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Theory of pressure-induced islands and self-healing in three-dimensional toroidal magnetohydrodynamic equilibria

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ABSTRACT

The role of singular currents in three-dimensonal toroidal equilibria and their resolution by magnetic island formation is discussed from both analytical and computational points of view. Earlier analytical results are extended to include small vacuum islands which may, in general, have different phases with respect to pressure-induced islands. In currentless stellarators, the formation of islands is shown to depend on the resistive parameter D_R as well as the integrated effect of global Pfirsch-Schlüter currents. It is demonstrated that the pressure-induced "self-healing" effect, recently discovered computationally, is also predicted by analytical theory.

keywords; helical system, Helias, 3-D MHD equilibrium, magnetic surface breaking, finite pressure effect, equilibrium beta limit, self-healing, magnetic island, Wendelstein 7-X

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I. INTRODUCTION

The computation of three-dimensional magnetohydrodynamic (MHD) toroidal equilibria for finite-pressure plasmas is one of the principal theoretical problems of stellarator research. The main purpose of this paper is to give a theory of pressure-induced islands in a stellarator extending earlier analytical work[1,2] on this subject, and to discuss the predictions of this theory in the light of recent computational results from the three-dimensional (3D) magnetohydrodynamic (MHD) equilibrium code HINT.[3] In particular, we give an analytical demonstration of the phenomenon of "self-healing," recently discovered computationally,[4,5] in which vacuum magnetic islands in a stellarator are reduced and eventually eliminated by the effect of plasma pressure-induced currents.

The following is a plan of this paper. In Section II, we discuss the role of current singularities ("current sheets") in three-dimensional toroidal equilibria. Even though the importance of current sheets is recognized in analytical theories,[1,2] they have not received quite the attention they deserve in computational studies of stellarators. These current sheets are resolved by allowing islands to open up at rational surfaces. Computational studies[4,5] using the HINT code have drawn attention to the importance of phase relations between the islands due to the vacuum field and the plasma pressure-induced perturbations. In Section III, we derive an expression for the island-width at a given rational surface, extending earlier work[2] to include vacuum islands. An important extension of this theory is the derivation of an expression for the phase of the island which predicts qualitatively the "phase-flip" seen in computational studies of pressure-induced self-healing. In Section III, we compare the predictions of the theory with numerical results from the HINT code. We conclude in Section IV with a summary.

II. ROLE OF CURRENT SHEETS

Unlike tokamaks, stellarators are intrinsically 3D devices, and in the absence of a direction of continuous symmetry, MHD equilibria with nested flux surfaces are not guaranteed to exist. Grad[6] argued that, strictly speaking, 3D toroidal MHD equilibria do not exist. However, if one extends the class of admissible solutions to include current sheets, then the question of the existence of equilibrium solutions with nested surfaces can be resolved, at least in a mathematical sense.

In a 3D toroidal device like the stellarator with nested flux surfaces, it is well-known that current sheets tend to occur at the rational surfaces on which field-lines close on themselves. The current sheet is forced into the equilibrium solution because the topological constraint of nested flux surfaces does not allow the solution enough room to escape from the singularity.

It is useful to bear in mind that there are related examples of current sheets in tokamaks as well. In fact, the first analytical demonstration of a δ–fuunction current singularity in a toroidal plasma equilibrium is due to Rosenbluth et al.[7] who showed that the ideal m=1, n=1 internal kink mode evolves from an initially axsymmetric equilibrium to a neighbouring helical equilibium with a curent sheet. Furthermore, if an external helical perturbation is imposed rapidly on the boundary of an axisymmetric plasma, a current sheet tends to occur at the rational surface resonant with the perturbation.[8-10] These examples from tokamak MHD help us, in a way, to see why current singularities are unavoidable in stellarators with nested flux surfaces. Since a 3D stellarator equilibrium can be viewed qualitatively as a 2-D equilibrium with an intrinsic symmetry-breaking perturbation, it is natural that a stellarator equilibrium will tend to develop current sheets for the same reason that a tokamak with an externally applied perturbation does (if the flux surfaces are forced to remain nested).

