R(\mathbf{x})$, for the self-organized quasi-steady relaxed state is then written as

$$\mathbf{B}_R(\mathbf{x}) \;=\; c_{m1}\mathbf{T}_m \;+\; c_{m2}\mathbf{S}_m. \tag{23}$$

Using eq.(23) and $\mu_o \mathbf{j} = \nabla \times \mathbf{B}$, we obtain the current density of the relaxed state, $\mathbf{j}_R(\mathbf{x})$, as follows,

$$\mathbf{j}_R(\mathbf{x}) \;=\; \frac{\lambda}{\mu_o}(c_{m1}\mathbf{S}_m \;+\; c_{m2}\mathbf{T}_m), \tag{24}$$

where eq.(21) and $\nabla \times \nabla \times \mathbf{T}_m = \lambda^2 \mathbf{T}_m$ are used. There are three unknown factors of $\{\lambda, c_{m1}, c_{m2}\}$ in eqs.(21)-(24). In order to determine the values of the three unknown factors $\{\lambda, c_{m1}, c_{m2}\}$, it is enough to use three measured values of the magnetic energy W_m, the toroidal magnetic flux Φ and the toroidal plasma current I inside the boundary at the time of the relaxed state, which are denoted here respectively by W_{mR}, Φ_R, and I_R. It is because that we obtain Φ_R and I_R by integrating eqs.(23) and (24) respectively across the poloida cross-section of the toroidal plasma.

Using eqs.(8),(23) and (24), we obtain the followings,

$$\nabla p \;=\; \mathbf{j}_R \times \mathbf{B}_R \;=\; \frac{\lambda}{\mu_o}({c_{m2}}^2 - {c_{m1}}^2)\mathbf{T}_m \times \mathbf{S}_m, \tag{25}$$

$$\frac{\lambda}{\mu_o}\Phi_R \;-\; I_R \;=\; \frac{\lambda}{\mu_o}(c_{m2} - c_{m1}) \int_{S_p} (\mathbf{S}_m \;-\; \mathbf{T}_m) \cdot \mathrm{ds}, \tag{26}$$

where $\int_{S_p}$ denotes the integral across the poloidal cross-section of the toroidal plasma. It is seen from eqs.(25) and (26) that the difference between c_{m1} and c_{m2} yields the non-force-free component which is balanced with the pressure gradient.

In the limit of the low β plasma, we come to have the profiles with $c_{m1} = c_{m2}$ from comparison between eq.(8) and eq.(25) because of two independent vector solutions of $\mathbf{T}_m$ and $\mathbf{S}_m$, and obtain the followings from eq.(23),

$$\mathbf{B}_R(\mathbf{x}) \;=\; c_{m1}(\mathbf{T}_m \;+\; \mathbf{S}_m), \tag{27}$$

which satisfies the following as was shown in ref.31,

$$\nabla \times \mathbf{B} \;=\; \pm\lambda \mathbf{B}\,. \tag{28}$$

We see from eqs.(27) and (28) that the force-free fields of $\nabla \times \mathbf{B} = \pm\lambda\mathbf{B}$, derived by Taylor based on "the minimum energy state under the time invariant of the total helicity",[10,11] can be derived generally as the low β plasma limit of the self-organized relaxed state which has the minimum dissipation rate profile and therefore is hardest to change its own profile for its own instantaneous amount of the containing magnetic energy, in the nonlinear and dissipative MHD system, without using the "helicity" and the "time invariant".

We show here some examples of axisymmetric plasmas in the cylindrical coordinates (r, θ, z). First, we consider simple cases of the straight axisymmetric plasmas such as the diffused Z pinch, the screw pinch, and the reversed field pinch (RFP) in the cylindrical approximation. The z direction is now the toroidal direction and we use the unit vector along the z direction, $\mathbf{e}_z$, for the fixed unit vector $\mathbf{e}$ in eq.(21). In this case, eq.(22) becomes one dimensional problem, and the solution of ψ_m is known to be the 0th order Bessel function written as $\psi_m = J_o(\lambda r)$, by solving eq.(22). Then the vector solutions of $\mathbf{T}_m$ and $\mathbf{S}_m$ in eq.(21) are obtained respectively as

$$\mathbf{T}_m \;=\; \lambda J_1(\lambda r)\mathbf{e}_\theta \;, \tag{29}$$

$$\mathbf{S}_m \;=\; \lambda J_o(\lambda r)\mathbf{e}_z \;, \tag{30}$$

where $J_1(\lambda r)$ is the 1st order Bessel function, and $\mathbf{e}_\theta$ is the unit vector of the θ direction. For the first example, we consider the self-organized relaxed state of the diffused Z pinch. Since the measured velue Φ_R of the toroidal flux for this Z pinch is zero, we obtain the followings from eqs.(23), (29) and (30),

$$\Phi_R = \int_{S_p} \mathbf{B}_R \cdot \mathrm{ds} \;=\; 2\pi c_{m2}\lambda \int_0^{r_w} J_o(\lambda r) r \mathrm{d}r \;=\; 0\;, \tag{31}$$

where r_w is the wall (boundary) radius. We therefore obtain $c_{m2} = 0$ from eq.(31) and find from eqs.(23), (24), (29) and (30) that the configurations of the relaxed state of the diffuse Z pinch are given by

$$\mathbf{B}_R \;=\; c_{m1}\lambda J_1(\lambda r)\mathbf{e}_\theta \;, \tag{32}$$

$$\mathbf{j}_R = \frac{c_{m1}\lambda^2}{\mu_o} J_o(\lambda r)\mathbf{e}_z . \quad (33)$$

The two factors of c_{m1} and λ are determined by using the other two measured values of W_{mR} and I_R. Substituting eqs.(32) and (33) into the equilibrium equation, we obtain the pressure gradient that leads to the pressure profile at the relaxed state as follows,

$$\nabla p = -\frac{{c_{m1}}^2\lambda^3}{\mu_o} J_o(\lambda r) J_1(\lambda r)\mathbf{e}_r , \quad (34)$$

where $\mathbf{e}_r$ is the unit vector of the r direction. We can expect from eq.(18) and eqs.(32)-(34) that the obtained profiles of $\mathbf{B}_R$, $\mathbf{j}_R$ and p at the relaxed state for the straight diffused Z pinch are followed by the self-similar decay phase. We should bear in mind, however, that the change of the spatial distribution of resistivity η caused such as by ohmic heating and also the second term of eq.(6) would result in some gradual deviation from the self-similar decay.

