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# Lagrangian direct-interaction approximation for homogeneous isotropic turbulence

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**Abstract** A perturbation method developed by Kraichnan [ J. Fluid Mech. 5 (1959) 497 ], which is based upon an extraction of direct interactions among Fourier modes of a velocity field, is applied to the Lagrangian velocity correlation and response functions. The resultant set of integro-differential equations for these functions obtained for homogeneous isotropic turbulence, which has no adjustable free parameters, happens to be identical to the one derived before by using a different perturbation method [ Kaneda, J. Fluid Mech. 107 (1981) 131 ]. The shape of the energy spectrum function is determined numerically in the universal range for stationary turbulence, and in the whole wavenumber range in a similarly evolving form for freely decaying case. The energy spectrum in the universal range takes same shape in the both cases, which also agrees excellently with many measurements of various kinds of real turbulence as well as numerical results obtained by Gotoh *et al.* [ J. Phys. Soc. Japan, 57 (1988) 866 ] as an initial value problem. Skewness factor of the longitudinal velocity derivative is calculated to be  $-0.66$  for stationary turbulence. Wavenumber dependence of the eddy-viscosity is also determined.

**Keywords** Lagrangian direct-interaction approximation, Isotropic turbulence, Kolmogorov spectrum

# 1 Introduction

One of the main objectives of statistical theory of turbulence is to derive the statistical averages of field quantities, such as the mean velocity distribution, the velocity correlation (or the energy spectrum) function, etc. systematically from the first principle, i.e. the Navier-Stokes equation. It has long been known, however, that the equations for velocity moments are never closed by simple averaging procedures because of non-linearity with respect to the velocity of the governing equations. There is a long history of attacking this closure problem. We refer the readers to standard textbooks (e.g. Leslie 1973, Monin & Yaglom 1975, Orszag 1977, Lesieur 1990) for details. We make here a few remarks only on some theories closely related to the current work.

There have been proposed so far a lot of closure theories which are based upon the quasi-normality of the one-point statistics of a turbulent velocity field. The zero-4th-order cumulant theory is one of the simplest approximations which unfortunately leads to a negative energy spectrum (Monin & Yaglom 1975). The EDQNM (eddy-damped quasi-normal Markovian) approximation (Orszag 1977) guarantees the positivity of the energy spectrum in the inertial range. This theory, however, has an adjustable free parameter to get a reasonable value of the Kolmogorov constant. An analytical theory of turbulence which does not include any free parameters was developed by Kraichnan (1959, 1965). He developed the DIA (direct-interaction approximation) and applied it to the Eulerian velocity field (Kraichnan 1959), which we call here the EDIA (Eulerian direct-interaction approximation). However, it does not give the proper  $-\frac{5}{3}$  power law of the energy spectrum in the inertial range but the  $-\frac{3}{2}$  power law. This failure may be attributed to the non-invariance under Galilean transformation of the Eulerian velocity covariance (see Orszag 1977). Later, he introduced the Lagrangian velocity field and rewrote the EDIA equations in terms of it and succeeded in obtaining the  $-\frac{5}{3}$  power energy spectrum (Kraichnan 1965), which we call Kraichnan's LDIA (Lagrangian direct-interaction approximation). The analysis is, however, too complicated for the present authors to understand. The EDIA equations can also be derived by a Reynolds number expansion of the Eulerian velocity field followed by a kind of renormalization (Leslie 1973). Kaneda (1981) applied this procedure to the Lagrangian velocity field. The resultant integro-differential equations, which are called LRA (Lagrangian renormalized approximation) equations by him, are much simpler than Kraichnan's LDIA equations and give the  $-\frac{5}{3}$  power inertial-range energy spectrum with Kolmogorov constant 1.722. This theory has also no free parameters. The properties of this set of equations have been extensively investigated by himself and his coworkers (Kaneda 1986, Gotoh, Kaneda & Bekki 1988).

The purpose of this paper is twofold. First, we apply Kraichnan's idea of DIA to the Lagrangian field quantities to obtain a set of integro-differential equations for the Lagrangian velocity and the response functions, which happens to be exactly same as the one derived by Kaneda (1981). Then, we seek for the shape of the energy spectrum function in two cases, i.e. stationary and freely decaying cases. In the former the energy spectrum is obtained in the universal range, while in the latter it is determined in the whole wavenumber range

in a similarly decaying form. We solve the governing equations by an iteration method instead of the initial value approach done before by Gotoh, Kaneda & Bekki (1988).

This paper is organized as follows. In the next section we introduce several basic quantities which are necessary in the subsequent analysis. In §3, we explain the idea, the assumption and the procedure of the present LDIA and derive a set of integro-differential equations for homogeneous isotropic turbulence. We then solve these equations for stationary case in §4. The skewness of the velocity derivative and the wavenumber dependence of the eddy-viscosity are also calculated. A freely decaying case is treated in §5. The shape of the energy spectrum and the time development are determined in a similarly evolving form. Section 6 is devoted to discussions. Details of some calculations are given in Appendices.

## 2 Preparations

### 2.1 Basic quantities

We deal with the motion of an incompressible viscous fluid which is described by Navier-Stokes equation

$$\frac{\partial}{\partial t} u_i(\mathbf{x}, t) + u_j(\mathbf{x}, t) \frac{\partial}{\partial x_j} u_i(\mathbf{x}, t) = -\frac{1}{\rho} \frac{\partial}{\partial x_i} p(\mathbf{x}, t) + \nu \frac{\partial^2}{\partial x_j \partial x_j} u_i(\mathbf{x}, t) \quad (i = 1, 2, 3) \quad (2.1)$$

and continuity equation

$$\frac{\partial}{\partial x_i} u_i(\mathbf{x}, t) = 0, \quad (2.2)$$

where  $u_i(\mathbf{x}, t)$  is the Eulerian velocity at position  $\mathbf{x}$  at time  $t$ ,  $\rho$  is the constant density,  $p(\mathbf{x}, t)$  is the pressure and  $\nu$  is the kinematic viscosity of fluid. Repeated subscripts are summed over 1—3.

Lagrangian position function

$$\psi(\mathbf{x}, t|\mathbf{x}', t') = \delta^3(\mathbf{x} - \mathbf{y}(t|\mathbf{x}', t')), \quad (2.3)$$

which was introduced before by Kaneda (1981) plays an important role in the present analysis, where  $\mathbf{y}(t|\mathbf{x}', t')$  stands for the Lagrangian coordinate (i.e. the position of a fluid element at time  $t$  which passed at position  $\mathbf{x}'$  at time  $t' (< t)$ ), and  $\delta^3$  is Dirac's delta function. Obviously, the position function obeys

$$\frac{\partial}{\partial t} \psi(\mathbf{x}, t|\mathbf{x}', t') = -u_j(\mathbf{x}, t) \frac{\partial}{\partial x_j} \psi(\mathbf{x}, t|\mathbf{x}', t') \quad (2.4)$$

with initial condition

$$\psi(\mathbf{x}, t'|\mathbf{x}', t') = \delta^3(\mathbf{x} - \mathbf{x}'). \quad (2.5)$$

Lagrangian velocity  $v_i(t|\mathbf{x}', t') = u_i(\mathbf{y}(t|\mathbf{x}', t'), t)$  and the Eulerian velocity are related with each other as

$$v_i(t|\mathbf{x}', t') = \iiint d\mathbf{x} u_i(\mathbf{x}, t) \psi(\mathbf{x}, t|\mathbf{x}', t'), \quad (2.6)$$

$$u_i(\mathbf{x}, t) = \iiint d\mathbf{x}' v_i(t|\mathbf{x}', t') \psi(\mathbf{x}, t|\mathbf{x}', t'). \quad (2.7)$$

As mentioned in the introduction, we will construct a system of equations for Lagrangian velocity function which is defined by

$$V_{ij}(\mathbf{r}, t, t') = \overline{v_i(t|\mathbf{x} + \mathbf{r}, t') u_j(\mathbf{x}, t')}. \quad (2.8)$$

Here and below, an overbar denotes an ensemble average. We have assumed that the velocity field is statistically homogeneous, so that  $V_{ij}$  is independent of position vector  $\mathbf{x}$ .

## 2.2 Fourier decomposition

For simplicity of description, we consider the motion of a fluid confined in a periodic cube of side  $L$ . Then we can expand  $u_i$ ,  $v_i$ ,  $\psi$  and  $V_{ij}$  into Fourier series as

$$u_i(\mathbf{x}, t) = \left(\frac{2\pi}{L}\right)^3 \sum_{\mathbf{k}} \tilde{u}_i(\mathbf{k}, t) \exp[\mathbf{i}\mathbf{k} \cdot \mathbf{x}], \quad (2.9)$$

$$v_i(t|\mathbf{x}', t') = \left(\frac{2\pi}{L}\right)^3 \sum_{\mathbf{k}} \tilde{v}_i(t|\mathbf{k}, t') \exp[\mathbf{i}\mathbf{k} \cdot \mathbf{x}'], \quad (2.10)$$

$$\psi(\mathbf{x}, t|\mathbf{x}', t') = \left(\frac{2\pi}{L}\right)^6 \sum_{\mathbf{k}} \sum_{\mathbf{k}'} \tilde{\psi}(\mathbf{k}, t|\mathbf{k}', t') \exp[\mathbf{i}(\mathbf{k} \cdot \mathbf{x} + \mathbf{k}' \cdot \mathbf{x}')] \quad (2.11)$$

and

$$V_{ij}(\mathbf{r}, t, t') = \left(\frac{2\pi}{L}\right)^3 \sum_{\mathbf{k}} \tilde{V}_{ij}(\mathbf{k}, t, t') \exp[\mathbf{i}\mathbf{k} \cdot \mathbf{r}], \quad (2.12)$$

respectively, where

$$\mathbf{k} = \frac{2\pi}{L} (n_1, n_2, n_3) \quad (n_1, n_2, n_3 = 0, \pm 1, \pm 2, \dots) \quad (2.13)$$

is the wavenumber vector. The summations are taken over triplets of integers  $n_1$ ,  $n_2$  and  $n_3$ . The Fourier inverse transformations are similarly written as

$$\tilde{u}_i(\mathbf{k}, t) = \left(\frac{1}{2\pi}\right)^3 \iiint d\mathbf{x} u_i(\mathbf{x}, t) \exp[-\mathbf{i}\mathbf{k} \cdot \mathbf{x}], \quad (2.14)$$

$$\tilde{v}_i(t|\mathbf{k}, t') = \left(\frac{1}{2\pi}\right)^3 \iiint d\mathbf{x} v_i(t|\mathbf{x}', t') \exp[-\mathbf{i}\mathbf{k} \cdot \mathbf{x}'], \quad (2.15)$$

$$\tilde{\psi}(\mathbf{k}, t|\mathbf{k}', t') = \left(\frac{1}{2\pi}\right)^6 \iiint d\mathbf{x} \iiint d\mathbf{x}' \psi(\mathbf{x}, t|\mathbf{x}', t') \exp[-\mathbf{i}(\mathbf{k} \cdot \mathbf{x} + \mathbf{k}' \cdot \mathbf{x}')] \quad (2.16)$$

and

$$\tilde{V}_{ij}(\mathbf{k}, t, t') = \left(\frac{1}{2\pi}\right)^3 \iiint d\mathbf{r} V_{ij}(\mathbf{r}, t, t') \exp[-\mathbf{i}\mathbf{k} \cdot \mathbf{r}] = \left(\frac{2\pi}{L}\right)^3 \overline{\tilde{v}_i(t|\mathbf{k}, t') \tilde{u}_j(-\mathbf{k}, t')}, \quad (2.17)$$

respectively. In these equations, integrations are carried out over the periodic cube. Relations between Eulerian velocity and Lagrangian velocity (2.6) and (2.7) are written, in the Fourier space, as

$$\tilde{v}_i(t|\mathbf{k}, t') = \frac{(2\pi)^6}{L^3} \sum_{\mathbf{k}'} \tilde{u}_i(\mathbf{k}', t) \tilde{\psi}(-\mathbf{k}', t|\mathbf{k}, t') \quad (2.18)$$

and

$$\tilde{u}_i(\mathbf{k}, t) = \frac{(2\pi)^6}{L^3} \sum_{\mathbf{k}'} \tilde{v}_i(t|\mathbf{k}', t') \tilde{\psi}(\mathbf{k}, t|-\mathbf{k}', t'), \quad (2.19)$$

respectively.