Some recent analytical theories[1,2] of pressure-induced islands in stellarators have emphasized the crucial role of current sheets. It is possible to demonstrate the existence of current singularities by representing the magnetic field **B** in the so-called "straight field-line" form,

$$\mathbf{B} = \nabla \Phi \times \nabla(\theta - \iota\phi) \quad , \tag{1}$$

where Φ is the toroidal flux function that labels flux surfaces, θ and ϕ are the poloidal and toroidal angles, respectively, parameterizing a given flux surface, and $\iota = \iota(\Phi)$ is the rotational transform. The Jacobian $\mathsf{J} = (\nabla\Phi . \nabla\theta \times \nabla\phi)^{-1}$ can be expanded in a Fourier series,

$$\mathsf{J} = \sum_{m,n} \mathsf{J}_{mn}(\Phi)\exp(im\theta - in\phi) \quad . \tag{2}$$

The magnetostatic equilibrium condition can be written as,

$$\mathbf{J} = Q\mathbf{B} + \mathbf{B}\times\nabla p / B^2 \equiv \mathbf{J}_{\parallel} + \mathbf{J}_{\perp} \quad , \tag{3}$$

where $\mathbf{J}$ is the current density and $p = p(\Phi)$ is the plasma pressure. If we represent the parallel current term $Q = \mathbf{J}.\mathbf{B} / B^2$ in a Fourier series, i.e.,

$$Q = \sum_{m,n} Q_{mn}\exp(im\theta - in\phi) \quad , \tag{4}$$

it follows from $\nabla.\mathbf{J} = \mathbf{0}$ that

$$Q_{mn} = -p'\frac{\mathsf{J}_{mn}}{\iota - n/m} + \hat{Q}_{mn}\delta(\Phi - \Phi_{mn}) \quad , \tag{5}$$

where $(m,n) \neq (0,0)$. Equation (5) incorporates the condition that the toroidal current within a flux surface is zero. The first and second terms on the right of (5) are the inhomogeneous and homogeneous parts of the solution for the magnetic differential equation $\mathbf{B}.\nabla Q = -\nabla.\mathbf{J}_{\perp}$, respectively.

It is clear from Eq. (5) that a δ–function current singularity is a generic feature of the general solution to the three-dimensional MHD equilibrium problem, although the procedure given above leaves the amplitude $\hat{Q}$ undetermined. The current sheets can be resolved by allowing islands to open up at rational surfaces. The determination of $\hat{Q}$, was the main goal of the boundary-layer theory of Cary and Kotschenreuther[1] (hereafter, CK) and Hegna and Bhattacharjee[2] (hereafter, HB). Earlier, Solov'ev and Shafranov[11] and Boozer[12] had considered the physical consequences of the first term on the right side of Eq. (5). In a self-consistent plasma equilibrium, the first term represents a part of the Pfirsch-Schlüter contribution to the parallel current. Reiman and Boozer[13] gave detailed asymptotic estimates of the island-width in a stellarator caused by this term, but they did not consider the effect of the current sheet. As shown by CK and HB, and discussed in Section III of this paper, this term plays a significant role in determining the island-size in a stellarator equilibrium.

We comment here on the implications of current sheets for numerical solutions of three-dimensional MHD codes[14-17] which assume the existence of nested flux surfaces. Though it is evident that current sheets are a part of the equilibrium solution in these codes, there has not been, as yet, much numerical effort to find them. (But see Ref. 17 for a useful discussion of the characteristics of the equilibrium equations and a possible numerical strategy to find current sheets in the context of the variational spectral methods.[15-17]) On the other hand, the part of the parallel current represented by the first term on the right of (5) has been numerically computed[18,19] by making explicit use of the analytic formula. While this procedure has drawn attention to the importance of near-singular currents in numerical equilibrium and stability studies for stellarators, it neglects *a priori* the current sheet contribution to $\mathbf{J}_{\parallel}$ (arising from the homogeneous part of the solution).

We remark that since three-dimensional codes such as BETA[14] and VMEC[15] use variants of the relaxation method which seeks minima of an energy functional, the global features of a numerically relaxed solution may not appear to be sensitive to the presence of current sheets which are extremely localized and thus need a fine spatial grid. Nonetheless, the presence of these current sheets in 3D codes may be reflected in the local force-balance condition which is generally satisfied less accurately in the vicinity of a rational surface than it is in the exterior region. This is due not only to the difficulty in obtaining adequate spatial resolution in the vicinity of the rational surfaces, but also because current sheets tend to form nonlinearly at a rather slow rate, with algebraic time-dependence.[9,10] Hence, it can take a long relaxation time (in 3D MHD equilibrium codes with artificial viscosity) to satisfy the magnetostatic equilibrium condition in the vicinity of rational surfaces.