For the second example, we consider the self-organized relaxed state of the straight screw pinch. We now express c_{m2} as $c_{m2} = c_{m1}$ - Δc. We then obtain the followings from eqs.(23)-(25) and eqs.(29) and (30),

$$\mathbf{B}_R = c_{m1}\lambda[\, J_1(\lambda r)\mathbf{e}_\theta + J_o(\lambda r)\mathbf{e}_z \,] - \Delta c \lambda J_o(\lambda r)\mathbf{e}_z , \quad (35)$$

$$\mathbf{j}_R = \frac{c_{m1}\lambda^2}{\mu_o}[\, J_1(\lambda r)\mathbf{e}_\theta + J_o(\lambda r)\mathbf{e}_z \,] - \frac{\Delta c \lambda^2}{\mu_o} J_1(\lambda r)\mathbf{e}_\theta , \quad (36)$$

$$\frac{\lambda}{\mu_o}\Phi_R - I_R = -\frac{2\pi\Delta c\lambda^2}{\mu_o}\int_0^{r_w} J_o(\lambda r) r \mathrm{d}r , \quad (37)$$

$$\nabla p = -\frac{\Delta c(2c_{m1} - \Delta c)\lambda^3}{\mu_o} J_o(\lambda r) J_1(\lambda r)\mathbf{e}_r . \quad (38)$$

The three factors of Δc, c_{m1}, and λ are determined by using the three measured values of W_{mR}, Φ_R, and I_R. The screw pinch is usually operated at the high toroidal field without the field reversal. We see from eqs.(37) and (38) that the value of Δc depends on the β value of the confined plasma. We find from eqs.(35) and (36) that the configurations of $\mathbf{B}_R$ and $\mathbf{j}_R$ at the relaxed state of the screw pinch containes the force-free field component of the Bessel function model, i.e. the first terms of eqs.(35) and (36), which would be fairly high compared with the non-force-free field component that depends on the β value of the confined plasma. We can also expect from eq.(18) and eqs.(35)-(38) that the obtained profiles of $\mathbf{B}_R$, $\mathbf{j}_R$ and p at the relaxed state for the screw pinch are followed by the self-similar decay phase with some gradual deviation, just as same as the diffused Z pinch shown above. In the experimental screw pinch plasma, the spatial distribution of the resistivity η would fairly modify the profiles of $\mathbf{B}_R$, $\mathbf{j}_R$ and p, especially in the boundary region.

For the third example, we consider the RFP plasma which has the toroidal field reversal. The profiles of $\mathbf{B}_R$, $\mathbf{j}_R$ and p at the relaxed state for the RFP are also shown by eqs.(35)-(38), just the same as for the screw pinch. In the limit of the low β plasma, Δc becomes zero from eq.(38), and we obtain the followings for the $\beta = 0$ RFP plasma from eqs.(35)-(37),

$$\mathbf{B}_R = c_{m1}\lambda[\, J_1(\lambda r)\mathbf{e}_\theta + J_o(\lambda r)\mathbf{e}_z \,] , \tag{39}$$

$$\mathbf{j}_R = \frac{c_{m1}\lambda^2}{\mu_o}[\, J_1(\lambda r)\mathbf{e}_\theta + J_o(\lambda r)\mathbf{e}_z \,] , \tag{40}$$

$$\frac{\lambda}{\mu_o}\Phi_R = I_R . \tag{41}$$

We easily recognize that eqs.(39)-(41) are the well known Bessel function model for the RFP plasma derived and discussed by Taylor based on the time invariant of the total helicity.[10,11)]

When we consider the finite β RFP plasma with $\Delta c > 0$, the pressure profile would be given basically by eq.(38). However, we notice from eq.(38) that the direction of ∇p reverses across the field reversal point of $J_o(\lambda r) = 0$. This result suggests that the RFP plasma outside the field reversal point at the relaxed state, based on the assumption of η = const., is unstable or tends to have uniform pressure profile in the field reversal region through the interaction with the boundary wall. In the experimental RFP plasma, the resultant spatial distribution of the resistivity η, affected by the plasma-wall interaction, would fairly modify the profiles of $\mathbf{B}_R$, $\mathbf{j}_R$ and p consequently, especially in the boundary region.[14,18]

We now consider the self-organized relaxed state of the field reversal configuration (FRC) plasma, where the toroidal direction is along the θ direction.[33] In the case of the FRC plasma, eq.(22) becomes two dimensional problem with respect to r and z, and the solution of ψ_m will be expressed as $\psi_m = \psi_m(r, z)$. The two vector solutions of $\mathbf{T}_m$ and $\mathbf{S}_m$ in eq.(21) are then written as follows,

$$\mathbf{T}_m = -\frac{\partial \psi_m}{\partial r}\,\mathbf{e}_\theta\,, \tag{42}$$

$$\mathbf{S}_m = \frac{1}{\lambda}\Big[\frac{\partial^2 \psi_m}{\partial r \partial z}\,\mathbf{e}_r - \Big(\frac{1}{r}\frac{\partial \psi_m}{\partial r} + \frac{\partial^2 \psi_m}{\partial r^2}\Big)\,\mathbf{e}_z\Big]. \tag{43}$$

Since the FRC plasma has no toroidal flux usually, we obtain the followings from eqs.(23),(42), and (43),

$$\Phi_R = \int_{S_p} \mathbf{B}_R \cdot \mathrm{d}\mathbf{s} = -c_{m1}\int_{S_p} \frac{\partial \psi_m}{\partial r}\mathrm{d}s = 0\,. \tag{44}$$

We therefore obtain $c_{m1} = 0$ from eq.(44) and find from eqs.(23), (24), (42) and (43) that the configurations of the relaxed state of the FRC plasma are given by

$$\mathbf{B}_R = c_{m2}\frac{1}{\lambda}\Big[\frac{\partial^2 \psi_m}{\partial r \partial z}\,\mathbf{e}_r - \Big(\frac{1}{r}\frac{\partial \psi_m}{\partial r} + \frac{\partial^2 \psi_m}{\partial r^2}\Big)\,\mathbf{e}_z\Big], \tag{45}$$

$$\mathbf{j}_R = -\frac{c_{m2}\lambda}{\mu_o}\frac{\partial \psi_m}{\partial r}\,\mathbf{e}_\theta\,. \tag{46}$$

The two factors of c_{m2} and λ are determined by using the other two measured values of W_{mR} and I_R. Substituting eqs.(45) and (46) into the equilibrium equation, eq.(25), we obtain the pressure gradient that leads to the pressure profile at the relaxed state as follows,

$$\nabla p \;=\; \frac{{c_{m2}}^2}{\mu_o}\Big[\,\frac{\partial\psi_m}{\partial r}\frac{\partial^2\psi_m}{\partial r\partial z}\,\mathbf{e}_z \;+\; \frac{\partial\psi_m}{\partial r}\Big(\,\frac{1}{r}\frac{\partial\psi_m}{\partial r} \;+\; \frac{\partial^2\psi_m}{\partial r^2}\,\Big)\,\mathbf{e}_r\,\Big]. \tag{47}$$