The governing equations of  $\tilde{u}_i$  and  $\tilde{\psi}$  are derived from (2.1), (2.2), (2.4) and (2.5) as

$$\left[ \frac{\partial}{\partial t} + \nu k^2 \right] \tilde{u}_i(\mathbf{k}, t) = -\frac{i}{2} \left( \frac{2\pi}{L} \right)^3 \tilde{P}_{ijm}(\mathbf{k}) \sum_{\mathbf{p}} \sum_{\mathbf{q}} \tilde{u}_j(-\mathbf{p}, t) \tilde{u}_m(-\mathbf{q}, t), \quad (2.20)$$

$$k_i \tilde{u}_i(\mathbf{k}, t) = 0, \quad (2.21)$$

and

$$\frac{\partial}{\partial t} \tilde{\psi}(\mathbf{k}, t | \mathbf{k}', t') = -i k_j \left( \frac{2\pi}{L} \right)^3 \sum_{\mathbf{p}} \sum_{\mathbf{q}} \tilde{u}_j(-\mathbf{p}, t) \tilde{\psi}(-\mathbf{q}, t | \mathbf{k}', t') \quad (2.22)$$

with initial condition

$$\tilde{\psi}(\mathbf{k}, t' | \mathbf{k}', t') = \frac{L^3}{(2\pi)^6} \delta_{\mathbf{k}+\mathbf{k}'}. \quad (2.23)$$

Here,

$$\tilde{P}_{ijm}(\mathbf{k}) = k_m \tilde{P}_{ij}(\mathbf{k}) + k_j \tilde{P}_{im}(\mathbf{k}), \quad \tilde{P}_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}, \quad (2.24)$$

where  $\delta_{\mathbf{k}}^3$  and  $\delta_{ij}$  are Kronecker's deltas ( $\delta_{\mathbf{k}}^3 = 0$  ( $\mathbf{k} \neq \mathbf{o}$ ),  $\delta_{\mathbf{k}}^3 = 1$  ( $\mathbf{k} = \mathbf{o}$ ),  $\delta_{ij} = 0$  ( $i \neq j$ ),  $\delta_{ij} = 1$  ( $i = j$ )).

The time-derivative of (2.18) then yields

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{v}_i(t | \mathbf{k}', t') &= -\frac{(2\pi)^6}{L^3} \nu \sum_{\mathbf{p}} p^2 \tilde{u}_i(\mathbf{p}, t) \tilde{\psi}(-\mathbf{p}, t | \mathbf{k}', t') \\ &\quad - i \frac{(2\pi)^9}{L^6} \sum_{\mathbf{p}} \sum_{\mathbf{q}} \sum_{\mathbf{r}} \frac{r_i r_m r_n}{r^2} \tilde{u}_m(\mathbf{p}, t) \tilde{u}_n(\mathbf{q}, t) \tilde{\psi}(\mathbf{r}, t | \mathbf{k}', t'), \end{aligned} \quad (2.25)$$

where use has been made of (2.20) and (2.22). By using (2.17), (2.20) and (2.25), we can derive the governing equations of the two-point Lagrangian velocity correlation function for a single time as

$$\begin{aligned} \left[ \frac{\partial}{\partial t} + 2\nu k^2 \right] \tilde{V}_{ij}(\mathbf{k}, t, t) &= -\frac{i}{2} \left( \frac{2\pi}{L} \right)^6 \tilde{P}_{imn}(\mathbf{k}) \sum_{\mathbf{p}} \sum_{\mathbf{q}} \overline{\tilde{u}_m(-\mathbf{p}, t) \tilde{u}_n(-\mathbf{q}, t) \tilde{u}_j(-\mathbf{k}, t)} \\ &\quad + (i \leftrightarrow j, \mathbf{k} \rightarrow -\mathbf{k}) \end{aligned} \quad (2.26)$$

and for two times as

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{V}_{ij}(\mathbf{k}, t, t') &= -\frac{(2\pi)^9}{L^6} \nu \sum_{\mathbf{p}} p^2 \overline{\tilde{u}_i(\mathbf{p}, t) \tilde{\psi}(-\mathbf{p}, t | \mathbf{k}, t') \tilde{u}_j(-\mathbf{k}, t')} \\ &\quad - i \frac{(2\pi)^{12}}{L^9} \sum_{\mathbf{p}} \sum_{\mathbf{q}} \sum_{\mathbf{r}} \frac{r_i r_m r_n}{r^2} \overline{\tilde{u}_m(\mathbf{p}, t) \tilde{u}_n(\mathbf{q}, t) \tilde{\psi}(\mathbf{r}, t | \mathbf{k}, t') \tilde{u}_j(-\mathbf{k}, t')}. \end{aligned} \quad (2.27)$$

Notice that the above two equations contain three- and four-point correlation functions. Similarly the governing equation of three-point velocity correlation function would contain four- and five-point correlation functions. Therefore the system of equations cannot be closed for ever, unless more-point correlations functions are expressed in terms of fewer-point ones at some stage.

For later convenience, we define here the energy spectrum

$$E(k, t) = \frac{1}{2} k^2 \oint d\Omega \tilde{V}_{ii}(\mathbf{k}, t, t), \quad (2.28)$$

where  $\oint d\Omega$  denotes the solid angle integration in the Fourier space, and the incompressible part of Lagrangian velocity correlation

$$\tilde{Q}_{ij}(\mathbf{k}, t, t') = \tilde{P}_{im}(\mathbf{k}) \tilde{V}_{mj}(\mathbf{k}, t, t') . \quad (2.29)$$

### 2.3 Response function

The response functions of the velocity and the position functions play a key role in the present formulation (Kraichnan 1959 ; Kaneda 1981). Eulerian velocity response function

$$\tilde{G}_{ij}^{(E)}(\mathbf{k}, t | \mathbf{k}', t') = \frac{\delta \tilde{u}_i(\mathbf{k}, t)}{\delta \tilde{u}_j(\mathbf{k}', t')} \quad (2.30)$$

expresses the influence to  $\tilde{u}_i(\mathbf{k}, t)$  at time  $t$  due to an infinitesimal disturbance for  $\tilde{u}_j(\mathbf{k}', t')$  ( $t' \leq t$ ), where  $\delta$  denotes a functional derivative. By taking a functional derivative of (2.20), we obtain the governing equation for this function as

$$\left[ \frac{\partial}{\partial t} + \nu k^2 \right] \tilde{G}_{ij}^{(E)}(\mathbf{k}, t | \mathbf{k}', t') = -i \left( \frac{2\pi}{L} \right)^3 \tilde{P}_{imn}(\mathbf{k}) \sum_{\mathbf{p}} \sum_{\mathbf{q}} \tilde{u}_m(-\mathbf{p}, t) \tilde{G}_{nj}^{(E)}(-\mathbf{q}, t | \mathbf{k}', t') . \quad (2.31)$$

( $\mathbf{k} + \mathbf{p} + \mathbf{q} = \mathbf{o}$ )

The initial condition is given by

$$\tilde{G}_{ij}^{(E)}(\mathbf{k}, t | \mathbf{k}', t') = \frac{L^3}{(2\pi)^6} \delta_{ij} \delta_{\mathbf{k} + \mathbf{k}'}^3 . \quad (2.32)$$

Similarly, the governing equations of Lagrangian velocity response function

$$\tilde{G}_{ij}^{(L)}(t | \mathbf{k}, \mathbf{k}', t') = \frac{\delta \tilde{u}_i(t | \mathbf{k}, t')}{\delta \tilde{u}_j(\mathbf{k}', t')} \quad (2.33)$$

and the position response function

$$\tilde{\Psi}_i(\mathbf{k}, t | \mathbf{k}', \mathbf{k}'', t') = \frac{\delta \tilde{\psi}(\mathbf{k}, t | \mathbf{k}', t')}{\delta \tilde{u}_i(\mathbf{k}'', t')} \quad (2.34)$$

are obtained from (2.25) and (2.22), respectively, as

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{G}_{ij}^{(L)}(t | \mathbf{k}, \mathbf{k}', t') = & \\ & -\nu \frac{(2\pi)^6}{L^3} \sum_{\mathbf{k}''} k''^2 \left[ \tilde{G}_{ij}^{(E)}(\mathbf{k}'', t | \mathbf{k}', t') \tilde{\psi}(-\mathbf{k}'', t | \mathbf{k}, t') + \tilde{u}_i(\mathbf{k}'', t) \tilde{\Psi}_j(-\mathbf{k}'', t | \mathbf{k}, \mathbf{k}', t') \right] \\ & -i \frac{(2\pi)^9}{L^6} \sum_{\mathbf{p}} \sum_{\mathbf{q}} \sum_{\mathbf{r}} \frac{r_i r_m r_n}{r^2} \left[ 2\tilde{u}_m(\mathbf{p}, t) \tilde{G}_{nj}^{(E)}(\mathbf{q}, t | \mathbf{k}', t') \tilde{\psi}(\mathbf{r}, t | \mathbf{k}, t') + \tilde{u}_m(\mathbf{p}, t) \tilde{u}_n(\mathbf{q}, t) \tilde{\Psi}_j(\mathbf{r}, t | \mathbf{k}, \mathbf{k}', t') \right] \end{aligned} \quad (2.35)$$

with initial condition

$$\tilde{G}_{ij}^{(L)}(t | \mathbf{k}, \mathbf{k}', t') = \frac{L^3}{(2\pi)^6} \delta_{ij} \delta_{\mathbf{k} + \mathbf{k}'}^3 , \quad (2.36)$$

and

$$\frac{\partial}{\partial t} \tilde{\Psi}_i(\mathbf{k}, t | \mathbf{k}', \mathbf{k}'', t') =$$

$$-i k_a \left( \frac{2\pi}{L} \right)^3 \sum_{\substack{\mathbf{p} \\ (\mathbf{k}+\mathbf{p}+\mathbf{q}=\mathbf{o})}} \sum_{\mathbf{q}} \left[ \tilde{u}_a(-\mathbf{p}, t) \tilde{\Psi}_i(-\mathbf{q}, t|\mathbf{k}', \mathbf{k}'', t') + \tilde{G}_{\text{oi}}^{(E)}(-\mathbf{p}, t|\mathbf{k}'', t') \tilde{\psi}(-\mathbf{q}, t|\mathbf{k}', t') \right] \quad (2.37)$$

with initial condition

$$\tilde{\Psi}_i(\mathbf{k}, t'|\mathbf{k}', \mathbf{k}'', t') = 0. \quad (2.38)$$

A functional derivative of (2.19) gives a relation among the Eulerian velocity response, the Lagrangian velocity response and the position response functions as

$$\begin{aligned} \tilde{G}_{ij}^{(E)}(\mathbf{k}, t|\mathbf{k}', t') &= \frac{(2\pi)^6}{L^3} \sum_{\mathbf{k}''} \tilde{G}_{ij}^{(L)}(t|\mathbf{k}'', \mathbf{k}', t') \tilde{\psi}(\mathbf{k}, t|-\mathbf{k}'', t') \\ &+ \frac{(2\pi)^{12}}{L^6} \sum_{\mathbf{k}''} \sum_{\mathbf{k}'''} \tilde{u}_i(\mathbf{k}''', t) \tilde{\psi}(-\mathbf{k}''', t|\mathbf{k}'', t') \tilde{\Psi}_j(\mathbf{k}, t|-\mathbf{k}'', \mathbf{k}', t'). \end{aligned} \quad (2.39)$$

For later use, we define here the incompressible part of the Lagrangian velocity response function by

$$\tilde{G}_{ij}(\mathbf{k}, t, t') = \frac{(2\pi)^6}{L^3} \overline{\tilde{G}_{im}^{(L)}(t|\mathbf{k}, -\mathbf{k}, t')} \tilde{P}_{mj}(\mathbf{k}). \quad (2.40)$$

#### 2.4 Homogeneous isotropic turbulence

If turbulence is isotropic as well as homogeneous, second-order tensors  $\tilde{Q}_{ij}$  and  $\tilde{G}_{ij}$  are represented by a single scalar as

$$\tilde{Q}_{ij}(\mathbf{k}, t, t') = \frac{1}{2} \tilde{P}_{ij}(\mathbf{k}) Q(k, t, t'), \quad (2.41)$$

and

$$\tilde{G}_{ij}(\mathbf{k}, t, t') = \tilde{P}_{ij}(\mathbf{k}) G(k, t, t'). \quad (2.42)$$

Then, the energy spectrum is represented, from (2.28), (2.29) and (2.41), as

$$E(k, t) = 2\pi k^2 Q(k, t, t). \quad (2.43)$$

The energy spectrum at large wavenumbers for (locally) homogeneous isotropic and stationary turbulence at large Reynolds numbers was considered by Kolmogorov (1941). By employing a dimensional analysis in terms of wavenumber  $k$ , kinematic viscosity  $\nu$  and the mean of energy dissipation rate  $\epsilon$ , he derived a similarity form of the energy spectrum function at large wavenumbers as

$$E(k, t) = \nu^{\frac{5}{4}} \epsilon^{\frac{1}{4}} F(k/k_d), \quad (2.44)$$

$$k_d = \epsilon^{\frac{1}{4}} \nu^{-\frac{3}{4}}, \quad (2.45)$$

where  $F$  is a non-dimensional function and  $k_d$  is the Kolmogorov wavenumber. The wavenumber range in which similarity law (2.44) is observed is called the universal range. This similarity law has been supported



by many measurements of different kinds of turbulence (see Chapman 1979, Saddoughi & Veeravelli 1994 and also figure 2). In the inertial subrange ( $k \ll k_d$ ), the spectrum may not depend on  $\nu$  and take a power form as

$$E(k, t) = K \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}, \quad (2.46)$$

where  $K$  is the Kolmogorov constant, which is evaluated experimentally to be  $1.62 \pm 0.17$  (Sreenivasan 1995). It is one of the main objectives of the current closure theory of turbulence to determine the universal function  $F$  with resort to the Navier-Stokes equation.

### 3 Lagrangian Direct-Interaction Approximation

#### 3.1 Direct-interaction decomposition

Following an idea developed by Kraichnan (1959), we introduce a direct-interaction decomposition. Recall that the RHS of Navier-Stokes equation (2.20) is composed of a sum of an infinite number of quadratic nonlinear terms, each of which represents direct-interactions among three Fourier components with wavenumbers  $\mathbf{k}$ ,  $\mathbf{p}$  and  $\mathbf{q}$  which construct a triangle ( $\mathbf{k} + \mathbf{p} + \mathbf{q} = \mathbf{o}$ ).