III. ISLAND WIDTHS IN 3D EQUILIBRIA

The current sheets, discussed in Section II, can be resolved by allowing magnetic islands to open up at rational surfaces. In this Section, we generalize the results of HB to allow for vacuum islands which may not, in general, have the same phase as pressure-induced islands. To resolve the current singularities near the rational surfaces, it is neccessary to solve the magnetostatic equilibrium condition (3) in the vicinity of rational surfaces by relaxing the requirement that the magnetic field lines lie on surfaces of constant Φ. The analytical calculation of CK and HB involves asymptotic matching of two regions : the exterior region in which the parallel current is described by (5), and the interior region where the singular solution is resolved and one essentially carries out an asymptotic analysis very similar to earlier work on resistive instabilities of axisymmetric plasmas.[20] An important technical step involved in the calculation is the derivation of a nonlinear island equation in the vicinity of a resonant surface by solving Ampere's law $\nabla \times (\nabla \times \mathbf{A}) = \mathbf{J}$, where A is the perturbed vector potential. In order to solve this equation in the exterior region, CK and HB assume that the flux surfaces are nearly circular and the plasma has low β. This simplifies the metric coefficients involved in the analytical inversion of the operator $\nabla \times (\nabla \times$ and decouples the different helicities. In the interior region near a rational surface, where islands open up to resolve the singularity in the current, it is assumed that the "single-harmonic approximation" holds, i.e., the contribution of the resonant part of the vector potential dominates all other contributions. This is a reasonable approximation if island widths are smaller than the characteristic distance between rational surfaces. In this approximation, it is also possible to neglect the spatial variation of the equilibrium quantities and the perturbed fields across the width of the island, analogus to the "constant Φ" approximation of resistive MHD theory.

We consider an island at the rational surface $\iota = n_r / m_r$, and transform to angle coordinates, $\alpha = \theta - (n_r / m_r)\phi$ and $\zeta = \phi$. In the new coordinates, the perturbed vector potential can be written as

$$\mathbf{A} = \sum_{m,n} \left[A_{\zeta mn}(\Phi)\nabla\zeta + A_{\alpha mn}(\Phi)\nabla\alpha \right] \exp[im\alpha + i\{m(n_r / m_r) - n\}\zeta]] \quad , \quad (6)$$

where we have chosen the gauge condition $A_\Phi = 0$. Since at a given rational surface, the resonant component of A_ζ gives the largest contribution to the island, it is useful to define an average over the angle ζ that selects only the resonant term.[1] Representing the averaged resonant component by A, the helical flux function in the vicinity of a given rational surface, $\Phi = \Phi_r$, can be written as[1,2]

$$\Psi = (\iota' / 2)(\Phi - \Phi_r)^2 - A \equiv (\iota' / 2)x^2 - A \ , \quad (7)$$

where ι' is the derivative of ι with respect to Φ at the rational surface.

In the exterior region, assuming near-circular flux surfaces, Ampere's law can be written as

$$\left(\frac{\partial}{\partial r} \frac{r / R_0}{1 + (mr / R_0)^2} \frac{\partial}{\partial r} + \frac{1}{rR_0} \frac{\partial^2}{\partial \theta^2} \right) A = \frac{dp}{dr} \frac{J_{mn}}{\iota - n / m} - \hat{Q}\delta(r - r_{mn}) \ , \quad (8)$$

where we have dropped the subscript r on the integers m and n for notational convenience. Equation (8) can be solved using the Greene's function $G(r,r')$. In the presence of a resonant vacuum perturbation, the real part of the general solution can be written as

$$|A|\cos(m\theta - n\zeta + \phi_0) = A_v \cos(m\theta - n\zeta) +$$

$$\int dr' G(r,r') \left[\frac{dp}{dr} \frac{|J_{mn}|\cos(m\theta - n\xi + \phi_1)}{\iota - n / m} - |\hat{Q}|\cos(m\theta - n\zeta + \phi_0)\delta(r - r_{mn}) \right] \ , \quad (9)$$

assuming that a single harmonic is dominant. Without loss of generality, we take the phase of the vacuum perturbation to be zero. The angle ϕ_0 is the phase of the current sheet as well as the island that resolves it, whereas ϕ_1 is the phase of the island caused by the Pfirsch-Schlüter currents. Asymptotic evaluation of the Greene's function integral[1,2] yields

$$|A|\cos(m\theta - n\xi + \phi_0) \cong A_v \cos(m\theta - n\xi) - \frac{R_0}{2m^2} \frac{dp}{dr} \left(\frac{d\iota}{dr} \right)^{-1} |J_{mn}|\cos(m\theta - n\xi + \phi_1)$$
$$+ \frac{R_0}{2m} |\hat{Q}|\cos(m\theta - n\xi + \phi_0) \quad . \quad (10)$$