We can expect from eq.(18) and eqs.(45)-(47) that the obtained profiles of $\mathbf{B}_R$, $\mathbf{j}_R$ and p at the relaxed state for the FRC plasma are followed by the self-similar decay phase, similarly to the cases of the diffused Z pinch, the screw pinch, and the RFP shown above. Recently, it was reported that low toroidal field was observed experimentally in the translated FRC plasma in FIX machine.[33] This configuration of the FRC plasma with low toroidal field is corresponding to the case with finite value of c_{m1} in eq.(44). By adding the term of this small c_{m1} in eqs.(23) and (24), the configurations of eqs.(45)-(47) are rewritten easily for the relaxed state of the translated FRC plasma with the low toroidal field.[33]

Using eq.(12), we next discuss the mode transition point of the relaxed state, for example from the cylindrical mode to the mixed helical one in the cylindrical plasma.[10,11,16] We consider here the following associated eigenvalue problem for critical perturbations $\delta\mathbf{B}$ that make $\delta^2 F$ in eq.(12) vanish:

$$\nabla\times(\eta\nabla\times\delta\mathbf{B}_i) \;-\; \frac{\mu_o\alpha_i}{2}\delta\mathbf{B}_i \;=\; 0, \tag{48}$$

with the boundary conditions of $\delta\mathbf{B}\cdot\mathrm{ds}=0$, and $(\eta\delta\mathbf{j}\times\delta\mathbf{B})\cdot\mathrm{ds}=0$ at the boundary, where α_i and $\delta\mathbf{B}_i$ denote the eigenvalue and the eigensolution, respectively. Substituting the eigensolution $\delta\mathbf{B}_i$ into eq.(12) and using eq.(48), we obtain the following:

$$\delta^2 F \;=\; \frac{1}{\mu_o}(\alpha_i \;-\; \alpha)\int\delta\mathbf{B}_i\cdot\delta\mathbf{B}_i\;\mathrm{d}v \;>\; 0. \tag{49}$$

Since eq.(49) is required for all eigenvalues, we obtain the following condition for the self-organized relaxed state with the minimum $|dW_m/dt|$,

$$0 < \alpha < \alpha_1 , \tag{50}$$

where α_1 is the smallest of the positive eigenvalues, and α is assumed to be positive, as was assumed at eq.(20). When the value of α corresponding to W_{mR} goes out of the condition of eq.(50), like as $\alpha_1 < \alpha$, then the mixed mode, which has the value of W_{mR} and consists of the basic mode by the solution of eq.(13) with $\alpha = \alpha_1$ and the lowest eigenmode by eq.(48), becomes the self-organized relaxed state with the minimum value of $|dW_m/dt|$. By using definitions of $\eta(\mathbf{x}) = \eta_o g(\mathbf{x})$ and $|\lambda| = \sqrt{\alpha\mu_o/2\eta_o}$, the condition of eq.(50) can be rewritten to other form similar to the mode transition condition shown in refs.16 and 18, where η_o is the value of η at the magnetic axis. The mode transition condition of eq.(50) is the generalization of the mode transition condition by Taylor.[10,11,16]

It is easy to show from eqs.(19)-(28) that in the case of the low β plasma limit with η = const., the eigenvalue problem of eq.(48) includes the following eigenvalue problem as a force-free branch,

$$\nabla \times \delta\mathbf{B}_i = \pm\lambda_i \,\delta\mathbf{B}_i \tag{51}$$

with the boundary condition of $\delta\mathbf{B} \cdot d\mathbf{s} = 0$ at the boundary, where λ_i is the eigenvalue, and this eigensolution $\delta\mathbf{B}_i$ makes the surface integral term of eq.(12) vanish automatically. Substituting the eigensolution $\delta\mathbf{B}_i$ into eq.(12) and using eq.(51), we obtain the following:

$$\delta^2 F = \frac{2\eta}{\mu_o^2}(\lambda_i^2 - \lambda^2)\int \delta\mathbf{B}_i \cdot \delta\mathbf{B}_i \, dv > 0, \tag{52}$$

where eq.(20) is used. Since eq.(52) is required for all eigenvalues, we obtain the following condition for the relaxed state with the minimum $|dW_m/dt|$,

$$\lambda_{-1} < \lambda < \lambda_1 , \tag{53}$$

where λ_{-1} and λ_1 are the largest of the negative and the smallest of the positive eigenvalues, respectively. This mode transition condition is the same as that in Taylor's theory.[10,11,16]

The experimental relaxation phenomena in the simple toroidal Z pinch in the ZP-2 device[22–24] can be explained by the present theory, because of no need of "helicity" and "invariant". Furthermore, since the present theory permits the quasi-steady energy flow through the boundary surface by the Poynting vector, as is indeed the case in most experiments, the present result reveals that the relaxations to the state of $\nabla \times \mathbf{B} = \pm \lambda \mathbf{B}$ are more general phenomena that take place in low β plasmas even within nonideally conducting boundary.

§ 4. Application to Incompressible Viscous Fluids

We next apply the set of general thoughts { [I], [II] } with eqs.(1) - (4) to the incompressible viscous fluid which is described by the Navier-Stokes equation

$$\rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\nabla p + \nu \nabla^2 \mathbf{u}, \tag{54}$$

where ρ, $\mathbf{u}$, and p are the fluid mass density, the fluid velocity, and the pressure, respectively, ν is the coefficient of viscosity, and $\nabla \cdot \mathbf{u} = 0$. We pick up here the flow energy $W_f = \int (\rho u^2/2) \mathrm{d}v$ of the system and look for the self-organized quasi-steady relaxed state with to W_f, which is therefore the global quantity W in the set of general thoughts { [I], [II] }. Using eq.(54), the vector formula of $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b}$, $\nabla \cdot \mathbf{u} = 0$, and the Gauss theorem, we obtain the following,

$$\frac{\mathrm{d}W_f}{\mathrm{d}t} = -\int \nu\, \omega \cdot \omega \mathrm{d}v + \oint \{ \nu(\mathbf{u} \times \omega) - p\mathbf{u} \} \cdot \mathrm{d}\mathbf{s}, \tag{55}$$

where $\omega = \nabla \times \mathbf{u}$ is the vorticity. We assume here ν to be constant, for simplicity. Substituting W_f and $|\mathrm{d}W_f/\mathrm{d}t|$ respectively to W and $|\mathrm{d}W/\mathrm{d}t|$ in the set of general thoughts { [I], [II] } with eqs.(1)-(4) to find the internal structures of the self-organized relaxed state, we obtain the followings,[29]

$$\delta F \;=\; \int (2\nu\, \delta\omega \cdot \omega \;-\; \alpha\rho\delta\mathbf{u} \cdot \mathbf{u})\mathrm{d}v \;=\; 0, \tag{56}$$

$$\delta^2 F \;=\; \int (2\nu\, \delta\omega \cdot \delta\omega \;-\; \alpha\rho\delta\mathbf{u} \cdot \delta\mathbf{u})\mathrm{d}v \;>\; 0, \tag{57}$$