Here, we choose arbitrarily a triangular triplet of wavenumbers, say  $\mathbf{k}_0$ ,  $\mathbf{p}_0$  and  $\mathbf{q}_0$  ( $\mathbf{k}_0 + \mathbf{p}_0 + \mathbf{q}_0 = \mathbf{o}$ ). We imagine then a fictitious field which does not contain the direct-interactions among these special wavenumbers. This fictitious field is called the NDI-field (non-direct-interaction field), and is denoted by  $\tilde{u}_i^{(0)}(\mathbf{k}, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)$ . Furthermore, we define the DI-field (direct-interaction field)  $\tilde{u}_i^{(1)}(\mathbf{k}, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)$  by the difference between the true field and the NDI-field, namely,

$$\tilde{u}_i(\mathbf{k}, t) = \tilde{u}_i^{(0)}(\mathbf{k}, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) + \tilde{u}_i^{(1)}(\mathbf{k}, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0). \quad (3.1)$$

Hence, the governing equation for  $\tilde{u}_i^{(0)}$  is written as

$$\begin{aligned} \left[ \frac{\partial}{\partial t} + \nu k^2 \right] \tilde{u}_i^{(0)}(\mathbf{k}, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\ = -\frac{i}{2} \left( \frac{2\pi}{L} \right)^3 \tilde{P}_{ijm}(\mathbf{k}) \sum_{\mathbf{p}} \sum'_{\mathbf{q}} \tilde{u}_j^{(0)}(-\mathbf{p}, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{u}_m^{(0)}(-\mathbf{q}, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0), \end{aligned} \quad (3.2)$$

$(\mathbf{k} + \mathbf{p} + \mathbf{q} = \mathbf{o})$

where  $\Sigma\Sigma'$  stands for summation without the interactions among chosen three wavenumbers  $\mathbf{k}_0$ ,  $\mathbf{p}_0$  and  $\mathbf{q}_0$ . Subtracting the above equation from the Navier-Stokes equation, we obtain the time-evolution equation of the DI-field as

$$\begin{aligned} \left[ \frac{\partial}{\partial t} + \nu k^2 \right] \tilde{u}_i^{(1)}(\mathbf{k}, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\ = -i \left( \frac{2\pi}{L} \right)^3 \tilde{P}_{ijm}(\mathbf{k}) \sum_{\mathbf{p}} \sum'_{\mathbf{q}} \tilde{u}_j(-\mathbf{p}, t) \tilde{u}_m^{(1)}(-\mathbf{q}, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\ - \delta_{\mathbf{k}-\mathbf{k}_0}^3 i \left( \frac{2\pi}{L} \right)^3 \tilde{P}_{ijm}(\mathbf{k}_0) \tilde{u}_j^{(0)}(-\mathbf{p}_0, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{u}_m^{(0)}(-\mathbf{q}_0, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \end{aligned}$$

$$\begin{aligned}
& + \delta_{\mathbf{k}+\mathbf{k}_0}^3 i \left( \frac{2\pi}{L} \right)^3 \tilde{P}_{i,j,m}(\mathbf{k}_0) \tilde{u}_j^{(0)}(\mathbf{p}_0, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{u}_m^{(0)}(\mathbf{q}_0, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
& + (\mathbf{p}_0 \rightarrow \mathbf{q}_0 \rightarrow \mathbf{k}_0 \rightarrow \mathbf{p}_0) .
\end{aligned} \tag{3.3}$$

Here we have neglected the higher-order terms of  $\tilde{u}^{(1)}$  (assumption-1 written in the next section). The direct-interaction decompositions for Eulerian velocity response function (2.31), position function (2.22) and position response function (2.37) are made similarly. The details of the calculation will be described in Appendix A.

By comparing (3.3) with (A.4) in Appendix A, we find that  $\tilde{G}_{ij}^{(E0)}$  serves as Green's function of  $\tilde{u}_i^{(1)}$ . A solution to (3.3), therefore, is expressed in terms of  $\tilde{G}_{ij}^{(E0)}$  and  $\tilde{u}_i$  as

$$\begin{aligned}
\tilde{u}_i^{(1)}(\mathbf{k}, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) = & i \frac{(2\pi)^9}{L^6} \tilde{P}_{abc}(\mathbf{k}) \int_{t_0}^t dt' \tilde{G}_{ia}^{(E0)}(\mathbf{k}, t | -\mathbf{k}, t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
& \times \left[ \begin{aligned}
& - \delta_{\mathbf{k}-\mathbf{k}_0}^3 \tilde{u}_b^{(0)}(-\mathbf{p}_0, t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{u}_c^{(0)}(-\mathbf{q}_0, t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
& - \delta_{\mathbf{k}+\mathbf{k}_0}^3 \tilde{u}_b^{(0)}(\mathbf{p}_0, t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{u}_c^{(0)}(\mathbf{q}_0, t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
& + (\mathbf{p}_0 \rightarrow \mathbf{q}_0 \rightarrow \mathbf{k}_0 \rightarrow \mathbf{p}_0)
\end{aligned} \right], \tag{3.4}
\end{aligned}$$

where  $\tilde{u}_i^{(1)}$  has been assumed to vanish at time  $t_0$ . The DI-fields of other quantities are similarly represented in terms of the NDI-fields and the true fields (see Appendix B).

### 3.2 Methods of approximation

For the purpose of easy reference, we summarize here assumptions and procedures to construct a system of equations for statistical quantities (i.e. the Lagrangian velocity correlation and the Lagrangian velocity response functions). We make the following three assumptions (Kraichnan 1959) :

**Assumption-1**  $\tilde{X}^{(1)}$  is much smaller than  $\tilde{X}^{(0)}$  in magnitude, hence we can neglect the higher-order terms of them. Here,  $\tilde{X}$  stands for any physical quantity (for example  $\tilde{u}_i$ ).

**Assumption-2**  $\tilde{u}_i^{(0)}(\mathbf{k}_0, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)$ ,  $\tilde{u}_j^{(0)}(\mathbf{p}_0, t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)$  and  $\tilde{u}_k^{(0)}(\mathbf{q}_0, t'' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)$  are statistically independent of each other.

**Assumption-3**  $\{\tilde{u}\}$ ,  $\{\tilde{G}\}$ ,  $\{\tilde{\psi}\}$  and  $\{\tilde{\Psi}\}$  are statistically independent of each other.

We construct a set of integro-differential equations for the velocity correlation and the response functions by employing the following four procedures :

**Procedure-1** Substitute the direct-interaction decompositions, (3.1) and (A.1)—(A.3), into the RHS of the governing equations of statistical quantities. Thanks to assumption-1, we can neglect higher-order terms of DI-field.

**Procedure-2** Eliminate physical quantities of DI-field by making use of (3.4), (B.1), (B.2) and (B.3).

**Procedure-3** Eliminate  $\overline{\tilde{G}_{ij}^{(E0)}}$ ,  $\overline{\tilde{\psi}^{(0)}}$  and  $\overline{\tilde{\Psi}_i^{(0)}}$  respectively by making use of

$$\overline{\tilde{G}_{ij}^{(E0)}(\mathbf{k}, t|\mathbf{k}', t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)} = \overline{\tilde{G}_{ij}^{(L0)}(t|\mathbf{k}, \mathbf{k}', t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)}, \quad (3.5)$$

$$\overline{\tilde{\psi}^{(0)}(\mathbf{k}, t|\mathbf{k}', t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)} = \frac{L^3}{(2\pi)^6} \delta_{\mathbf{k}+\mathbf{k}'}^3 \quad (3.6)$$

and

$$\overline{\tilde{\Psi}_i^{(0)}(\mathbf{k}, t|\mathbf{k}', \mathbf{k}'', t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)} = -\frac{i k_a}{(2\pi)^3} \int_{t'}^t dt'' \overline{\tilde{G}_{ai}^{(L0)}(t''|\mathbf{k} + \mathbf{k}', \mathbf{k}'', t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)}. \quad (3.7)$$

**Procedure-4** Replace  $\overline{\tilde{u}_i^{(0)}\tilde{u}_j^{(0)}}$  and  $\overline{\tilde{G}_{im}^{(L0)}}$  by  $\tilde{Q}_{ij}$  and  $\tilde{G}_{ij}$  through

$$\overline{\tilde{u}_i^{(0)}(\mathbf{k}, t|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)\tilde{u}_j^{(0)}(-\mathbf{k}, t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)} = \left(\frac{L}{2\pi}\right)^3 \tilde{Q}_{ij}(\mathbf{k}, t, t') \quad (3.8)$$

and

$$\frac{(2\pi)^6}{L^3} \overline{\tilde{G}_{im}^{(L0)}(t|\mathbf{k}, -\mathbf{k}, t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)} \tilde{P}_{mj}(\mathbf{k}) = \tilde{G}_{ij}(\mathbf{k}, t, t'), \quad (3.9)$$

which follows from (2.40) under assumption-1. For derivations of relations (3.5)–(3.8), see Appendix C.

### 3.3 LDIA equations

Applying the methods of LDIA described in the preceding section, we can construct a system of equations for statistical quantities. We start with two-point-one-time velocity correlation function. By substituting direct-interaction decomposition (3.1) into the RHS of (2.26), we obtain

$$\begin{aligned} \left[ \frac{\partial}{\partial t} + 2\nu k^2 \right] \tilde{V}_{ij}(\mathbf{k}, t, t) = & -\frac{i}{2} \left( \frac{2\pi}{L} \right)^6 \tilde{P}_{imn}(\mathbf{k}) \\ & \times \sum_{\mathbf{p}} \sum_{\mathbf{q}} \left[ \overline{\tilde{u}_m^{(0)}(-\mathbf{p}, t|\mathbf{k}, \mathbf{p}, \mathbf{q})\tilde{u}_n^{(0)}(-\mathbf{q}, t|\mathbf{k}, \mathbf{p}, \mathbf{q})\tilde{u}_j^{(0)}(-\mathbf{k}, t|\mathbf{k}, \mathbf{p}, \mathbf{q})} \right. \\ & + 2 \overline{\tilde{u}_m^{(0)}(-\mathbf{p}, t|\mathbf{k}, \mathbf{p}, \mathbf{q})\tilde{u}_n^{(1)}(-\mathbf{q}, t|\mathbf{k}, \mathbf{p}, \mathbf{q})\tilde{u}_j^{(0)}(-\mathbf{k}, t|\mathbf{k}, \mathbf{p}, \mathbf{q})} \\ & \left. + \overline{\tilde{u}_m^{(0)}(-\mathbf{p}, t|\mathbf{k}, \mathbf{p}, \mathbf{q})\tilde{u}_n^{(0)}(-\mathbf{q}, t|\mathbf{k}, \mathbf{p}, \mathbf{q})\tilde{u}_j^{(1)}(-\mathbf{k}, t|\mathbf{k}, \mathbf{p}, \mathbf{q})} \right] \\ & + (i \leftrightarrow j, \mathbf{k} \rightarrow -\mathbf{k}). \end{aligned} \quad (3.10)$$

Here a set of removed wavenumbers have been selected as  $(\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) = (\mathbf{k}, \mathbf{p}, \mathbf{q})$  in the summand of the RHS of the above equation, and higher-order terms of DI-field have been neglected (assumption-1). Applying procedures-2–4 to (3.10), we obtain the time-evolution equation of the two-point velocity correlation function

as

$$\begin{aligned} \left[ \frac{\partial}{\partial t} + 2\nu k^2 \right] \tilde{Q}_{ij}(\mathbf{k}, t, t) = & \frac{1}{2} \frac{(2\pi)^9}{L^6} \tilde{P}_{ia}(\mathbf{k}) \\ & \times \left\{ \tilde{P}_{amn}(\mathbf{k}) \sum_{\mathbf{p}} \sum_{\mathbf{q}} \int_{t_0}^t dt' \tilde{Q}_{mb}(-\mathbf{p}, t, t') \left[ 2 \left( q_b \tilde{G}_{nc}(-\mathbf{q}, t, t') + q_c \tilde{G}_{nb}(-\mathbf{q}, t, t') \right) \tilde{Q}_{jc}(-\mathbf{k}, t, t') \right. \right. \end{aligned}$$

$$\begin{aligned}
& + \left( k_b \tilde{G}_{jc}(-\mathbf{k}, t, t') + k_c \tilde{G}_{jb}(-\mathbf{k}, t, t') \right) \tilde{Q}_{nc}(-\mathbf{q}, t, t') \Big] \\
& + (a \leftrightarrow j, \mathbf{k} \rightarrow -\mathbf{k}) \Big\}, \tag{3.11}
\end{aligned}$$

(see Appendix D for derivation).