The island half-width, for a given A, can be written $w = 2\sqrt{|A/\iota'|}$ which, expressed as an extent in the rotational transform, is $\delta\iota = |\iota'|w$. We have

$$\delta\iota = 2\sqrt{|\iota' A|} \quad . \tag{11}$$

Multiplying Eq. (10) by ι', we get

$$\delta\iota^2 \cos(m\theta - n\zeta + \phi_0) = \delta\iota_v^{\,2} \cos(m\theta - n\xi) + C\cos(m\theta - n\xi + \phi_1)$$
$$+ D\cos(m\theta - n\xi + \phi_0) \quad , \tag{12}$$

where $\delta\iota_v = 2\sqrt{|\iota' A_v|}$ is the half-width of the vacuum island,

$$C \equiv \left(-\frac{2}{B_0^{\,2}} \frac{dp}{dr} \right) \frac{R_0^{\,2}}{r} \frac{1}{m^2} \left| \frac{J_{mn}}{J_{00}} \right| \quad , \tag{13a}$$

and

$$D \equiv \frac{2}{m} \frac{R_0}{B_0 r} \frac{d\iota}{dr} |\hat{Q}| \quad . \tag{13b}$$

As shown by CK and HB, the amplitude of the current sheet, $|\hat{Q}|$, is determined by the physics of the interior region. The solution of the interior region equations, using the Ohm's law of resistive MHD, has been carried out by HB under the approximations discussed at the beginning of this section. (We do not repeat the calculation here, but refer the reader to Ref. 2.) Following the analysis given in HB, we obtain

$$D = G\delta\iota \equiv \frac{D_R}{m} \left| r \frac{d\iota}{dr} \right| \delta\iota \quad , \tag{14}$$

where D_R is the well-known resistive stability parameter due to Glasser et al.[21] In Eq. (14), the pressure profile is determined by assuming that it is constant within the island separatrix, and that the particle flux due to finite resistivity is constant outside the separatrix. Away from the island, the pressure gradient is matched to its value in the exterior region. Equating terms proportional to $\sin(m\theta - n\zeta)$ and $\cos(m\theta - n\zeta)$ in Eq.(12), we get,

$$\left(\delta\iota^2 - G\delta\iota \right) \cos\phi_0 = \delta\iota_v^{\,2} + C\cos\phi_1 \quad , \tag{15}$$

and

$$\left(\delta\iota^2 - G\delta\iota \right) \sin\phi_0 = C\sin\phi_1 \quad , \tag{16}$$

respectively. If we set $\delta\iota_v = 0$, we obtain $\phi_0 = \phi_1$ and the result,

$$\delta\iota = G/2 + \sqrt{(G/2)^2 + |C|} \qquad , \qquad (17)$$

given earlier by HB. More generally, solving (15) and (16) for $\delta\iota$ and ϕ_0, we get

$$\delta\iota = G/2 + \sqrt{(G/2)^2 + (\delta\iota_v{}^4 + C^2 + 2\delta\iota_v{}^2 C\cos\phi_1)^{1/2}} \qquad , \qquad (18)$$

and

$$\tan\phi_0 = \frac{C\sin\phi_1}{\delta\iota_v{}^2 + C\cos\phi_1} \qquad . \qquad (19)$$

In Eqs. (18) and (19) , the quantity C represents the globally integrated effect of the first term in (5). Far from an island, this term describes the effect of slowly varying exterior currents that produce resonant fields and islands. In the terminology of Hayashi et al.[22], this is a "global" effect, and though induced self-consistently by plasma pressure, it is similar in ways to the effect of an external coil on an axisymmetric equilibrium. Reiman and Boozer's[13] treatment of islands deals mainly with this so-called "global" effect. The principal result of the analysis of CK and HB is that this "global" effect accounts for only a part of the expression of the island width, and that another crucial part of the expression comes from the second term in (5) which is much more singular than the first term. In fact, it is the current sheet in the exterior region solution that brings in the term G (proportional to the parameter $D_R = E + F + H^2$ defined by Glasser et al.[21]) in the expression (6) for the island width. [Strictly speaking, HB use a β-ordering in which $E + F >> H^2$, but we have included H^2 in the formula (6) for greater generality.] In the terminology of Ref. 22, the term G represents a "local" effect. The principal difference between CK and HB lies in the description of the term G. Whereas CK identify G with the so-called "well-hill" property, HB identify G with the parameter D_R, as discussed above.

Equations (18) and (19) permit a number of interesting possibilities, of which Eq. (17), explored earlier by HB, is one. Equation (17) describes the case in which one begins with nearly perfect vacuum surfaces, and the magnitude of G dominates C to the extent that G controls the quality of the magnetic surfaces almost entirely in a finite-beta equilibrium. Since this case has been discussed extensively by CK and HB, we move on here to consider other possibilities that are of considerable practical interest.