where the values of quantities on the boundary surface in $\mathrm{d}W_f/\mathrm{d}t$ are assumed to be given so that the surface integral terms vanish in both δF and $\delta^2 F$ by the boundary conditions of eq.(4), for simplicity. The boundary conditions are given here as { $\delta\mathbf{u} = 0,\ \mathbf{u} \cdot \mathrm{d}\mathbf{s} = 0$; at the boundary }. Using $\delta\omega = \nabla \times \delta\mathbf{u}$, $\omega = \nabla \times \mathbf{u}$, $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b}$, and the Gauss theorem again, we obtain the followings from eqs.(56) and (57),

$$\delta F \;=\; 2\nu \int \delta\mathbf{u} \cdot (\nabla \times \nabla \times \mathbf{u} - \frac{\alpha\rho}{2\nu}\mathbf{u})\mathrm{d}v \;=\; 0\,, \tag{58}$$

$$\delta^2 F \;=\; 2\nu \int \delta\mathbf{u} \cdot (\nabla \times \nabla \times \delta\mathbf{u} - \frac{\alpha\rho}{2\nu}\delta\mathbf{u})\mathrm{d}v \;>\; 0, \tag{59}$$

where the surface integral terms vanish in both δF and $\delta^2 F$ by the same boundary conditions used at eqs.(56) and (57). We then obtain the Euler-Lagrange equation for arbitrary variations of $\delta\mathbf{u}$ from eq.(58) as follows,[29]

$$\nabla \times \nabla \times \mathbf{u} \;=\; \kappa^2 \mathbf{u}\,, \tag{60}$$

$$|\,\kappa\,| = \sqrt{\frac{\alpha\rho}{2\nu}}\,, \tag{61}$$

where the Lagrange multiplier α is assumed to be positive. Equation (60) is the same type with eq.(19) which is used for the classical spheromak.[31,32]

We now have found that the self-organized quasi-steady relaxed state has the peculiar internal structure which satisfies eq.(60). Using the vector formula of $\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times \nabla \times \mathbf{u}$, $\nabla \cdot \mathbf{u} = 0$, and eq.(60), we obtain

$$\nabla^2 \mathbf{u} = -\kappa^2 \mathbf{u} . \tag{62}$$

Using the vector formula of $\nabla u^2 = 2\mathbf{u} \times (\nabla \times \mathbf{u}) + 2(\mathbf{u} \cdot \nabla)\mathbf{u}$, and substituting eq.(62) into eq.(54), we obtain the following,

$$\rho \frac{\partial \mathbf{u}}{\partial t} = - \frac{\alpha \rho}{2} \mathbf{u} - \nabla p - \frac{\rho}{2} \nabla u^2 + \rho(\mathbf{u} \times \omega), \tag{63}$$

where eq.(61) is used. When we deal with a system where the self-organized relaxed state has still high flow fields of $\mathbf{u}$, we may assume the following for the right hand side of eq.(63),

$$\frac{\alpha \rho}{2} \mathbf{u} \gg \nabla p + \frac{\rho}{2} \nabla u^2 - \rho(\mathbf{u} \times \omega). \tag{64}$$

Using eq.(64), we obtain the following from eq.(63),

$$\frac{\partial \mathbf{u}}{\partial t} \cong -\frac{\alpha}{2} \mathbf{u}. \tag{65}$$

Equation (65) gives us the following solution

$$\mathbf{u}(\mathbf{x}, t) \cong \mathbf{u}_R(\mathbf{x}) e^{-\frac{\alpha}{2} t}, \tag{66}$$

where $\mathbf{u}_R(\mathbf{x})$ is the solution of eq.(60) for the self-organized quasi-steady relaxed state. We see from eq.(66) that the flow pattern of $\mathbf{u}$ just after the realization of the self-organized relaxed state has the self-similar decay phase without significant change the spatial structure. The right hand side of eq.(64) and boundary conditions would lead to finite deviation from the self-similar decay gradually. We may recognize from eq.(60) for the self-organized quasi-steady relaxed state and eq.(66) for the time evolution of the relaxed $\mathbf{u}$ field that the present nonlinear dynamical system relaxes to the state that has attained such a peculiar internal spatial structure that yields the minimum decay rate of W_f and thereafter leads to the self-similar decay phase without significant change of the spatial structure.

Using the same procedure from eq.(19) to eq.(24), we obtain the general solutions of $\mathbf{u}_R(\mathbf{x})$ and $\omega_R(\mathbf{x}) = \nabla \times \mathbf{u}_R$ as follows,

$$\mathbf{u}_R(\mathbf{x}) \;=\; c_{f1}\mathbf{T}_f \;+\; c_{f2}\mathbf{S}_f, \tag{67}$$

$$\omega_R(\mathbf{x}) \;=\; \kappa(c_{f1}\mathbf{S}_f \;+\; c_{f2}\mathbf{T}_f), \tag{68}$$

$$\mathbf{T}_f \;= \nabla \times (\mathrm{e}\psi_f), \;\text{ and }\; \mathbf{S}_f \;=\; \frac{1}{\kappa}\nabla \times \mathbf{T}_f\,, \tag{69}$$

$$\nabla^2\psi_f \;+\; \kappa^2\psi_f \;=\; 0\,. \tag{70}$$

Three unknown factors of $\{\kappa, c_{f1}, c_{f2}\}$ are determined by three measured values of the flow energy W_{fR}, the toroidal flow flux Φ_{fR} and the toroidal vorticity flux Ω_R inside the boundary. It is because that we obtain Φ_{fR} and Ω_R by integrating eqs.(67) and (68) respectively across the poloidal cross-section of the toroidal fluids.

Using eqs.(67)-(70), we obtain

$$\mathbf{u}_R \times \omega_R \;=\; \kappa({c_{f2}}^2 - {c_{f1}}^2)\mathbf{T}_f \times \mathbf{S}_f, \tag{71}$$

$$\kappa\Phi_{fR} \;-\; \Omega_R \;=\; \kappa(c_{f2} - c_{f1})\int_{S_p}(\mathbf{S}_f \;-\; \mathbf{T}_f)\cdot \mathrm{ds}, \tag{72}$$

where $\int_{S_p}$ denotes the integral across the poloidal cross-section of the toroidal fluids. When the right hand side of eq.(64) is assumed to be negligibly small, we obtain the equilibrium equation at the relaxed state which is written by $\nabla p + \frac{\rho}{2}\nabla u^2 \;\cong\; \rho(\mathbf{u}\times\omega)$. And further if it is assumed that ρ = const. and $p + (\rho u^2/2)$ = const. so that ∇p + $\frac{\rho}{2}\nabla u^2 = 0$, then we may put $c_{f1} \cong c_{f2}$ from the comparison of the equilibrium equation shown above with eq.(71). In this case, we may write eq.(67) as follows,

$$\mathbf{u}_R(\mathbf{x}) = c_{f1}(\mathbf{T}_f + \mathbf{S}_f), \tag{73}$$

which satisfies the following,[31] in the same way at eq.(28),

$$\nabla \times \mathbf{u}_R = \pm \kappa \mathbf{u}_R . \tag{74}$$

Since eq.(74) has the same form with the force-free field $\nabla \times \mathbf{B} = \pm\lambda\mathbf{B}$, it is easy to show that the Bessel Function model (BFM) for the reversed field pinch (RFP) plasma is also applicable to the flow profiles of $\mathbf{u}_R$ for eq.(74). Equation (74) may represent the region of the helical motion after the "turbulent puff".[34]