In a similar way, we can express the governing equation of the two-point-two-time Lagrangian velocity correlation function in terms of only the two-point Lagrangian velocity correlation and the response functions. Employing procedures-1—4 for (2.27), we obtain

$$\left[ \frac{\partial}{\partial t} + \nu k^2 \right] \tilde{Q}_{ij}(\mathbf{k}, t, t') = -2 \left( \frac{2\pi}{L} \right)^3 \tilde{P}_{id}(\mathbf{k}) \sum_{\mathbf{p}} \sum_{\mathbf{q}} \frac{q_a q_b q_c q_d}{q^2} \int_{t'}^t dt'' \tilde{Q}_{ab}(\mathbf{p}, t, t'') \tilde{Q}_{cj}(\mathbf{k}, t, t'). \tag{3.12}$$

Finally, for the Lagrangian velocity response function, it follows from (2.35) that

$$\begin{aligned}
\left[ \frac{\partial}{\partial t} + \nu k^2 \right] \tilde{G}_{ij}(\mathbf{k}, t, t') = & \\
& -2 \left( \frac{2\pi}{L} \right)^3 \sum_{\mathbf{p}} \sum_{\mathbf{q}} \frac{k_i k_m k_n}{k^2} \int_{t'}^t dt'' \left[ q_b \tilde{G}_{nc}(-\mathbf{q}, t, t') + q_c \tilde{G}_{nb}(-\mathbf{q}, t, t') \right] \tilde{G}_{cj}(\mathbf{k}, t'', t') \tilde{Q}_{mb}(-\mathbf{p}, t, t'') \\
& -2 \left( \frac{2\pi}{L} \right)^3 \sum_{\mathbf{p}} \sum_{\mathbf{q}} \frac{q_a q_b q_c q_i}{q^2} \int_{t'}^t dt'' \tilde{Q}_{ab}(\mathbf{p}, t, t'') \tilde{G}_{cj}(\mathbf{k}, t, t'). \tag{3.13}
\end{aligned}$$

Equations (3.11)—(3.13) constitute a closed system of equations for the time-evolution of the Lagrangian velocity correlation and the response functions. It should be remarked that this system of equations agrees exactly with that obtained before by the use of LRA (Kaneda 1981). It is interesting to observe that completely different methods lead to an identical result.

Equations (3.11)—(3.13) were derived for a homogeneous turbulence. If turbulence is isotropic as well as homogeneous, they reduce to

$$\begin{aligned}
\left[ \frac{\partial}{\partial t} + 2\nu k^2 \right] Q(k, t, t) = 2\pi \iint_{\Delta_k} dpdq k p q \hat{b}(k, p, q) \\
\times \int_{t_0}^t dt' Q(q, t, t') \left[ G(k, t, t') Q(p, t, t') - G(p, t, t') Q(k, t, t') \right], \tag{3.14}
\end{aligned}$$

$$\left[ \frac{\partial}{\partial t} + \nu k^2 + \hat{\eta}(k, t, t') \right] Q(k, t, t') = 0, \tag{3.15}$$

$$\left[ \frac{\partial}{\partial t} + \nu k^2 + \hat{\eta}(k, t, t') \right] G(k, t, t') = 0 \tag{3.16}$$

and

$$G(k, t, t) = 1 \tag{3.17}$$

respectively (for derivation, see Kaneda 1981), where we have taken the limit that the size  $L$  of a periodic box is infinity <sup>†</sup>. Functions  $\widehat{b}(k, p, q)$  and  $\widehat{\eta}(k, t, t')$  are respectively defined by

$$\widehat{b}(k, p, q) = \frac{1}{8k^4 p^2 q^2} \left[ k^2 - (p - q)^2 \right] \left[ (p + q)^2 - k^2 \right] \left[ (2p^2 - q^2) k^2 - (p^2 - q^2) q^2 \right] \quad (3.18)$$

and

$$\widehat{\eta}(k, t, t') = \frac{4}{3} \pi k^5 \int_0^\infty dp p^{\frac{10}{3}} J(p^{\frac{2}{3}}) \int_{t'}^t dt'' Q(kp, t, t'') \quad (3.19)$$

with

$$J(p) = \frac{3}{32p^5} \left[ \frac{(1 - p^3)^4}{2p^{\frac{2}{3}}} \log \frac{(1 + p^{\frac{2}{3}})}{|1 - p^{\frac{2}{3}}|} - \frac{1 + p^3}{3} (3p^6 - 14p^3 + 3) \right]. \quad (3.20)$$

It follows from (3.15)—(3.17) that

$$Q(k, t, t') = Q(k, t', t') G(k, t, t'). \quad (3.21)$$

Once single-time velocity correlation function  $Q(k, t, t)$  and response function  $G(k, t, t')$  are determined, two-time velocity correlation function  $Q(k, t, t')$  follows from the above equation. In homogeneous isotropic turbulence, therefore, it is sufficient to deal with the system of equations for  $Q(k, t, t)$  and  $G(k, t, t')$ . Note that using (3.14) and (3.21) we can write the equation for  $Q(k, t, t)$  as

$$\left[ \frac{\partial}{\partial t} + 2\nu k^2 \right] Q(k, t, t) = 2\pi \iint_{\Delta_k} dpdq kpq \widehat{b}(k, p, q) \times \int_{t_0}^t dt' G(k, t, t') G(p, t, t') G(q, t, t') Q(q, t', t') \left[ Q(p, t', t') - Q(k, t', t') \right]. \quad (3.22)$$

## 4 Stationary Turbulence

In this section, we consider how LDIA equations (3.16) and (3.22) behave in homogeneous isotropic stationary turbulence. Under the assumption of stationarity, it is shown that  $Q$  and  $G$  depend only on the difference between  $t$  and  $t'$ , so that we put

$$Q(k, t, t') = \check{Q}(k, t - t') \quad (4.1)$$

$$G(k, t, t') = \check{G}(k, t - t'). \quad (4.2)$$

Then, the single-time velocity correlation function is written as

$$Q(k, t, t) = \check{Q}(k, 0), \quad (4.3)$$

and (3.21) as

$$\check{Q}(k, t) = \check{Q}(k, 0) \check{G}(k, t). \quad (4.4)$$

<sup>†</sup>The content in the rest of this section and §4.1 is essentially same as that by Kaneda (1981, 1986). However, we recapitulate it because it is necessary for further analysis in the subsequent sections. Equations (4.5) and (4.6) are basic in §4.2, and (3.16), (3.22) and (4.13) are essential for the analysis in §5.

Introductions of the above relations into (3.16) yields

$$\frac{\partial}{\partial t} \log \check{Q}(k, t) = -\nu k^2 - \frac{3}{4} \pi k^5 \int_0^\infty dp p^{\frac{10}{3}} J(p^{\frac{2}{3}}) \int_0^t dt' \check{Q}(kp, t'). \quad (4.5)$$

On the other hand, integration from  $k_0$  to infinity with respect to  $k$  of (3.22) multiplied by  $2\pi k^2$  leads to

$$4\pi\nu \int_0^{k_0} dk k^4 \check{Q}(k, 0) = \epsilon - 4\pi^2 \int_{k_0}^\infty dk \iint_{\Delta_k} dpdq k^3 pq \widehat{b}(k, p, q) \\ \times \int_0^\infty dt' \check{Q}(k, t') \check{Q}(p, t') \check{Q}(q, t') \left[ \check{Q}(k, 0)^{-1} - \check{Q}(p, 0)^{-1} \right], \quad (4.6)$$

where use has been made of

$$4\pi\nu \int_0^\infty dk k^4 \check{Q}(k, 0) = \epsilon, \quad (4.7)$$

and the initial time has been put at  $t_0 = -\infty$ . Equations(4.5) and (4.6) constitute the basic equations in the stationary turbulence.

#### 4.1 Kolmogorov constant

To examine the behavior of these equations in the inertial subrange, we take the limit of  $\nu \rightarrow 0$ . Then, (4.5) and (4.6) are reduced respectively to

$$\frac{\partial}{\partial t} \log \check{Q}(k, t) = -\frac{3}{4} \pi k^5 \int_0^\infty dp p^{\frac{10}{3}} J(p^{\frac{2}{3}}) \int_0^t dt' \check{Q}(kp, t') \quad (4.8)$$

and

$$\epsilon = 4\pi^2 \int_{k_0}^\infty dk \iint_{\Delta_k} dpdq k^3 pq \widehat{b}(k, p, q) \\ \times \int_0^\infty dt' \check{Q}(k, t') \check{Q}(p, t') \check{Q}(q, t') \left[ \check{Q}(k, 0)^{-1} - \check{Q}(p, 0)^{-1} \right]. \quad (4.9)$$

These equations permit such similar solutions as

$$\check{Q}(k, t) = \frac{K}{2\pi} \epsilon^{\frac{2}{3}} k^{-\frac{11}{3}} \check{Q}^\dagger(K^{\frac{1}{2}} \epsilon^{\frac{1}{3}} k^{\frac{2}{3}} t), \quad (4.10)$$

with

$$\check{Q}^\dagger(0) = 1. \quad (4.11)$$

In terms of  $\check{Q}^\dagger$ , we can rewrite (4.8) and (4.9) as

$$\frac{d}{dt} \log \check{Q}^\dagger(t) = - \int_0^\infty dp J(p) \int_0^t dt' \check{Q}^\dagger(pt') \quad (4.12)$$

and

$$K^{-\frac{2}{3}} = \int_1^\infty dk \int_0^1 dp \int_{\max\{k-p, p\}}^{k+p} dq k^3 pq \int_0^\infty dt \check{Q}^\dagger(k^{\frac{2}{3}} t) \check{Q}^\dagger(p^{\frac{2}{3}} t) \check{Q}^\dagger(q^{\frac{2}{3}} t) \\ \times \left\{ \left[ \widehat{b}(k, p, q) + \widehat{b}(k, q, p) \right] (pq)^{-\frac{11}{3}} - \left[ \widehat{b}(t, p, q) q^{-\frac{11}{3}} + \widehat{b}(t, q, p) p^{-\frac{11}{3}} \right] k^{-\frac{11}{3}} \right\}. \quad (4.13)$$

The energy spectrum is represented, from (2.43) and (4.3), by †

$$E(k) = 2\pi k^2 \check{Q}(k, 0) = K \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}. \quad (4.14)$$

Hence,  $K$  is actually the Kolmogorov constant (cf. (2.46)). We solved (4.12) numerically with boundary condition (4.11). The result is shown in figure 1, which is same as figure 1 in Kaneda (1986). Here,  $G(k, t, t') = \check{Q}^\dagger(K^{\frac{1}{2}} \epsilon^{\frac{1}{3}} k^{\frac{2}{3}}(t - t'))$ . From (4.13) and the above numerical result for  $\check{Q}^\dagger$ , we can evaluate Kolmogorov constant to be 1.722 (Kaneda 1986).

## 4.2 Energy spectrum

Here, we solve (4.5) and (4.6) in the entire universal range. We express  $\check{Q}$  in terms of non-dimensional functions  $\check{Q}^\dagger$  of non-dimensional wavenumber  $\kappa$  and non-dimensional time  $\tau$  as

$$\check{Q}(k, t) = \frac{1}{2\pi} K \epsilon^{\frac{2}{3}} k^{-\frac{11}{3}} \check{Q}^\dagger(\kappa, \tau) \quad (4.15)$$

with

$$\check{Q}^\dagger(0, 0) = 1, \quad (4.16)$$

$$\kappa = K^{-\frac{3}{8}} \epsilon^{-\frac{1}{4}} \nu^{\frac{3}{4}} k, \quad \tau = K^{\frac{1}{2}} \epsilon^{\frac{1}{3}} k^{\frac{2}{3}} t. \quad (4.17)$$

Then, (4.5) is converted into

$$\frac{\partial^2}{\partial \tau^2} \log \check{Q}^\dagger(\kappa, \tau) = - \int_0^\infty dp J(p) \check{Q}^\dagger(\kappa p^{\frac{3}{2}}, \tau p) \quad (4.18)$$

with

$$\frac{\partial}{\partial \tau} \log \check{Q}^\dagger(\kappa, \tau) \Big|_{\tau=0} = -\kappa^{\frac{4}{3}}, \quad (4.19)$$

while (4.6) is written as

$$\begin{aligned} \check{Q}^\dagger(\kappa, 0) &= \frac{1}{2} \iint_{\Delta_\kappa} dpdq (pq)^{-\frac{8}{3}} \kappa^{-1} \widehat{b}(\kappa, p, q) \\ &\times \int_0^\infty dt' \check{Q}^\dagger(\kappa, \kappa^{\frac{2}{3}} t') \check{Q}^\dagger(p, p^{\frac{2}{3}} t') \check{Q}^\dagger(q, q^{\frac{2}{3}} t') \left[ \kappa^{\frac{11}{3}} \check{Q}^\dagger(\kappa, 0)^{-1} - p^{\frac{11}{3}} \check{Q}^\dagger(p, 0)^{-1} \right]. \end{aligned} \quad (4.20)$$

Note that the energy spectrum is expressed as

$$E(k) = K \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}} \check{Q}^\dagger(\kappa, 0). \quad (4.21)$$

Hence, by searching numerically † the solution of  $\check{Q}^\dagger$  which satisfies (4.18)—(4.20) and (4.16), we can determine the functional form of the energy spectrum through (4.21). The result is shown in figures 2 for (a) the compensated three-dimensional energy spectrum  $E(k, t)/(\epsilon^{\frac{2}{3}} k^{-\frac{5}{3}})$ , (b) the one-dimensional longitudinal energy spectrum

$$E_{\parallel}(k) = \frac{1}{2} \int_k^\infty dk' \left( 1 - \frac{k^2}{k'^2} \right) \frac{E(k')}{k'} \quad (4.22)$$

†In this section we omit the time argument in  $E(k, t)$  for brevity, which is independent of time.