A second possibility is when the resonant vacuum perturbation has the same phase as the perturbation caused by the Pfirsch-Schlüter currents. In our phase convention, this corresponds to $\phi_1 = 0$, whence Eq.(18) gives

$$\delta\iota = G/2 + \sqrt{(G/2)^2 + |C| + \delta\iota_v^{\,2}} \quad . \tag{20}$$

We consider, for simplicity, a case in which there are small vacuum islands but the device has been so optimized in the presence of plasma pressure that G (i.e., D_R) is negative and $C << |G|$. Then the analytical formula (20) predicts that the vacuum islands tend to heal in the sense that the magnetic island width decreases as the plasma beta increases (and D_R becomes more negative).[23] On the other hand, if D_R is positive, then Eq. (20) predicts increasing island-size with increasing plasma pressure.

A third possibility is the case $\phi_1 = \pi$ when the vacuum perturbation and the perturbation caused by the Pfirsch-Schlüter currents are exactly out of phase. Then Eq. (18) becomes

$$\delta\iota = (G/2) + \sqrt{(G/2)^2 + \left|\delta\iota_v^{\,2} - C\right|} \quad . \tag{21}$$

We first consider an equilibrium with $G \le 0$ (i.e., $D_R \le 0$). For $\beta = 0$, we have $\delta\iota = \delta\iota_v$ and $\phi_0 = 0$. As β increases, the term C causes a reduction in the island-size. A critical value of beta (denoted by β_c) is determined by the condition $\delta\iota_v^{\,2} = C$. At $\beta = \beta_c$, the two terms inside the modulus sign in Eq. (21) cancel each other exactly and the island-width obtains its smallest value, i.e., $\delta\iota = 0$. As β increases beyond β_c, the term inside the modulus sign in (21) increases with β. Furthermore, Eq. (19) predicts that there is a "phase-flip" of the island from $\phi_0 = 0$ for $\beta < \beta_c$ to $\phi_0 = \pi$ for $\beta > \beta_c$. This phase-flip was discovered by Hayashi in a computational study [4] that provided the stimulus for the present paper.

On the other hand, if $G > 0$ (i.e., $D_R > 0$), Eq. (21) predicts that the island width cannot be reduced to zero for any value of the plasma β. If $G >> 0$ and $G >> C$, the island-width increases monotonically with increasing β. If $G>0$ and $G > C$, the island-width typically exhibits a non-monotonic dependence on β, in that the width at first decreases to a non-zero, minimum value, and then increases with increasing values of β. In contrast to the case $G \leq 0$ (i.e., $D_R \leq 0$), it is impossible to achieve complete self-healing if $G > 0$ (i.e., $D_R > 0$).

The analytical results discussed above are depicted schematically in Fig. 1, where we plot the island width $\delta\iota$ as a function of the plasma β using Eq. (21). We take $\delta\iota_v = 0.05$, $C = 0.05\beta$ and let G take three values equal to -0.5β, 0 and 0.5β , respectively. (The value of G is taken to be somewhat larger than neccessary for the sake of clarity in the graphs, but the same qualitative behavior is obtained for lower values of G.) Self-healing occurs in the first two cases, but not in the third case. The island goes through a phase-flip in all cases. We note that at any given value of β, the island-width is the least in the case $G < 0$.

IV. COMPARISON WITH NUMERICAL RESULTS

In Fig. 2, we show a case-study of the Helias device[18] from the HINT code. In Fig. 2(a), we show the Poincare section of the vacuum field, where a chain of magnetic islands exists on the $\iota = 5/6$ rational surface in the region of closed magnetic surfaces. In Fig. 2(b), we show the flux contours in a finite-beta equilibrium with a peak value of $\beta(0) = 9\%$ on the magnetic axis. (This corresponds to a volume-averaged beta of approximately 3%.) It is seen that the 5/6 island chain in the vacuum field heals almost completely. When the plasma beta is increased further to $\beta(0) = 12\%$ (corresponding to a volume-averaged beta of approximately 4%), the 5/6 island reappears again in Fig. 2(c), though now in a somewhat different radial location, and with a different phase than the islands in Fig. 2(a). Inspection of Figs. 2(a) and 2(c) indicates a phase-flip, with X-points of the vacuum islands in 2(a) replaced by O-points in 2(c).

To test the analytical results discussed in Section III, it is neccessary at first to calculate D_R for the equilibria described by Fig. 2. The parameter D_R is defined for equilibria with nested surfaces. The appropriate place to calculate the relevant D_R is in the exterior region, outside the island. (We note that in the analytical theory as well as the HINT code, the pressure profile is flat inside the island region.)