We next discuss the mode transition point of the relaxed state, by using eq.(59), in the same way at eqs.(48)-(53) for the resistive MHD plasma.[10,11,16,29] We consider here the following associated eigenvalue problem for the critical perturbation $\delta\mathbf{u}$ that makes $\delta^2 F$ in eq.(59) become zero:

$$\nabla \times \nabla \times \delta\mathbf{u}_i - \kappa_i^2\, \delta\mathbf{u}_i = 0, \tag{75}$$

with the boundary condition of $\delta\mathbf{u} = 0$ at the boundary, where κ_i and $\delta\mathbf{u}_i$ denote the eigenvalue and the eigensolution, respectively. It is easy to show from eqs.(60) and (74) that the eigenvalue problem of eq.(75) includes the eigenvalue problem of $\nabla \times \delta\mathbf{u}_i = \pm\kappa_i\, \delta\mathbf{u}_i$ as a branch. Substituting the eigensolution $\delta\mathbf{u}_i$ into eq.(59) and using eq.(75), we obtain the following:

$$\delta^2 F = 2\nu(\kappa_i^2 - \kappa^2) \int \delta\mathbf{u}_i \cdot \delta\mathbf{u}_i\, \mathrm{d}v > 0, \tag{76}$$

where eq.(61) is used. Since eq.(76) is required for all eigenvalues, we obtain the following condition for the relaxed state with the minimum $|\mathrm{d}W_f/\mathrm{d}t|$,[29]

$$\kappa_{-1} < \kappa < \kappa_1 , \tag{77}$$

where κ_{-1} and κ_1 are the largest of the negative and the smallest of the positive eigenvalues, respectively. When the value of κ corresponding to W_{fR} goes out of the condition eq.(77), like as $\kappa_1 < \kappa$, then the mixed mode, which has W_{fR} and consists with the basic mode by eq.(60) with $\kappa = \kappa_1$ and the lowest eigenmode by eq.(75), becomes the relaxed state with the minimun value of $|dW_f/dt|$, in the similar way to the case of the resistive MHD plasma discussed at eq.(50). Since the mathematical structures of eqs.(54) - (77) for the fluid flow velocity **u** are similar to those of equations for the magnetic field **B** used in the resistive MHD plasma (one of which is the RFP plasma), some common phenomena are expected to be observed in the RFP plasma and in the incompressible fluid, like as the saw tooth oscillation in the former and its corresponding phenomenon such as the turbulent puff[34)] in the latter.

§ 5. Application to Incompressible Viscous MHD Fluids

We next show another example of the application of the set of general thoughts { [I], [II] } with eqs.(1) - (4) to the incompressible viscous MHD fluid which is described by the following extended Navier-Stokes equation and the equation for the magnetic field,

$$\rho\frac{d\mathbf{u}}{dt} = \mathbf{j}\times\mathbf{B} - \nabla p + \nu\nabla^2\mathbf{u}, \tag{78}$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B}) - \nabla\times(\eta\mathbf{j}). \tag{79}$$

We pick up here the magnetic energy, W_m, and the flow energy, W_f, of the system and look for the relaxed state with respect to $W = W_m + W_f = \int(B^2/2\mu_o + \rho u^2/2)dv$. (If the internal energy is assumed to be negligible compared to this

W, then the relaxed state of W is equivalent to the relaxed state of energy of the system.) Using eqs.(78) and (79), Maxwell's equations, Ohm's law of $\eta\mathbf{j} = \mathbf{E} + \mathbf{u}\times\mathbf{B}$, $\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot\nabla\times\mathbf{a} - \mathbf{a}\cdot\nabla\times\mathbf{b}$, and the Gauss theorem again, we obtain the following

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\int(\,\eta\,\mathbf{j}\cdot\mathbf{j} + \nu\,\omega\cdot\omega\,)\mathrm{d}v + \oint\{\,\nu(\mathbf{u}\times\omega) - p\mathbf{u} - \mathbf{E}\times\mathbf{H}\,\}\cdot\mathrm{d}\mathbf{s}. \quad (80)$$

Substituting W and $|\mathrm{d}W/\mathrm{d}t|$ shown above respectively to W and $|\mathrm{d}W/\mathrm{d}t|$ in the set of general thoughts { [I], [II] } with eqs.(1)-(4) and using the same procedure used from eq.(9) to eq.(12) and from eq.(56) to eq.(59), we obtain the followings,

$$\delta F = \int\{\frac{2}{\mu_o}\delta\mathbf{B}\cdot[\,\nabla\times(\eta\mathbf{j}) - \frac{\alpha}{2}\mathbf{B}] + 2\nu\delta\mathbf{u}\cdot(\nabla\times\nabla\times\mathbf{u} - \frac{\alpha\rho}{2\nu}\mathbf{u})\,\}\mathrm{d}v - \frac{2}{\mu_o}\oint(\,\eta\mathbf{j}\times\delta\mathbf{B}\,)\cdot\mathrm{d}\mathbf{s} = 0\,, \quad (81)$$

$$\delta^2 F = \int\{\frac{2}{\mu_o}\delta\mathbf{B}\cdot[\,\nabla\times(\eta\delta\mathbf{j}) - \frac{\alpha}{2}\delta\mathbf{B}] + 2\nu\delta\mathbf{u}\cdot(\nabla\times\nabla\times\delta\mathbf{u} - \frac{\alpha\rho}{2\nu}\delta\mathbf{u})\,\}\mathrm{d}v - \frac{2}{\mu_o}\oint(\,\eta\delta\mathbf{j}\times\delta\mathbf{B}\,)\cdot\mathrm{d}\mathbf{s} > 0. \quad (82)$$