†We have used an iteration method for (4.18) and (4.19), and the Newton-Raphson method for (4.20).

and (c) the compensated one-dimensional longitudinal energy spectrum  $E_{\parallel}(k)/(\epsilon^{2/3}k^{-5/3})$ . Agreement with a measurement of tidal channel (Grant *et al.* 1962), which is plotted by solid and white circles, and other experimental data is excellent.

### 4.3 Skewness factor of velocity derivative

Although only the second-order moments are dealt with in the present closure theory, the skewness factor of longitudinal derivative of velocity, which is a third-order moment, can be calculated with the help of Kármán-Howarth equation for homogeneous isotropic turbulence as

$$S = \frac{\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^3}}{\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2}^{\frac{3}{2}}} = -\frac{3\sqrt{30}\nu}{7} \int_0^\infty dk k^4 E(k) \left/ \left( \int_0^\infty dk k^2 E(k) \right)^{\frac{3}{2}} \right. . \quad (4.23)$$

Using the result of numerical computation for  $\tilde{Q}^\dagger$ , we can perform the integrations in the RHS of (4.23) to find

$$S = -0.66 . \quad (4.24)$$

Comparable results have been obtained by other statistical theories (see Gotoh, Kaneda & Bekki 1988 for references). Note that this factor is independent of Reynolds number. Many turbulence measurements, on the contrary, show that it may increase in magnitude with Reynolds number which expresses the intermittency of turbulence, though it is not conclusive because fluctuations in the data are quite large. It varies from  $-0.6$  to  $-1$  in range  $10^3 < R_\lambda < 2 \times 10^4$  (see Van Atta & Antonia 1980). Present result (4.24) is consistent with observation within this range of the Reynolds number.

### 4.4 Eddy-viscosity

One of the main difficulties in analysing the structure of developed turbulence at high Reynolds numbers may be attributed to the enormously wide range of relevant active motions. The ratio between the largest and smallest motions, i.e. the energy-containing and the energy-dissipation scales increases in proportion to the power of three quarters of the Reynolds number. It is hard to resolve the smallest excited scales of developed turbulence of practical interest even on a present-day supercomputer. The so-called large-eddy simulation (see Lesieur & Métais 1996 for a review) may be one of the most promising methods to analyse the turbulence dynamics in which only large-scale components of motion are explicitly simulated, while the effects on the resolved-scale components of the sub-grid-scale motions are implicitly taken into account as eddy-viscosity. It is the purpose of the present section to examine the property of the eddy-viscosity in the framework of closure theories (Kraichnan 1976).

For the discussion of the eddy-viscosity it is convenient to rewrite energy equation (3.22) as

$$\frac{\partial}{\partial t} E(k, t) = -2\nu k^2 E(k, t) + T(k, t) , \quad (4.25)$$

where

$$T(k, t) = \iint_{\Delta_k} dpdq T^{(3)}(k, p, q, t) , \quad (4.26)$$



and

$$\begin{aligned}
T^{(3)}(k, p, q, t) = & 2\pi^2 k^3 pq \int_{-\infty}^t dt' G(k, t, t') G(p, t, t') G(q, t, t') \\
& \times \left[ \left( \widehat{b}(k, p, q) + \widehat{b}(k, q, p) \right) Q(p, t', t') Q(p, t', t') \right. \\
& \quad \left. - \left( \widehat{b}(k, p, q) Q(q, t', t') + \widehat{b}(k, q, p) Q(p, t', t') \right) Q(k, t', t') \right] \quad (4.27)
\end{aligned}$$

is the triad energy transfer function. Here, the initial time has been set at  $t_0 = -\infty$ . The first term on the RHS of (4.25) represents the dissipation of energy by molecular viscosity, while the second the energy transfer to the modal energy of wavenumber  $k$  from all the other modal energies through non-linear interactions. Eddy-viscosity is defined as the molecular viscosity counterpart when the contribution from the sub-grid scale components is expressed like the first term (see (4.29) below). In figure 3, the triad energy transfer function is shown in the case that all of three wavenumbers,  $k, p, q$  are in the inertial range. Positive regions are shaded. Sharp peaks at the corners of the rectangular domain represent strong non-local triad interactions (see Ohkitani & Kida 1992).

Let us denote by  $k_c$  the cut-off wavenumber which is a reciprocal of the dividing length of the resolved and the sub-grid scales (e.g. the mesh-size in a numerical simulation of turbulence). We divide energy transfer function  $T(k, t)$ , which is composed of many triad interactions, into two parts as

$$T(k, t) = T^<(k, t|k_c) + T^>(k, t|k_c), \quad (4.28)$$

where  $T^<$  denotes the contribution from the resolved scales, i.e. an integral of (4.26) over  $p$  and  $q \leq k_c$ , while  $T^>$  that from the sub-grid scales, i.e. an integral for  $p$  or  $q > k_c$ . If we write the second term of (4.28) formally as

$$T^>(k, t|k_c) = -2k^2 \nu_T(k, t|k_c) E(k, t), \quad (4.29)$$

then  $\nu_T(k, t|k_c)$  is regarded as the eddy-viscosity since it is a molecular viscosity counterpart (cf. the first term of (4.25)). Notice that the eddy-viscosity varies depending upon the relevant wavenumber contrary to the molecular viscosity. In the following we will estimate the wavenumber dependence of the eddy-viscosity for both wavenumbers  $k$  and  $k_c$  lying in the inertial range of a stationary turbulence.

On substitution of (4.10) and (4.14) in (4.29), we obtain

$$\nu_T(k, t|k_c) = K^{\frac{1}{2}} \epsilon^{\frac{1}{3}} k_c^{-\frac{4}{3}} I(k/k_c) \quad (4.30)$$

with

$$\begin{aligned}
I(k/k_c) = & -\frac{1}{3} (k/k_c)^{-\frac{4}{3}} \int_0^\infty dt \int_{(k/k_c)^{-1}t}^\infty dp \int_{p-t}^p dq t^3 pq \bar{Q}^\dagger(t^{\frac{2}{3}}) \bar{Q}^\dagger(p^{\frac{2}{3}}) \bar{Q}^\dagger(q^{\frac{2}{3}}) \\
& \times \left[ (pq)^{-\frac{11}{3}} \left( \widehat{b}(t, p, q) + \widehat{b}(t, q, p) \right) - (tp)^{-\frac{11}{3}} \widehat{b}(t, q, p) - (tq)^{-\frac{11}{3}} \widehat{b}(t, p, q) \right]. \quad (4.31)
\end{aligned}$$

Integration of (4.31) is carried out using the numerical solution of  $\tilde{Q}^\dagger$  obtained already in §4.1. At two extreme values of  $k/k_c$  we find  $I(0) = 0.170$  and  $I(1) = 0.970$ . The eddy-viscosity thus determined is shown in figure 4. At wavenumbers much smaller than the cut-off wavenumber ( $k \ll k_c$ ) the eddy-viscosity is nearly constant  $\nu_\tau \approx 0.224\epsilon^{\frac{1}{3}}k_c^{-\frac{4}{3}}$ . The variation is less than 50% up to  $k \approx 0.6k_c$ . However, as the wavenumber concerned approaches at the cut-off wavenumber, the eddy-viscosity increases more and more rapidly. At the cut-off wavenumber it is 5.7 times as large as that at small wavenumbers. This sharp rising-up of the eddy-viscosity near the cut-off wavenumber, which is caused by strong non-local triad interactions such as  $p \ll k \approx q$  or  $q \ll k \approx p$  (see figure 3), is also observed in other closure theories (Kraichnan 1976), large-eddy simulations (Lesieur and Rogallo 1989), and direct numerical simulations (Domaradzki *et al.* 1993).

## 5 Decaying Turbulence

In this section, we consider a homogeneous isotropic free decaying turbulence with LDIA equations. The initial value problem of LDIA equations, which are identical to LRA equations, was investigated numerically by Gotoh *et al.* (1988). Here, instead, we seek for solutions of these equations in a similarity form. It can be shown that there is in general no similar solutions with a single similarity law that hold over the entire wavenumber range. Therefore, as was done by one of the present authors for the modified zero-fourth order cumulant approximation (Kida 1981), we seek for similar solutions which obey different similarity laws in two wavenumber ranges; namely, they are characterized by total energy  $\mathcal{E}(t)$  and energy dissipation rate  $\epsilon(t)$  in the energy-containing range, while by  $\epsilon(t)$  and  $\nu$  in the universal range.

### 5.1 Similarity form in energy-containing range

It is easy to show that LDIA equations in the universal range are same irrespective of decaying or stationary turbulence (see Appendix F). LDIA equations in energy-containing range are derived as follows. We start with introducing index  $\zeta$  which characterizes large-scale structure by

$$E(k, t) \propto k^\zeta \quad (\text{as } k \rightarrow 0). \quad (5.1)$$

If we require that the velocity correlation tensor  $\tilde{V}_{ij}$  does not diverge at the origin, then  $\zeta$  is limited from below by 2. On the other hand, even if  $\zeta$  is greater than 4 at the initial instant,  $\zeta$  changes immediately to 4 (see (5.9) below). We restrict, therefore, ourselves in range

$$2 \leq \zeta \leq 4. \quad (5.2)$$

It is convenient for the following analysis to introduce a new variable  $\hat{E}_\zeta(k, t)$  by

$$E(k, t) = \hat{E}_\zeta(k, t) k^\zeta, \quad (5.3)$$

where

$$0 < \hat{E}_\zeta(0, t) < \infty. \quad (5.4)$$

Then, (2.43) and (3.22) lead to

$$\left[ \frac{\partial}{\partial t} + 2\nu k^2 \right] \widehat{E}_\zeta(k, t) = \iint_{\Delta_k} dpdq k^{3-\zeta} p q^{\zeta-1} \widehat{b}(k, p, q) \\ \times \int_0^t dt' G(k, t, t') G(p, t, t') G(q, t, t') \widehat{E}_\zeta(q, t') \left[ p^{\zeta-2} \widehat{E}_\zeta(p, t') - k^{\zeta-2} \widehat{E}_\zeta(k, t') \right], \quad (5.5)$$

where we have put  $t_0 = 0$ . Equation (3.16) for the response function is rewritten as

$$\left[ \frac{\partial}{\partial t} + 2\nu k^2 + \frac{2}{3} k^{\zeta+3} \int_0^\infty dp p^{\frac{4}{3}+\zeta} J(p^{\frac{2}{3}}) \int_{t'}^t dt'' \widehat{E}_\zeta(kp, t'') G(kp, t, t'') \right] G(k, t, t') = 0. \quad (5.6)$$

If we demand that the turbulence in the energy-containing range is characterized only by  $k$ ,  $\mathcal{E}(t)$  and  $\epsilon(t)$ , functions  $\widehat{E}_\zeta$  and  $G$  may be written, by the dimensional analysis, as

$$\widehat{E}_\zeta(k, t) = \mathcal{E}(t)^{\frac{1}{2}(3\zeta+5)} \epsilon(t)^{-\zeta-1} E_\zeta^\dagger \left( \mathcal{A} \mathcal{E}(t)^{\frac{3}{2}} \epsilon(t)^{-1} k \right), \quad (5.7)$$

$$G(k, t, t') = G^\dagger \left( \mathcal{A} \mathcal{E}(t)^{\frac{3}{2}} \epsilon(t)^{-1} k, \mathcal{A} \mathcal{E}(t')^{\frac{3}{2}} \epsilon(t')^{-1} k \right), \quad (5.8)$$

where  $\mathcal{A}$  is a non-dimensional constant, which will be determined later so that the final expression may be simple (see (5.19) below). By substituting these similarity forms into (5.5) and (5.6), we find that the viscous term is smaller than the time-derivative and the nonlinear terms by factor  $Re^{-1}$ , where  $Re$  is the Reynolds number ( $Re = \mathcal{E}^2/\epsilon\nu$ ). Hence, in the limit of large Reynolds numbers, the viscous terms can be neglected.

By neglecting the viscous term and taking the limit of  $k \rightarrow 0$  in (5.5), we obtain <sup>†</sup>

$$\frac{d}{dt} \widehat{E}_\zeta(0, t) = \begin{cases} 0 & (2 \leq \zeta < 4), \\ \frac{14}{15} \int_0^\infty dp \int_0^t dt' \left[ p^3 G(p, t, t') \widehat{E}_4(p, t') \right]^2 & (\zeta = 4). \end{cases} \quad (5.9a)$$

$$\frac{d}{dt} \widehat{E}_\zeta(0, t) = \begin{cases} 0 & (2 \leq \zeta < 4), \\ \frac{14}{15} \int_0^\infty dp \int_0^t dt' \left[ p^3 G(p, t, t') \widehat{E}_4(p, t') \right]^2 & (\zeta = 4). \end{cases} \quad (5.9b)$$

Substitution of similarity forms (5.7) and (5.8) into (5.9a) leads to that  $\mathcal{E}(t)$  and  $\epsilon(t)$  must be power functions of  $t$  as

$$\mathcal{E}(t) = \mathcal{E}_0 t^{-\sigma}, \quad (5.10)$$

where

$$\sigma = \frac{2(\zeta+1)}{\zeta+3} \quad (2 \leq \zeta < 4). \quad (5.11)$$

Here, we have used relation

$$\epsilon(t) = -\frac{d\mathcal{E}}{dt}. \quad (5.12)$$

For  $\zeta = 4$ , on the other hand, it may not be able to be proved that  $\mathcal{E}(t)$  is a power function of  $t$ . However, if we assume (5.10), then  $\sigma$  may be evaluated by

$$\frac{(2-\sigma)(10-7\sigma)}{2\sigma^2} E_4^\dagger(0) = \frac{28}{15} \mathcal{A}^{-7} \int_0^\infty dp \int_0^1 dt t^{(20-13\sigma)/(2-\sigma)} \left[ p^3 G^\dagger(p, pt) E_4^\dagger(pt) \right]^2 \quad (\zeta = 4). \quad (5.13)$$

Inversely, it is easy to show that (5.10), (5.11) and (5.13) are sufficient conditions for the existence of similar solutions (5.7) and (5.8).