The calculation of D_R is done by mapping the equilibria computed by HINT to the VMEC code. In Fig. 3, we show the result of this calculation for a sequence of equilibria with increasing values of β. Figs. 3(a) and 3(b) show D_R for equilibria prior to the healed state, and the healed state (Fig. 2(b)) itself, respectively. We note that in the vicinity of the 5/6 surface, D_R is negative. Thus, the self-healing effect is qualitatively consistent with the prediction of formula (21). Fig. 3(c) shows D_R for the equilibrium in Fig. 2(c). It is found that D_R still has a negative sign, but its magnitude is much smaller than it is in Figs. 2(a) and 2(b). As the system approaches the marginal point with respect to D_R, the so-called "global" effects embodied by the term C become increasingly important.

The analytical formula (21) predicts complete self-healing at a critical value of beta. Beyond this critical value, the islands reappear but with a flipped phase. This is accord with the behavior seen in Figs. 2(b) and 2(c).

Though the comparative study given above is helpful in establishing a connection between analytical theory and numerical computation, there are some caveats. The computation of D_R for low-shear systems is a difficult numerical problem. This problem is complicated the presence of singular currents in equilibria computed with codes such as BETA or VMEC (which assume that the magnetic field-lines lie on nested flux surfaces). Careful convergence studies are neccessary to calculate D_R accurately under these conditions. Figure 3 is the result of such a study, but we cannot be sure that the HINT code can resolve the "local" spatial scales accurately. It should be noted that even if we set D_R=0, Eq. (21) predicts complete healing (see Fig. 1). Thus, it is possible that HINT finds complete healing with no sensitivity to D_R (or the current sheets that bring D_R to play in Eq. (18)). This calls for some caution in the interpretation of the results on self-healing from HINT if D_R is large and positive (which is *not* the case in the example discussed in Ref. 5). The analytical theory shows that complete self-healing would not be possible under these conditions, but computational results that do not resolve the near-singular currents (and neglect the effect of D_R) may arrive at more optimistic predictions.

Another difficult theoretical issue is to find a rigorous basis for mapping equilibria from codes such as HINT which allow for islands and stochastic regions to codes such as VMEC which are neccessary to calculate the Mercier parameter D_I or the resistive parameter D_R. This issue has deeper underpinnings than we have addressed here because it has to do with the fundamental problem of defining near-invariant KAM torii in cases of weak departures from integrability. Fortunately, this is an area of ongoing research in which some progress has been made recently. [24]

V. SUMMARY

In this paper, we have extended earlier theoretical results[1,2] on 3D MHD stellarator equilibria to allow for vacuum islands and different phase relations that might exist between vacuum and pressure-induced islands. We have compared the predictions of theory with recent computational results[4,5] on pressure-induced islands in currentless stellarator equilibria. We have established that the presence of current sheets at rational surfaces (which are resolved by the presence of islands) as well as the integrated global effect of Pfirsch-Schlüter currents, play an important role in determining the island widths in three-dimensional equilibria. In particular, the phenomenon of self-healing, discovered computationally, is also shown to be predicted by analytical theory. These results are likely to be of relevance to the interpretation of experimental results, as work by Nakamura et al. indicates.[25] It should be borne in mind, however, that long mean-free-path effects,[26] outside the scope of the analytical and computational methods used in this paper, may change qualitatively some of the conclusions arrived at here.

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23. Such a possibility was, in fact, suggested by CK (see Sec. VI of Ref. 1), but in the context of the "well-hill" criterion which is valid for beta-values much smaller than are of interest for present-day stellarators.

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FIGURE CAPTIONS

Figure 1 : Plot of island width versus β given by Eq. (21). The solid curve corresponds to $G=-0.5\beta$, the - - - curve to $G=0$, and the --- curve to $G=0.5\beta$. Note that the island heals completely in the first two cases, but not in the third.

Figure 2 : Poincaré plots of magnetic field lines showing the process of self-healing for Helias from the HINT code. (a) A vacuum field with a 5/6 island-chain. (b) A self-healed state is realized as β increases, with $\beta_0=9\%$. (c) As β increases further, the 5/6 island- chain reappears (at a different radial location), but now with a flipped phase with respect to the vacuum island ($\beta_0=12\%$).

Figure 3 : The radial profile of D_R for a sequence of equilibria with increasing values of β. (a) corresponds an equilibrium prior to the healed state (Fig. 2(b)), (b) for the healed state, and (c) for an equilibrium after the healed state.