We then obtain the Euler-Lagrange equations for arbitrary variations of $\delta\mathbf{u}$ and $\delta\mathbf{B}$ from the volume integral terms of eq.(81), as follows,[29]

$$\nabla\times(\eta\mathbf{j}) = \frac{\alpha}{2}\mathbf{B}, \quad (83)$$

$$\nabla\times\nabla\times\mathbf{B} = \lambda^2\mathbf{B} \quad (\text{ for } \eta = \text{const}), \quad (84)$$

$$\nabla\times\nabla\times\mathbf{u} = \kappa^2\mathbf{u}\,, \quad (85)$$

$$|\,\lambda\,| = \sqrt{\frac{\alpha\mu_o}{2\eta}}\,, \quad (86)$$

$$|\,\kappa\,| = \sqrt{\frac{\alpha\rho}{2\nu}}\,, \quad (87)$$

Using the same procedure at eqs.(62) and (63), we obtain the following from eqs.(78) and (83)-(85),

$$\rho\frac{\partial \mathbf{u}}{\partial t} = -\frac{\alpha\rho}{2}\mathbf{u} + \mathbf{j}\times\mathbf{B} - \nabla p - \frac{\rho}{2}\nabla u^2 + \rho(\mathbf{u}\times\omega). \qquad (88)$$

When we deal with the system where the self-organized relaxed state has still high flow fields of $\mathbf{u}$, we may assume the following for the right hand side of eq.(88),

$$\frac{\alpha\rho}{2}\mathbf{u} \gg \nabla p + \frac{\rho}{2}\nabla u^2 - \mathbf{j}\times\mathbf{B} - \rho(\mathbf{u}\times\omega). \qquad (89)$$

If the right hand side of eq.(89) is assumed to be negligibly small at the relaxed state, we may have the following equilibrium equation at the relaxed state,

$$\nabla p + \frac{\rho}{2}\nabla u^2 = \mathbf{j}\times\mathbf{B} + \rho(\mathbf{u}\times\omega). \qquad (90)$$

Equations (88) and (89) lead us to the following,

$$\frac{\partial \mathbf{u}}{\partial t} \cong -\frac{\alpha}{2}\mathbf{u} \qquad (91)$$

Substituting eq.(83) into eq.(79), we obtain the following,

$$\frac{\partial \mathbf{B}}{\partial t} = -\frac{\alpha}{2}\mathbf{B} + \nabla\times(\mathbf{u}\times\mathbf{B}) \qquad (92)$$

When we deal with the system where the self-organized relaxed state has still high enough fields of $\mathbf{B}$ and the term of $(\mathbf{u}\times\mathbf{B})$ is nearly irrotational or the flow pattern of $\mathbf{u}$ has become almost parallel to $\mathbf{B}$, we may assume the following for the right hand side of eq.(92),

$$\frac{\alpha}{2}\mathbf{B} \gg \nabla\times(\mathbf{u}\times\mathbf{B}). \qquad (93)$$

In this case, we obtain the following from eq.(92),

$$\frac{\partial \mathbf{B}}{\partial t} \cong -\frac{\alpha}{2}\mathbf{B}. \qquad (94)$$

Equations (91) and (94) give us the following solutions,

$$\mathbf{u}(\mathbf{x},t) \cong \mathbf{u}_R(\mathbf{x})\mathrm{e}^{-\frac{\alpha}{2}t}, \tag{95}$$

$$\mathbf{B}(\mathbf{x},t) \cong \mathbf{B}_R(\mathbf{x})\mathrm{e}^{-\frac{\alpha}{2}t}, \tag{96}$$

where $\mathbf{u}_R(\mathbf{x})$ and $\mathbf{B}_R(\mathbf{x})$ are the solutions for the self-organized relaxed state. We see from eqs.(95) and (96) again that the flow pattern of $\mathbf{u}$ and the field profile of $\mathbf{B}$ just after the realization of the self-organized relaxed state have the self-similar decay phase without significant change of their spatial structures. We should bear in mind that the right hand sides of eqs.(89) and (93) and boundary conditions would lead to finite deviation from the self-similar decay gradually. We may recognize from eqs.(83)-(87) for the self-organized quasi-steady relaxed state and eqs.(95) and (96) for the time evolution of the relaxed $\mathbf{u}$ and $\mathbf{B}$ fields that the present nonlinear dynamical system relaxes to the state that has attained such a peculiar internal spatial structure, which yields the minimum dissipation rate of W and therefore is hardest to change its own spatial structure, and thereafter which leads to the self-similar decay phase without significant change of the spatial structure.

We now assume here the resistivity η to be constant, for simplicity. Using similar procedure from eq.(19) to eq.(24) and from eq.(67) to eq.(69), we obtain the general solutions of $\mathbf{u}_R(\mathbf{x})$, $\omega_R(\mathbf{x})$, $\mathbf{B}_R(\mathbf{x})$, and $\mathbf{j}_R(\mathbf{x})$, as follows,

$$\mathbf{u}_R(\mathbf{x}) = c_{f1}\mathbf{T}_f + c_{f2}\mathbf{S}_f, \tag{97}$$

$$\omega_R(\mathbf{x}) = \kappa(c_{f1}\mathbf{S}_f + c_{f2}\mathbf{T}_f), \tag{98}$$

$$\mathbf{B}_R(\mathbf{x}) = c_{m1}\mathbf{T}_m + c_{m2}\mathbf{S}_m, \tag{99}$$

$$\mathbf{j}_R(\mathbf{x}) = \frac{\lambda}{\mu_o}(c_{fm}\mathbf{S}_m + c_{m2}\mathbf{T}_m). \tag{100}$$

Six unknown factors of $\{\kappa, c_{f1}, c_{f2}\}$ and $\{\lambda, c_{m1}, c_{m2}\}$ are determined by six measured values of W_{mR}, Φ_{mR}, I_R, W_{fR}, Φ_{fR} and Ω_R inside the boundary.

From eqs.(97)-(100), we obtain the followings which are the same as eq.(25) and eq.(71),

$$\mathbf{j}_R \times \mathbf{B}_R = \frac{\lambda}{\mu_o}({c_{m2}}^2 - {c_{m1}}^2)\mathbf{T}_m \times \mathbf{S}_m, \tag{101}$$

$$\mathbf{u}_R \times \omega_R = \kappa({c_{f2}}^2 - {c_{f1}}^2)\mathbf{T}_f \times \mathbf{S}_f. \tag{102}$$

If it is assumed that $\rho \cong$ const. and $p + (\rho u^2/2) \cong$ const. so that $\nabla p + \frac{\rho}{2}\nabla u^2 \cong 0$, then we may put $c_{m1} \cong c_{m2}$ and $c_{f1} \cong c_{f2}$ from the comparison eq.(90) with eqs.(101) and (102). In this case, we may write eqs.(97) and (99) as follows,

$$\mathbf{u}_R(\mathbf{x}) = c_{f1}(\mathbf{T}_f + \mathbf{S}_f), \text{ and } \mathbf{B}_R(\mathbf{x}) = c_{m1}(\mathbf{T}_m + \mathbf{S}_m), \tag{103}$$

which satisfy the followings,[31] in the same way at eq.(28),

$$\nabla \times \mathbf{u}_R = \pm\kappa\mathbf{u}_R, \text{ and } \nabla \times \mathbf{B}_R = \pm\lambda\mathbf{B}_R. \tag{104}$$