<sup>†</sup>This equation tells us that the Birkhoff constant, which is  $\widehat{E}_2(0, t)$ , is invariant in time, but the Loitsiansky integral, which is equal to  $\widehat{E}_4(0, t)$ , is not, as pointed out before in other closure theories of turbulence (Lesieur & Schertzer 1978 ; Kida 1981).

By making use of (5.10) and (5.12), we can rewrite (5.7) and (5.8) as

$$\widehat{E}_\zeta(k, t) = \mathcal{E}_0^{(\zeta+3)/2} \sigma^{-\zeta-1} t^{(-\sigma\zeta-3\sigma+2\zeta+2)/2} E_\zeta^\dagger \left( \mathcal{A} \mathcal{E}_0^{\frac{1}{2}} \sigma^{-1} t^{-\frac{\sigma}{2}+1} k \right), \quad (5.14)$$

$$G(k, t, t') = G^\dagger \left( \mathcal{A} \mathcal{E}_0^{\frac{1}{2}} \sigma^{-1} t^{-\frac{\sigma}{2}+1} k, \mathcal{A} \mathcal{E}_0^{\frac{1}{2}} \sigma^{-1} t'^{-\frac{\sigma}{2}+1} k \right). \quad (5.15)$$

In order to make the final equations simpler, we further replace  $E^\dagger$ ,  $G^\dagger$  and  $\sigma$  with

$$E_\zeta^\dagger(x) = \mathcal{A}^{\zeta+3} \left( \frac{b}{2-3b} \right)^2 x^{-\zeta-\frac{5}{3}} E^\dagger(x^{\frac{2}{3}}), \quad (5.16)$$

$$G^\dagger(x, x') = G^\dagger(x^{\frac{2}{3}}, x'^{\frac{2}{3}}) \quad (5.17)$$

and

$$\sigma = 2 - 3b, \quad (5.18)$$

respectively. If we choose

$$\mathcal{A} = \frac{2-3b}{\mathcal{E}_0^{\frac{1}{2}} b^{\frac{5}{3}}}, \quad (5.19)$$

then (5.14) and (5.15) are rewritten as

$$\widehat{E}_\zeta(k, t) = t^{2(b-1)} k^{-\zeta-\frac{5}{3}} E^\dagger(b^{-1} t^b k^{\frac{2}{3}}), \quad (5.20)$$

$$G(k, t, t') = G^\dagger(b^{-1} t^b k^{\frac{2}{3}}, b^{-1} t'^b k^{\frac{2}{3}}), \quad (5.21)$$

and (5.5) and (5.6) as

$$\begin{aligned} \frac{\partial}{\partial t} \left( t^{2-\frac{2}{3}} E^\dagger(t) \right) &= t^{\frac{1}{3}-1} \iint_{\Delta_1} dpdq pq \widehat{b}(1, p, q) \int_0^t dt' t'^{3-\frac{2}{3}} \\ &\times G^\dagger(t, t') G^\dagger(p^{\frac{2}{3}} t, p^{\frac{2}{3}} t') G^\dagger(q^{\frac{2}{3}} t, q^{\frac{2}{3}} t') q^{-\frac{11}{3}} E^\dagger(q^{-\frac{11}{3}} t') \left[ q^{-\frac{11}{3}} E^\dagger(p^{-\frac{11}{3}} t') - E^\dagger(t') \right] \end{aligned} \quad (5.22)$$

and

$$\frac{\partial}{\partial t} \log G^\dagger(t, t') = -t^{\frac{1}{3}-1} \int_0^\infty dp J(p) \int_{t'}^t dt'' t''^{1-\frac{1}{3}} E^\dagger(pt'') G^\dagger(pt, pt''), \quad (5.23)$$

respectively. Conditions (5.11) and (5.13) for  $\sigma$  are also rewritten as

$$b = \begin{cases} \frac{4}{3(\zeta+3)} & (2 \leq \zeta < 4), \quad (5.24a) \\ \left[ \frac{21}{4} - \frac{7}{10 E_0^\dagger} \int_0^\infty dk k^{-\frac{11}{3}} \int_0^1 dt t^{3-\frac{2}{3}} \left( G^\dagger(k, kt) Q^\dagger(kt) \right)^2 \right]^{-1} & (\zeta = 4). \quad (5.24b) \end{cases}$$

Here,  $E_0^\dagger$  is defined by

$$E^\dagger(x) = E_0^\dagger x^{\zeta+\frac{5}{3}} \quad (\text{as } x \rightarrow 0), \quad (5.25)$$

which follows from (5.4) and (5.16).

Finally, we consider the boundary conditions for  $E^\dagger$  and  $G^\dagger$ . Energy spectrum  $E(k, t)$  is expressed, from (5.3) and (5.20), as

$$E(k, t) = t^{2(b-1)} k^{-\frac{5}{3}} E^\dagger(b^{-1} t^b k^{\frac{2}{3}}). \quad (5.26)$$

If we demand that  $E(k, t)$  at large wavenumbers in the inertial range be connected smoothly with the  $k^{-5/3}$  spectrum which is realized at lower wavenumbers in the universal range, then  $E^\dagger$  must be approach to a constant, which we can choose, without loss of generality, as unity, i.e.

$$E^\dagger(\infty) = 1 . \quad (5.27)$$

Then, we have

$$E(k, t) = k^{-\frac{5}{3}} t^{-2+2b} = \left( \mathcal{E}_0 (2 - 3b) \right)^{-\frac{2}{3}} \epsilon(t)^{\frac{2}{3}} k^{-\frac{5}{3}} \quad \left( = K \epsilon(t)^{\frac{2}{3}} k^{-\frac{5}{3}}, \text{ say} \right) \quad (\text{as } k \rightarrow \infty) , \quad (5.28)$$

where use have been made of (5.10), (5.12), (5.18), (5.19) and (5.26). Integration of (5.26) with respect to  $k$  together with  $\mathcal{E}(t) = \int_0^\infty dk E(k, t)$  gives

$$\mathcal{E}_0 = \frac{3}{2b} \int_0^\infty dx x^{-2} E^\dagger(x) . \quad (5.29)$$

Hence, the Kolmogorov constant is represented, from (5.28) and (5.29), as

$$K = \left[ \frac{3}{2} \left( \frac{2}{b} - 3 \right) \int_0^\infty dx x^{-2} E^\dagger(x) \right]^{-\frac{2}{3}} . \quad (5.30)$$

A boundary condition for  $G^\dagger$  follows from initial condition (3.17) as

$$G^\dagger(x, x) = 1 . \quad (5.31)$$

Thus, we have obtained a system of integro-differential equations (5.22)—(5.24) to be solved with boundary conditions (5.27) and (5.31).

## 5.2 Kolmogorov constant

In the limit that  $t' \rightarrow \infty$ , equation (5.23) of the response function becomes

$$\frac{\partial}{\partial t} \log G^\dagger(t, t') = - \int_0^\infty dp J(p) \int_{t'}^t dt'' G^\dagger(pt, pt'') \quad (\text{as } t' \rightarrow \infty) , \quad (5.32)$$

where use has been made of

$$\int_{t'}^t dt'' t''^{1-\frac{1}{b}} E^\dagger(pt'') G^\dagger(pt, pt'') = t^{1-\frac{1}{b}} \int_{t'}^t dt'' G^\dagger(pt, pt'') \quad (\text{as } t' \rightarrow \infty) . \quad (5.33)$$

Equation (5.33) with (5.31) permits such a solution as

$$G^\dagger(t, t') = G_\infty^\dagger(t - t') , \quad (5.34)$$

where  $G_\infty^\dagger$  obeys

$$\frac{d}{dt} \log G_\infty^\dagger(t) = - \int_0^\infty dp J(p) \int_0^t dt' G_\infty^\dagger(pt') , \quad (5.35)$$

with boundary condition

$$G_\infty^\dagger(0) = 1 . \quad (5.36)$$

Notice that (5.35) is identical to (4.12). The functional form of  $G_{\infty}^{\dagger}$  does, therefore, coincide with that of  $\tilde{Q}^{\dagger}$ , which is the response function for stationary case (figure 1) <sup>†</sup>.

Energy equation (5.22), on the other hand, is reduced to (see Appendix G for derivation)

$$\frac{3}{2} \left( \frac{2}{b} - 3 \right) \int_0^{\infty} dx x^{-2} E^{\dagger}(x) = \int_1^{\infty} dt \int_0^1 dp \int_{\max\{t-p, p\}}^{t+p} dq t^3 pq \int_0^{\infty} dt' G_{\infty}^{\dagger}(t^{\frac{2}{3}} t') G_{\infty}^{\dagger}(p^{\frac{2}{3}} t') G_{\infty}^{\dagger}(q^{\frac{2}{3}} t') \\ \times \left\{ \left[ \widehat{b}(t, p, q) + \widehat{b}(t, q, p) \right] (pq)^{-\frac{11}{3}} - \left[ \widehat{b}(t, p, q) q^{-\frac{11}{3}} + \widehat{b}(t, q, p) p^{-\frac{11}{3}} \right] t^{-\frac{11}{3}} \right\}. \quad (5.37)$$

Hence, it follows from (5.30) and (5.37) that Kolmogorov constant  $K$  is represented as

$$K^{-\frac{2}{3}} = \int_1^{\infty} dt \int_0^1 dp \int_{\max\{t-p, p\}}^{t+p} dq t^3 pq \int_0^{\infty} dt' G_{\infty}^{\dagger}(t^{\frac{2}{3}} t') G_{\infty}^{\dagger}(p^{\frac{2}{3}} t') G_{\infty}^{\dagger}(q^{\frac{2}{3}} t') \\ \times \left\{ \left[ \widehat{b}(t, p, q) + \widehat{b}(t, q, p) \right] (pq)^{-\frac{11}{3}} - \left[ \widehat{b}(t, p, q) q^{-\frac{11}{3}} + \widehat{b}(t, q, p) p^{-\frac{11}{3}} \right] t^{-\frac{11}{3}} \right\}. \quad (5.38)$$

By remembering that  $G_{\infty}^{\dagger}$  is identical to  $\tilde{Q}^{\dagger}$  and comparing (4.13) with (5.38), we can conclude that Kolmogorov constant in decaying turbulence is same as that in stationary turbulence, i.e.  $K = 1.722$ .

### 5.3 Two-similarity-range solution

Equations (5.22)—(5.24) for the energy-containing range are solved numerically under boundary conditions (5.27) and (5.31) for two extreme cases of index  $\zeta$ , i.e.  $\zeta = 2$  and 4. In the case of  $\zeta = 2$ , parameter  $b$  is fixed by (5.24a) as  $b = \frac{4}{15}$ . Equations (5.22) and (5.23) are then solved iteratively as described in Appendix H. The response function and the energy spectrum function thus obtained are plotted in figures 5(a) and (b), respectively <sup>‡</sup>. The almost equi-distance in the contours of logarithmic levels in figure 5(a) indicates that the response function decays exponentially with response time  $\tau - \tau'$ . The characteristic decay time of the response function takes a non-zero finite value at initial time  $\tau' = 0$  and decreases with time  $\tau'$  in this non-dimensional response time  $\tau - \tau'$ . If we see it in the original physical time  $t$ , however, it is a monotonously increasing function of time starting with zero at the initial instant (figures are omitted). The energy spectrum, shown in figure 5(b), has asymptotic forms,  $\propto k^2$  and  $\propto k^{-\frac{5}{3}}$ , at small and large wavenumbers respectively as imposed as the boundary conditions.

In the case of  $\zeta = 4$ , on the other hand, parameter  $b$  is not known a priori but must be determined iteratively together with  $E^{\dagger}$  and  $G^{\dagger}$  (cf. (5.24b)). We obtained numerically that

$$b = 0.207 \quad (\text{for } \zeta = 4), \quad (5.39)$$

<sup>†</sup>Here, note the difference in the arguments of  $\tilde{Q}^{\dagger}$  and  $G^{\dagger}$ . The former is  $K^{\frac{1}{2}} \epsilon^{\frac{1}{3}} k^{\frac{2}{3}} (t - t')$ , while, the later  $K^{\frac{1}{2}} k^{\frac{2}{3}} (\epsilon(t)^{\frac{1}{3}} t - \epsilon(t')^{\frac{1}{3}} t')/b$  (see (4.10) and a footnote in §5.3). However, these two agree with each other in the limit of  $t' \rightarrow \infty$  because

$$K^{\frac{1}{2}} k^{\frac{2}{3}} (\epsilon(t)^{\frac{1}{3}} t) - \epsilon(t')^{\frac{1}{3}} t' / b = k^{\frac{2}{3}} (t^b - t'^b) / b,$$

which is shown from (5.10), (5.12), (5.18) and (5.26).