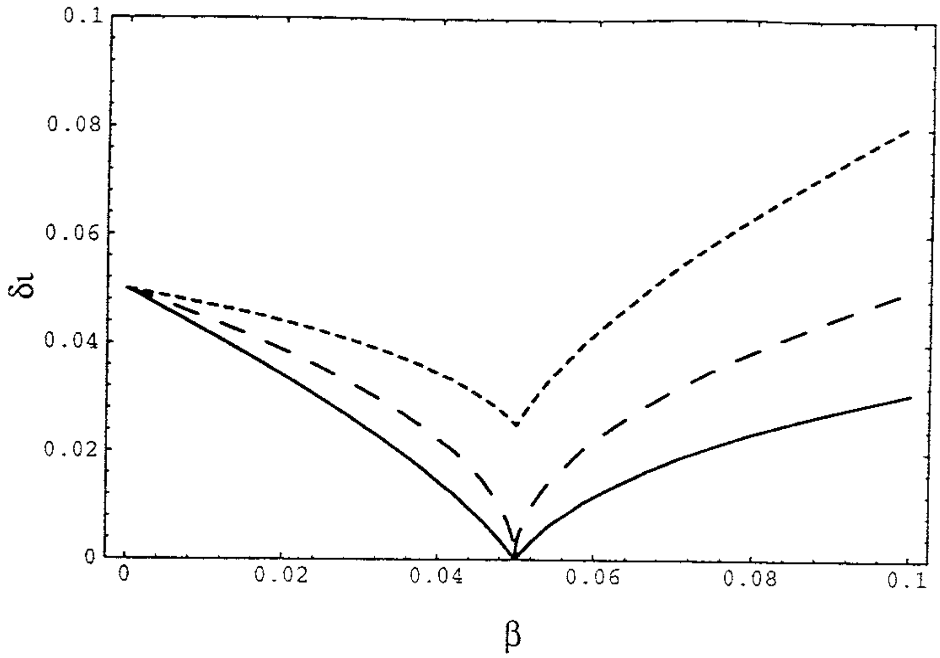

Figure 1

‘Self-healing’ of magnetic islands

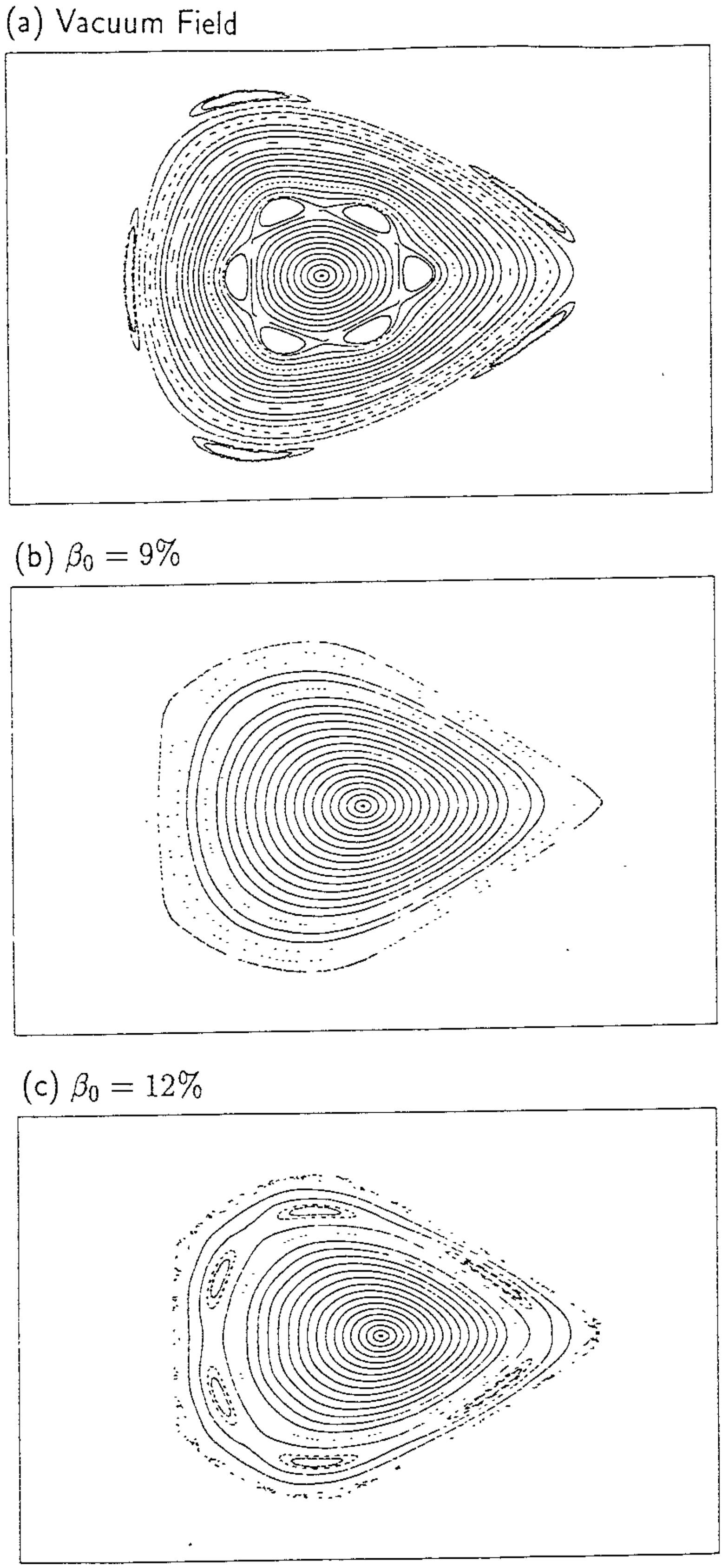