We can also discuss the mode transition points of the relaxed state for **B** and **u**, by using eq.(82) in the same way from eq.(48) to eq.(53) and from eq.(75) to eq.(77).[16,29] Various solutions for eqs.(97)-(100) are expected to be observed as the field profiles of **B** and the flow patterns of **u** in the relaxed states of energy of the incompressible viscous MHD fluid, corresponding to the amounts of six measured values of W_{mR}, Φ_{mR}, I_R, W_{fR}, Φ_{fR} and Ω_R inside the boundary. Some of them would represent the field profiles of **B** and the flow patterns of **u** realized in the magmas as the result of the earth dynamo.[30] When we consider the flow pattern **u** of the electron fluid in the experimental RFP plasma, gross features of **B** and **u** in the relaxed state are expected to be given by eq.(104) with some necessary corrections by the compressibility. One of the simplest solutions for eq.(104) is the solution for the case with $\mathbf{B}_R = \gamma \mathbf{u}_R$ with a constant value of γ. In this case, the two equations in eq.(104) become to be equivalent, and there exists a relation of $\mu_o/\eta = \rho/\nu$ from eqs.(86) and (87).

§ 6. Concluding Remarks

The thought analysis in Sec.2 on relaxation due to nonlinear processes with dissipation leads us to the followings: The fact that the nonlinear dynamical system of interest is dissipative with respect to W means that the system is an open system with respect to W. If the internal spatial distribution of the system is unstable against keeping or sustaining the instantanious amount of containing quantity of W, drastic change of the internal spatial distribution will be induced and develop nonlinearly to release and dissipate W rapidly, through driving elements of the system. This rapid decay phase of W with the nonlinear drastic change of internal spatial structure is recognized and called as "the relaxation phase". The relaxation phase will continue itself until and terminate itself at the time when the internal spatial distribution has come to have a peculiar internal spatial structure such that yields the minimum dissipation rate of W and therefore is hardest to change its own spatial distribution for the instantanious amount of the containing W. The state with this peculiar internal spatial structure yielding the minimum dissipation rate of W for the instantanious amount of W is recognized and called as "the self-organized relaxed state". These are summarized to the set of general thoughts { [I], [II] } with the mathematical expressions of eqs.(1)-(4), in order to find out the internal structures of the self-organized relaxed states. Since the self-organized relaxed state has the peculiar internal spatial structure such that is hardest to change its own spatial distribution, the relaxed state should be followed by the self-similar decay phase without significant change of the spatial structure. We should bear in mind, however, that the dissipation and being open of the system with respect to W will still lead to some gradual deviation from the self-similar decay. All of thoughts shown above would be applicable to all dynamical systems including physical systems, biological systems, and/or economical systems in general. The realization of the internal spatial structure of the self-organized relaxed state comes essentially from the fact that the dissipative nonlinear dynamical system of interest is the open system with respect to the global quantity W subject to the dissipation. The realization of the self-organization is a global property that is embedded in the laws ruling the dissipative, open, and nonlinear dynamical system of interest. This thought is connected to the well known thought of "the structure due to the dissipation" by Prigogine[35,36]

In Sec.3, we have applied the set of general thoughts, { [I], [II] } with eqs.(1)-(4) to the magnetic energy relaxation of the resistive MHD plasma and have led the self-organized relaxed state of eq.(13) and equivalent eq.(16), and also the mode transition condition of eq.(50) in the more general case and eq.(53) in the low β plasma limit with the spatially uniform resistivity η at the relaxed state, without using "helicity" and "invariant". We have proved that the self-organized relaxed states with the internal spatial structure of eq.(13) for the more general case are followed by the self-similar decay phase without significant change of its own spatial distribution, as was shown at eq.(18). For the simple case with the spatially uniform resistivity η at the relaxed state, we have shown the general solutions for the field $\mathbf{B}_R$ and the current $\mathbf{j}_R$, eqs.(23) and (24), and the equilibrium equation, eq.(25), at the self-organized relaxed state. We have also shown that the force-free fields of $\nabla \times \mathbf{B} = \pm\ \lambda\mathbf{B}$, derived by Taylor based on the minimum energy state under the time invariant of the total helicity,[10,11] can be derived generally as the low β plasma limit of the self-organized relaxed state from the present theory.

We have shown some typical examples of the axisymmetric plasmas in the self-organized relaxed state, such as the diffused Z pinch plasma, the screw pinch plasma, the RFP plasma, and the FRC plasma as were shown at eqs.(29)-(47), all of which are followed by the self-similar decay phase shown by eq.(18). These typical examples by the present theory would be supported by the exprimental fact that all of these plasmas are observed to have the quasi-steady relaxed state phase, i.e. the self-similar decay phase without significant change of their own spatial distribution.

The present theory provides various schemes of current driving or current sustaining by supplying the magnetic energy to the self-organized relaxed state plasma directly or indirectly, in order to recover the resistive decay of the magnetic energy; for example, by the power input through the Poynting vector $\mathbf{E} \times \mathbf{H}$ on the boundary surface, discussed at eqs.(9) and (10), or by injection of some compact magnetized plasmas such as the compact FRC plasma, the compact spheromak plasma, or the magnetized gun plasmas. Several experimental investigations of current drive by the so-called helicity injection have been reported, some of which failed in obtaining net current drive and others were successful. The present theory gives us quite usual and familiar physical explanations for these experimental results as follows; no net magnetic energy was injected to the relaxed state plasmas in the former, and net mangnetic energy was successfuly injected in the latter. The present theory takes us back to rather familiar physical picture of "the transportation of energy", compared with "the transportation of magnetic helicity". At any rate, we cannot deal with any new quantities agaist the law of the energy equation.

In Sec.4, we have applied the set of general thoughts, { [I], [II] } with eqs.(1)-(4) to the flow energy relaxation of the incompressible viscous fluid and have led the self-organized relaxed state flow of eq.(60), and also the mode transition condition of eq.(77), which have the same form with the case of the resistive MHD plasma with the spatially uniform resistivity in Sec.3 We have also proved that the self-organized relaxed state flow with the internal spatial structure of eq.(60) are followed by the self-similar decay phase without significant change of its own spatial distribution, as was shown at eq.(66). The solution of eq.(74) for the case of the axisymmetric cylindrical flow, which is given by the Bessel function model [the same type of solution with eq.(39)], would represent the self-organized relaxed state flow after the turbulent phase of flow in the incompressible viscous fluid within such a rotating cylindrical wall. The common mathematical structures of eqs.(55) - (77) for the fluid flow velocity $\mathbf{u}$ and the equations for the magnetic field $\mathbf{B}$ used in the resistive MHD plasma suggest that some common phenomena are expected to be observed in the RFP plasma and in the incompressible fluid, like as the saw tooth oscillation in the former and its corresponding phenomenon such as the turbulent puff[34)] in the latter.