<sup>‡</sup>The response function and the energy spectrum function are respectively written in terms of solutions of (5.22) and (5.23) as

$$G(k, t, t') = G^{\dagger}(\tau/b, \tau'/b), \quad E(k, t) / (\mathcal{E}^{\frac{5}{2}} \epsilon^{-1}) = K (\mathcal{E}^{\frac{2}{3}} \epsilon^{-1} k)^{-\frac{5}{3}} E^{\dagger}(K^{\frac{1}{2}} (2b^{-1} - 3) (\mathcal{E}^{\frac{2}{3}} \epsilon^{-1} k)^{\frac{2}{3}}),$$

where use have been made of (4.17), (5.10), (5.12), (5.18), (5.21), (5.26) and (5.28). Here  $\tau' = K^{\frac{1}{2}} \epsilon(t')^{\frac{1}{3}} k^{\frac{2}{3}} t'$ .

which gives, through (5.18), the power exponent of the energy

$$\sigma = 1.38 \quad (\text{for } \zeta = 4) . \quad (5.40)$$

Interestingly enough, this value is exactly the same as the one predicted by the EDQNM theory (Lesieur & Schertzer 1978) as well as by the modified zero-fourth order cumulant approximation (Kida 1981). As shown in figure 6(a) and (b), the shape of the response function and the energy spectrum function are qualitatively same as for case  $\zeta = 2$ .

As mentioned in the beginning of §5, there are in general no overall similarity solutions with a single similarity law valid over the entire wavenumber range. Instead, the energy spectrum obeys different similarity laws in the energy-containing range and in the universal range, and it is connected smoothly between them. It follows from (5.10) and (5.12) that the normalized energy and wavenumber depend on time, in the respective wavenumber ranges, as

$$\left\{ \begin{array}{l} \frac{E(k, t)}{\mathcal{E}^{\frac{3}{2}} \epsilon^{-1}} \propto t^{\frac{3}{2}\sigma-1} \quad , \quad \frac{k}{\mathcal{E}^{-\frac{3}{2}} \epsilon} \propto t^{-\frac{1}{2}\sigma+1} \quad (\text{in the energy-containing range}) , \\ \frac{E(k, t)}{\epsilon^{\frac{1}{2}} \nu^{\frac{5}{4}}} \propto t^{\frac{1}{2}(\sigma+1)} \quad , \quad \frac{k}{\epsilon^{\frac{1}{4}} \nu^{-\frac{1}{4}}} \propto t^{\frac{1}{4}(\sigma+1)} \quad (\text{in the universal range}) . \end{array} \right. \quad (5.41a)$$

$$\left\{ \begin{array}{l} \frac{E(k, t)}{\mathcal{E}^{\frac{3}{2}} \epsilon^{-1}} \propto t^{\frac{3}{2}\sigma-1} \quad , \quad \frac{k}{\mathcal{E}^{-\frac{3}{2}} \epsilon} \propto t^{-\frac{1}{2}\sigma+1} \quad (\text{in the energy-containing range}) , \\ \frac{E(k, t)}{\epsilon^{\frac{1}{2}} \nu^{\frac{5}{4}}} \propto t^{\frac{1}{2}(\sigma+1)} \quad , \quad \frac{k}{\epsilon^{\frac{1}{4}} \nu^{-\frac{1}{4}}} \propto t^{\frac{1}{4}(\sigma+1)} \quad (\text{in the universal range}) . \end{array} \right. \quad (5.41b)$$

Note that the two similarity laws coincide with each other only for  $\sigma = 1$  in which the energy spectral density diverges at zero wavenumber, or the three-dimensional energy spectrum behaves as  $E(k, t) \propto k$  (as  $k \rightarrow 0$ ). In this case, the total energy decreases in inversely proportion to time, which has been observed often in grid-generated turbulence (see Batchelor 1953).

Time-evolution of the energy spectrum with two-similarity decay laws is depicted in the entire wavenumber range for cases  $\zeta = 2$  (figure 7(a)) and 4 (figure 7(b)). In each figure, two kinds of inserted panels ( which are identical to figures 5(b) and 2(a) for  $\zeta = 2$ , and 6(b) and 2(a) for  $\zeta = 4$  ) represent respectively the energy-containing and the universal ranges, which translate in time in the directions indicated by arrows. Notice that the direction of arrow in the energy-containing range is exactly in parallel to the asymptotic slope at the small wavenumber of the energy spectrum for case  $\zeta = 2$ , which exhibits the invariance of the Birkhoff constant, whereas they are slightly inclined with each other in case  $\zeta = 4$ , which implies that the invariance of the Loitsiansky integral is slightly broken.

## 6 Concluding Remarks

It is one of the most important (at least in a fundamental theoretical point of view) and unsolved problems in the analytical theories of turbulence to deduce (a part of) the statistical properties that turbulence might have from the basic equations, i.e. the Navier-Stokes equations. Complete information of the statistics is included in the Hopf's (1952) functional formulation of the probability distribution function of velocity, which is unfortunately formidable to be solved. On the other hand, if we restrict our interests within a few of lower-order moments of velocity which are practically more important, such as the mean velocity distribution, the

velocity correlation functions, and try to construct evolution equations for these moments, then we encounter a closure problem originating from the nonlinearity in the Navier-Stokes equations. This problem has long been preventing us from constructing any rigorous (in the limit of large Reynolds numbers) theories despite of lots of efforts by many researchers. There are quite a few phenomenological theories which predict practically useful results. Among others, the EDQNM theory, the  $k$ - $\epsilon$  and the Reynolds stress models may be named as most successful ones. Notice that all of these theories have one or more free parameters to be adjusted for a better agreement with observation.

The current work is one of such trials that construct an analytical theory without any adjustable parameters. The LDIA equations derived in this paper by a direct interaction method (Kraichnan 1959) is exactly same as those derived before by a Reynolds number expansion method (Kaneda 1981). It may be a hint of the robustness of this theory that two different approaches lead to the same results. Strictly speaking, the present theory cannot be said to be a rational approximation because it is based upon several unproved (but intuitively reasonable) assumptions and procedures summarized in §3.2. Nevertheless, the predicted energy spectrum for homogeneous isotropic turbulence agrees excellently with many measurements of real turbulence over the whole universal range without any adjustable parameters. It is therefore very interesting to check validity of the basic assumptions, e.g. by the direct numerical simulation of the Navier-Stokes equations.

Several useful findings obtained by the present theory are summarized as follows. We have proved within the current closure theory that the form of the energy spectrum in the universal range is common between a stationary and a decaying turbulence if we appropriately normalize the energy spectrum and the wavenumber in terms of time-dependent energy dissipation rate  $\epsilon(t)$ . Two-similarity-range solutions for decaying turbulence may serve as reference spectra to be compared with experiments if accurate measurements of decaying turbulence becomes possible in the future. Wavenumber dependence of the eddy-viscosity, which is often used in large-eddy-simulation of turbulence, is estimated to give comparable results with other closure theories, such as the test field model (Kraichnan 1976). The skewness factor of the velocity derivative is obtained to be equal to  $-0.66$  and is independent of the Reynolds number. This value is consistent with available experimental data in Reynolds number range  $10^3 < R_\lambda < 2 \times 10^4$  though it is not conclusive if there will appear a significant difference at larger Reynolds numbers. Incidentally, effects of intermittency of turbulence fluctuations seems to be irrelevant to the lower-order (e.g. second and third order) moments of velocity considered in the present theory.

It looks that the resultant integro-differential equations in the present closure theory are more involved than the original Navier-Stokes equations. Since, however, the averaged quantities, which are expected to be smooth both in space and in time, are employed in the statistical theory, we can take larger mesh-sizes both in space and in time in the statistical theory than in the direct numerical simulation which solves an individual velocity field of much more complicated spatio-temporal structure. An application to anisotropic turbulence as well as inhomogeneous turbulence may be tractable next targets.



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## Appendix A

In §3.1, we introduced a direct-interaction decomposition for the Eulerian velocity field and constructed the governing equations for its NDI-field and DI-field. Here, we will adopt it for the Eulerian velocity response function, the position function and the position response function, i.e.

$$\tilde{G}_{ij}^{(E)}(\mathbf{k}, t|\mathbf{k}', t') = \tilde{G}_{ij}^{(E0)}(\mathbf{k}, t|\mathbf{k}', t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) + \tilde{G}_{ij}^{(E1)}(\mathbf{k}, t|\mathbf{k}', t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0), \quad (\text{A.1})$$

$$\tilde{\psi}(\mathbf{k}, t|\mathbf{k}', t') = \tilde{\psi}^{(0)}(\mathbf{k}, t|\mathbf{k}', t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) + \tilde{\psi}^{(1)}(\mathbf{k}, t|\mathbf{k}', t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \quad (\text{A.2})$$

and

$$\tilde{\Psi}_i(\mathbf{k}, t|\mathbf{k}', \mathbf{k}'', t') = \tilde{\Psi}_i^{(0)}(\mathbf{k}, t|\mathbf{k}', \mathbf{k}'', t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) + \tilde{\Psi}_i^{(1)}(\mathbf{k}, t|\mathbf{k}', \mathbf{k}'', t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0). \quad (\text{A.3})$$

Now, we construct the governing equations for the six quantities defined by the above three decompositions. First, we can show that, from (2.31) and (2.32),  $\tilde{G}_{ij}^{(E0)}$  obeys

$$\begin{aligned} \left[ \frac{\partial}{\partial t} + \nu k^2 \right] \tilde{G}_{ij}^{(E0)}(\mathbf{k}, t|\mathbf{k}', t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) = \\ -i \left( \frac{2\pi}{L} \right)^3 \tilde{P}_{imn}(\mathbf{k}) \sum_{\mathbf{p}} \sum_{\mathbf{q}}' \tilde{u}_m(-\mathbf{p}, t) \tilde{G}_{nj}^{(E0)}(-\mathbf{q}, t|\mathbf{k}', t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \end{aligned} \quad (\text{A.4})$$

$(\mathbf{k}+\mathbf{p}+\mathbf{q}=\mathbf{o})$

with initial condition

$$\tilde{G}_{ij}^{(E0)}(\mathbf{k}, t|\mathbf{k}', t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) = \frac{L^3}{(2\pi)^6} \delta_{ij} \delta_{\mathbf{k}+\mathbf{k}'}, \quad (\text{A.5})$$

while  $\tilde{G}_{ij}^{(E1)}$  obeys

$$\begin{aligned} \left[ \frac{\partial}{\partial t} + \nu k^2 \right] \tilde{G}_{ij}^{(E1)}(\mathbf{k}, t|\mathbf{k}', t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) = \\ -i \left( \frac{2\pi}{L} \right)^3 \tilde{P}_{imn}(\mathbf{k}) \sum_{\mathbf{p}} \sum_{\mathbf{q}}' \tilde{u}_m(-\mathbf{p}, t) \tilde{G}_{nj}^{(E1)}(-\mathbf{q}, t|\mathbf{k}', t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\ -\delta_{\mathbf{k}-\mathbf{k}_0}^3 i \left( \frac{2\pi}{L} \right)^3 \tilde{P}_{imn}(\mathbf{k}_0) \tilde{u}_m^{(0)}(-\mathbf{p}_0, t|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{G}_{nj}^{(E0)}(-\mathbf{q}_0, t|\mathbf{k}', t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\ -\delta_{\mathbf{k}-\mathbf{k}_0}^3 i \left( \frac{2\pi}{L} \right)^3 \tilde{P}_{imn}(\mathbf{k}_0) \tilde{u}_m^{(0)}(-\mathbf{q}_0, t|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{G}_{nj}^{(E0)}(-\mathbf{p}_0, t|\mathbf{k}', t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\ +\delta_{\mathbf{k}+\mathbf{k}_0}^3 i \left( \frac{2\pi}{L} \right)^3 \tilde{P}_{imn}(\mathbf{k}_0) \tilde{u}_m^{(0)}(\mathbf{p}_0, t|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{G}_{nj}^{(E0)}(\mathbf{q}_0, t|\mathbf{k}', t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\ +\delta_{\mathbf{k}+\mathbf{k}_0}^3 i \left( \frac{2\pi}{L} \right)^3 \tilde{P}_{imn}(\mathbf{k}_0) \tilde{u}_m^{(0)}(\mathbf{q}_0, t|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{G}_{nj}^{(E0)}(\mathbf{p}_0, t|\mathbf{k}', t'|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\ + (\mathbf{p}_0 \rightarrow \mathbf{q}_0 \rightarrow \mathbf{k}_0 \rightarrow \mathbf{p}_0) \end{aligned} \quad (\text{A.6})$$

with initial condition

$$\tilde{G}_{ij}^{(E1)}(\mathbf{k}, t' | \mathbf{k}', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) = 0, \quad (\text{A.7})$$

where higher-order terms of DI-field have been neglected (assumption-1).