Figure 2

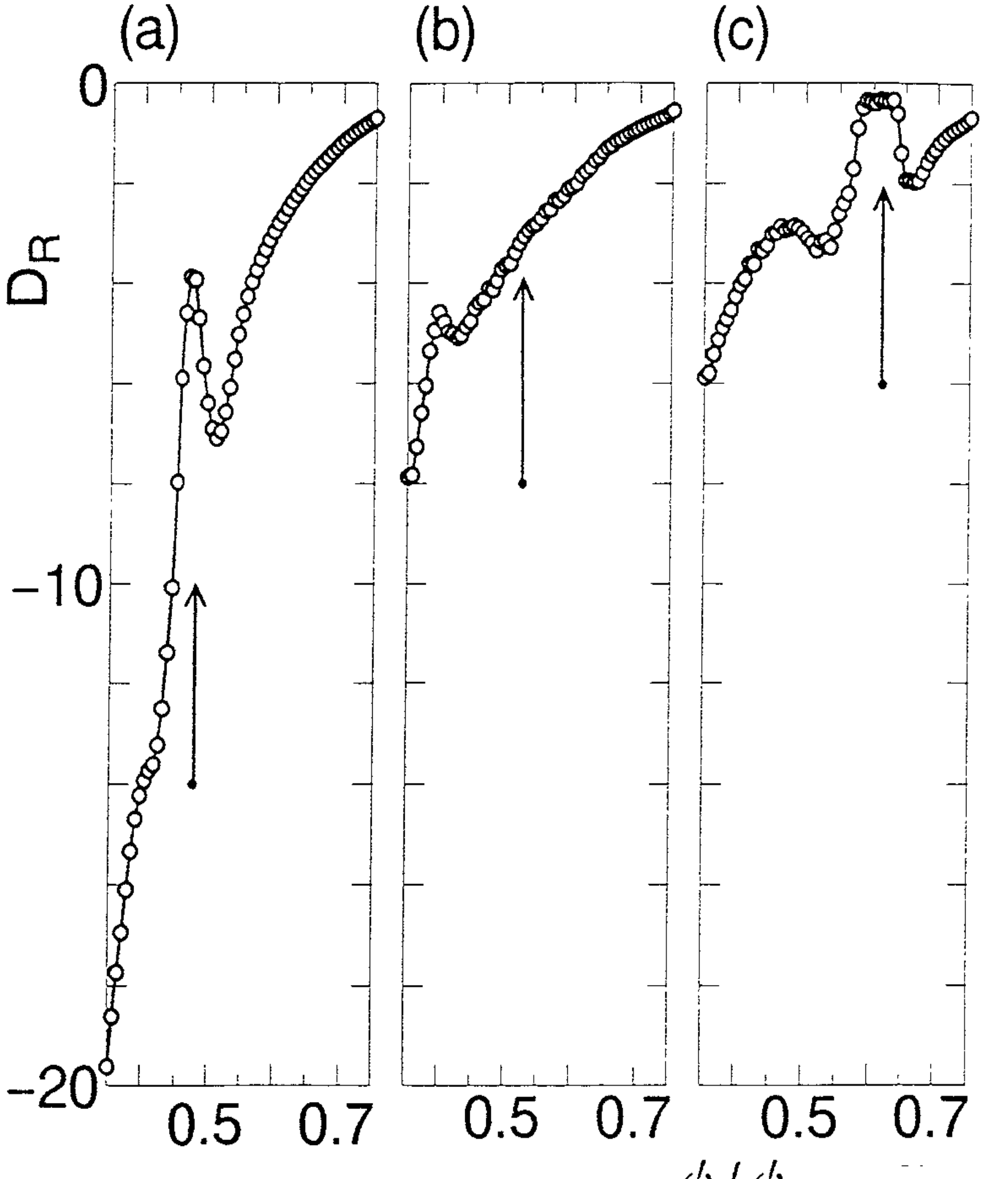

Figure 3

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