In Sec.5, we have applied the set of general thoughts, { [I], [II] } with eqs.(1)-(4) to the flow- and the magnetic energy relaxation of the incompressible viscous MHD fluid and have led the self-organized relaxed state flow and field, eqs.(83) and (85). The mode transition condition of the self-organized relaxed state of eqs.(83) and (85) can be also derived in the same way as for the resistive MHD plasma in Sec.3 and the incompressible viscous fluid in Sec.4. We have also proved that the self-organized relaxed state, which has the internal spatial structures of eqs.(83) and (85) and satisfies approximately the equilibrium equation, eq.(90), are followed by the self-similar decay phase without significant change of its own spatial distribution, as was shown at eqs.(95) and (96). The mathematical structures of the self-organized relaxed state for both the flow pattern of $\mathbf{u}$ and the field profile of $\mathbf{B}$ are shown to be common. The self-organized relaxed state described by eqs.(83) and (85) [or more simply by eq.(104)] in the incompressible MHD fluid would represent the field profiles of **B** and the flow patterns of **u** realized such as in the magmas as the result of the earth dynamo,[30] in and around the magnetized neutron stars, and others in the universe.

Every nonlinear and dissipative dynamical system may have "the self-organized quasi-steady relaxed states", if the laws ruling the elements of the system yields the equations which determine peculiar internal structures by applying the set of general thoughts, { [I], [II] } with eqs.(1)-(4), like as obtained in the present paper.

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References

1. H. A. B. Bodin and A. A. Newton: Nucl. Fusion **20** (1980) 1255.

2. T. Tamano, W. D. Bard, C. Chu, Y. Kondoh, R. J. LaHaye, P. S. Lee, M. Saito, M. J. Schaffer, P. L. Taylor: Phys. Rev. Lett. **59** (1987) 1444.

3. V. Antoni and S. Ortolani: Phys. Fluids **30** (1987) 1489.

4. S. Masamune, H. Oshiyama, A. Nagata, H. Arimoto, S. Yamada and K. Sato: J. Phys. Soc. Jpn. **56** (1987) 1278.

5. T. Shimada, Y. Yagi, Y. Hirano, K. Hattori, I. Hirota, Y. Kondoh, Y. Maejima, K. Saito, S. Shiina and K. Ogawa: in Proc. 12th Int. Conf. on Plasma Physics and Controlled Fuclear Fusion Research, Nice, France, Oct. 1988 (IAEA, Vienna), CN-50/C-2-4-2.

6. Y. Hirano, Y. Yagi, T. Shimada, K. Hattori, Y. Maejima, I. Hirota, Y. Kondoh, K. Saito and S. Shiina: in Proc. 13th Int. Conf. on Plasma Physics and Controlled Fuclear Fusion Research, Washington, D.C., U.S.A., Oct. 1990 (IAEA, Vienna), CN-53/C-4-12.

7. M. Yamada, H. P. Furth, W. Hsu, A. Janos, S. Jardin, M. Okabayashi, J. Sinnis, T. H. Stix and K. Yamazaki: Phys. Rev. Lett. **46** (1981) 188.

8. T. R. Jaboe, I. Henins, H. W. Hoida, R. K. Linford, J. Marshall, D. A. Platts and A. R. Sherwood: Phys. Rev. Lett. **45** (1980) 1264.

9. K. Watanabe, K. Ikegami, A. Ozaki, N. Satomi and T. Uyama: J. Phys. Soc. Jpn. **50** (1981) 1823.

10. J. B. Taylor: Phys. Rev. Lett. **33** (1974) 1139.

11. J. B. Taylor: Rev. Mod. Phys. **58** (1986) 741.

12. Y. Kondoh: Nucl. Fusion **21** (1981) 1607.

13. Y. Kondoh: Nucl. Fusion **22** (1982) 1372.

14. Y. Kondoh, T. Amano, A. Nagata, K. Ogawa, Y. Maejima, T. Shimada, Y. Hirano and S. Goto: J. Phys. Soc. Jpn. **53** (1984) 3427.

15. Y. Kondoh: J. Phys. Soc. Jpn. **54** (1985) 1813.

16. Y. Kondoh: J. Phys. Soc. Jpn. **58** (1989) 489.

17. Y. Kondoh, N. Takeuchi, S. Yamaguchi, Y. Yagi, T. Shimada and K. Ogawa: J. Phys. Soc. Jpn. **58** (1989) 887.

18. Y. Kondoh, N. Takeuchi, A. Matsuoka, Y. Yagi, Y. Hirano and T. Shimada: J. Phys. Soc. Jpn. **60** (1991) 2201.

19. A. Bhattacharjee and R. L. Dewar: Phys. Fluids **25** (1982) 887.

20. D. Montgomery and L. Phillips: Phys. Rev. bfA **38** (1988) 2953.

21. T. Kato and T. Furusawa: Proc. 2nd Int. Conf. on Plasma Physics and Controlled Nuclear Fusion, November 27-30, 1990, National Institute for Fusion Science, Toki, Japan, VIII-2.

22. K. Sugisaki: Jpn. J. Appl. **24** (1985) 328.

23. K. Sugisaki: J. Phys. Soc. Jpn. **56** (1987) 3176.

24. K. Sugisaki: J. Phys. Soc. Jpn. **57** (1988) 4175.

25. R. Horiuchi and T. Sato: Phys. Rev. Lett. **55** (1985) 211.

26. Y. Kondoh: "Thought Analysis on Relaxation and General Principle to Find Relaxed State", Research Rep., National Institute for Fusion Science, Nagoya, Japan, 1991, NIFS-109.

27. Y. Kondoh: "A Physical Thought Analysis for Maxwell's Electromagnetic Fundamental Equations", Rep. Electromagnetic Theory meeting of IEE Japan, 1972, EMT-72-18 (in Japanese).

28. Y. Kondoh: J. Phys. Soc. Jpn. **60** (1991) 2851.

29. Y. Kondoh: "Relaxed State of Energy in Incompressible Fluid and Incompressible MHD Fluid", Research Rep., National Institute for Fusion Science, Nagoya, Japan, 1991, NIFS-123.

30. A. Yoshizawa: Phys. Fluids **B 2** (1990) 1589.

31. S. Chandrasekhar and P. C. Kendall: Astrophys. J. **126** (1957) 457.

32. M. N. Rosenbluth and M. N. Bussac: Nucl. Fusion **19** (1979) 489.

33. A. Shiokawa, S. Okada, Y. Ito, and S. Goto: Jpn. J. Appl. Phys. **30** (1991) L1142.

34. K. Kose: J. Phys. D **23** (1990) 981.

35. I. Prigogine: Etude Thermodynamique des Phenomenes Irreversibles (Dunod, Paris, 1974).

36. P. Glansdorff and I. Prigogine: Thermodynamic Theory of Structure, Stability, and Fluctuations (Wiley-Interscience, New York, 1971).

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