Next, the time-evolution of the position function is derived from (2.22) and (2.23) as

$$\frac{\partial}{\partial t} \tilde{\psi}^{(0)}(\mathbf{k}, t | \mathbf{k}', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) = -i k_j \left( \frac{2\pi}{L} \right)^3 \sum_{\mathbf{p}} \sum_{\mathbf{q}}' \tilde{u}_j(-\mathbf{p}, t) \tilde{\psi}^{(0)}(-\mathbf{q}, t | \mathbf{k}', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \quad (\text{A.8})$$

( $\mathbf{k} + \mathbf{p} + \mathbf{q} = \mathbf{o}$ )

with initial condition

$$\tilde{\psi}^{(0)}(\mathbf{k}, t' | \mathbf{k}', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) = \frac{L^3}{(2\pi)^6} \delta_{\mathbf{k} + \mathbf{k}'}, \quad (\text{A.9})$$

and

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{\psi}^{(1)}(\mathbf{k}, t | \mathbf{k}', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) &= -i k_j \left( \frac{2\pi}{L} \right)^3 \sum_{\mathbf{p}} \sum_{\mathbf{q}}' \tilde{u}_j(-\mathbf{p}, t) \tilde{\psi}^{(1)}(-\mathbf{q}, t | \mathbf{k}', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\ &\quad - \delta_{\mathbf{k} - \mathbf{k}_0}^3 i k_{0j} \left( \frac{2\pi}{L} \right)^3 \tilde{u}_j^{(0)}(-\mathbf{p}_0, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{\psi}^{(0)}(-\mathbf{q}_0, t | \mathbf{k}', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\ &\quad - \delta_{\mathbf{k} - \mathbf{k}_0}^3 i k_{0j} \left( \frac{2\pi}{L} \right)^3 \tilde{u}_j^{(0)}(-\mathbf{q}_0, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{\psi}^{(0)}(-\mathbf{p}_0, t | \mathbf{k}', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\ &\quad + \delta_{\mathbf{k} + \mathbf{k}_0}^3 i k_{0j} \left( \frac{2\pi}{L} \right)^3 \tilde{u}_j^{(0)}(\mathbf{p}_0, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{\psi}^{(0)}(\mathbf{q}_0, t | \mathbf{k}', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\ &\quad + \delta_{\mathbf{k} + \mathbf{k}_0}^3 i k_{0j} \left( \frac{2\pi}{L} \right)^3 \tilde{u}_j^{(0)}(\mathbf{q}_0, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{\psi}^{(0)}(\mathbf{p}_0, t | \mathbf{k}', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\ &\quad + (\mathbf{p}_0 \rightarrow \mathbf{q}_0 \rightarrow \mathbf{k}_0 \rightarrow \mathbf{p}_0), \end{aligned} \quad (\text{A.10})$$

with initial condition

$$\tilde{\psi}^{(1)}(\mathbf{k}, t' | \mathbf{k}', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) = 0. \quad (\text{A.11})$$

Finally, for position response function, we obtain, from (2.37) and (2.38),

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{\Psi}_i^{(0)}(\mathbf{k}, t, | \mathbf{k}', \mathbf{k}'', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) &= -i k_a \left( \frac{2\pi}{L} \right)^3 \sum_{\mathbf{p}} \sum_{\mathbf{q}}' \tilde{u}_a(-\mathbf{p}, t) \tilde{\Psi}_i^{(0)}(-\mathbf{q}, t | \mathbf{k}', \mathbf{k}'', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\ &\quad - i k_a \left( \frac{2\pi}{L} \right)^3 \sum_{\mathbf{p}} \sum_{\mathbf{q}}' \tilde{G}_{ai}^{(E)}(-\mathbf{p}, t | \mathbf{k}'', t') \tilde{\psi}(-\mathbf{q}, t | \mathbf{k}', t') \end{aligned} \quad (\text{A.12})$$

( $\mathbf{k} + \mathbf{p} + \mathbf{q} = \mathbf{o}$ )

with

$$\tilde{\Psi}_i^{(0)}(\mathbf{k}, t' | \mathbf{k}', \mathbf{k}'', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) = 0 \quad (\text{A.13})$$

and

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{\Psi}_i^{(1)}(\mathbf{k}, t, | \mathbf{k}', \mathbf{k}'', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) &= \\ &\quad -i k_a \left( \frac{2\pi}{L} \right)^3 \sum_{\mathbf{p}} \sum_{\mathbf{q}}' \tilde{u}_a(-\mathbf{p}, t) \tilde{\Psi}_i^{(1)}(-\mathbf{q}, t | \mathbf{k}', \mathbf{k}'', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \end{aligned}$$

( $\mathbf{k} + \mathbf{p} + \mathbf{q} = \mathbf{o}$ )

$$\begin{aligned}
& -\delta_{\mathbf{k}-\mathbf{k}_0}^3 i k_{0\alpha} \left(\frac{2\pi}{L}\right)^3 \tilde{u}_\alpha^{(0)}(-\mathbf{p}_0, t \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{\Psi}_i^{(0)}(-\mathbf{q}_0, t | \mathbf{k}', \mathbf{k}'', t' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
& -\delta_{\mathbf{k}-\mathbf{k}_0}^3 i k_{0\alpha} \left(\frac{2\pi}{L}\right)^3 \tilde{u}_\alpha^{(0)}(-\mathbf{q}_0, t \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{\Psi}_i^{(0)}(-\mathbf{p}_0, t | \mathbf{k}', \mathbf{k}'', t' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
& +\delta_{\mathbf{k}+\mathbf{k}_0}^3 i k_{0\alpha} \left(\frac{2\pi}{L}\right)^3 \tilde{u}_\alpha^{(0)}(\mathbf{p}_0, t \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{\Psi}_i^{(0)}(\mathbf{q}_0, t | \mathbf{k}', \mathbf{k}'', t' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
& +\delta_{\mathbf{k}+\mathbf{k}_0}^3 i k_{0\alpha} \left(\frac{2\pi}{L}\right)^3 \tilde{u}_\alpha^{(0)}(\mathbf{q}_0, t \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{\Psi}_i^{(0)}(\mathbf{p}_0, t | \mathbf{k}', \mathbf{k}'', t' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
& + (\mathbf{p}_0 \rightarrow \mathbf{q}_0 \rightarrow \mathbf{k}_0 \rightarrow \mathbf{p}_0)
\end{aligned} \tag{A.14}$$

with

$$\tilde{\Psi}_i^{(1)}(\mathbf{k}, t | \mathbf{k}', \mathbf{k}'', t' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) = 0. \tag{A.15}$$

In the derivation of (A.10) and (A.14), higher-order terms of DI-field have been neglected (assumption-1).

## Appendix B

In performing procedure-2, DI-fields must be represented in terms of NDI-fields. Here, we will give the expressions for  $\tilde{G}_{ij}^{(E1)}$ ,  $\tilde{\psi}^{(1)}$  and  $\tilde{\Psi}_i^{(1)}$ .

First, by using (A.4) and (A.7), we can solve (A.6) to obtain

$$\begin{aligned}
\tilde{G}_{ij}^{(E1)}(\mathbf{k}, t | \mathbf{k}', t' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) &= i \frac{(2\pi)^9}{L^6} \tilde{P}_{abc}(\mathbf{k}) \int_{t'}^t dt'' \tilde{G}_{ia}^{(E0)}(\mathbf{k}, t | -\mathbf{k}, t'' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
&\times \left[ \begin{aligned}
& -\delta_{\mathbf{k}-\mathbf{k}_0}^3 \tilde{u}_b^{(0)}(-\mathbf{p}_0, t'' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{G}_{cj}^{(E0)}(-\mathbf{q}_0, t'' | \mathbf{k}', t' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
& -\delta_{\mathbf{k}-\mathbf{k}_0}^3 \tilde{u}_b^{(0)}(-\mathbf{q}_0, t'' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{G}_{cj}^{(E0)}(-\mathbf{p}_0, t'' | \mathbf{k}', t' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
& -\delta_{\mathbf{k}+\mathbf{k}_0}^3 \tilde{u}_b^{(0)}(\mathbf{p}_0, t'' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{G}_{cj}^{(E0)}(\mathbf{q}_0, t'' | \mathbf{k}', t' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
& -\delta_{\mathbf{k}+\mathbf{k}_0}^3 \tilde{u}_b^{(0)}(\mathbf{q}_0, t'' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{G}_{cj}^{(E0)}(\mathbf{p}_0, t'' | \mathbf{k}', t' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
& + (\mathbf{p}_0 \rightarrow \mathbf{q}_0 \rightarrow \mathbf{k}_0 \rightarrow \mathbf{p}_0)
\end{aligned} \right]. \tag{B.1}
\end{aligned}$$

Next, it follows from (A.8), (A.10) and (A.11) that

$$\begin{aligned}
\tilde{\psi}^{(1)}(\mathbf{k}, t | \mathbf{k}', t' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) &= i k_j \frac{(2\pi)^9}{L^6} \int_{t'}^t dt'' \tilde{\psi}^{(0)}(\mathbf{k}, t | -\mathbf{k}, t'' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
&\times \left[ \begin{aligned}
& -\delta_{\mathbf{k}-\mathbf{k}_0}^3 \tilde{u}_j^{(0)}(-\mathbf{p}_0, t'' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{\psi}^{(0)}(-\mathbf{q}_0, t'' | \mathbf{k}', t' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
& -\delta_{\mathbf{k}-\mathbf{k}_0}^3 \tilde{u}_j^{(0)}(-\mathbf{q}_0, t'' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{\psi}^{(0)}(-\mathbf{p}_0, t'' | \mathbf{k}', t' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
& -\delta_{\mathbf{k}+\mathbf{k}_0}^3 \tilde{u}_j^{(0)}(\mathbf{p}_0, t'' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{\psi}^{(0)}(\mathbf{q}_0, t'' | \mathbf{k}', t' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
& -\delta_{\mathbf{k}+\mathbf{k}_0}^3 \tilde{u}_j^{(0)}(\mathbf{q}_0, t'' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{\psi}^{(0)}(\mathbf{p}_0, t'' | \mathbf{k}', t' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
& + (\mathbf{p}_0 \rightarrow \mathbf{q}_0 \rightarrow \mathbf{k}_0 \rightarrow \mathbf{p}_0)
\end{aligned} \right]. \tag{B.2}
\end{aligned}$$

Finally, it is shown from (A.12), (A.14) and (A.15) that

$$\begin{aligned}
\tilde{\Psi}_i^{(1)}(\mathbf{k}, t | \mathbf{k}', \mathbf{k}'', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) &= i k_a \frac{(2\pi)^3}{L^6} \int_{t'}^t dt'' \tilde{\psi}^{(0)}(\mathbf{k}, t | -\mathbf{k}, t'' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
&\times \left[ \begin{aligned}
&-\delta_{\mathbf{k}-\mathbf{k}_0}^3 \tilde{u}_a^{(0)}(-\mathbf{p}_0, t'' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{\Psi}_i^{(0)}(-\mathbf{q}_0, t'' | \mathbf{k}', \mathbf{k}'', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
&-\delta_{\mathbf{k}-\mathbf{k}_0}^3 \tilde{u}_a^{(0)}(-\mathbf{q}_0, t'' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{\Psi}_i^{(0)}(-\mathbf{p}_0, t'' | \mathbf{k}', \mathbf{k}'', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
&-\delta_{\mathbf{k}+\mathbf{k}_0}^3 \tilde{u}_a^{(0)}(\mathbf{p}_0, t'' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{\Psi}_i^{(0)}(\mathbf{q}_0, t'' | \mathbf{k}', \mathbf{k}'', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
&-\delta_{\mathbf{k}+\mathbf{k}_0}^3 \tilde{u}_a^{(0)}(\mathbf{q}_0, t'' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{\Psi}_i^{(0)}(\mathbf{p}_0, t'' | \mathbf{k}', \mathbf{k}'', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \\
&+ (\mathbf{p}_0 \rightarrow \mathbf{q}_0 \rightarrow \mathbf{k}_0 \rightarrow \mathbf{p}_0)
\end{aligned} \right]. \tag{B.3}
\end{aligned}$$

## Appendix C

We derive here relations (3.5)–(3.8). First, by taking an ensemble average of (A.8) and using assumption-3, we obtain

$$\frac{\partial}{\partial t} \overline{\tilde{\psi}^{(0)}(\mathbf{k}, t | \mathbf{k}', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)} = 0, \tag{C.1}$$

which leads to (3.6) under initial condition (A.9) of  $\tilde{\psi}^{(0)}$ . Next, an ensemble average of (2.39) gives, under assumptions-1 and 3, that

$$\overline{\tilde{G}_{ij}^{(E0)}(\mathbf{k}, t | \mathbf{k}', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)} = \frac{(2\pi)^6}{L^3} \sum_{\mathbf{k}''} \overline{\tilde{G}_{ij}^{(L0)}(t | \mathbf{k}'', \mathbf{k}', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)} \overline{\tilde{\psi}^{(0)}(\mathbf{k}, t | -\mathbf{k}'', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)}, \tag{C.2}$$

where  $\tilde{G}_{ij}^{(L0)}$  denotes NDI-field of  $\tilde{G}_{ij}^{(L)}$ . By substituting (3.6) into the above equation, we obtain (3.5). As for (3.7), we take an ensemble average of (A.12) to obtain

$$\begin{aligned}
\frac{\partial}{\partial t} \overline{\tilde{\Psi}_i^{(0)}(\mathbf{k}, t, | \mathbf{k}', \mathbf{k}'', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)} &= \\
&-i k_a \left( \frac{2\pi}{L} \right)^3 \sum_{\mathbf{p}} \sum_{\mathbf{q}} \overline{\tilde{G}_{ai}^{(E0)}(-\mathbf{p}, t | \mathbf{k}'', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)} \overline{\tilde{\psi}^{(0)}(-\mathbf{q}, t | \mathbf{k}', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)}, \tag{C.3}
\end{aligned}$$

where assumptions-1 and 3 have been employed. Equation (3.7) follows by substituting (3.5) and (3.6) into (C.3) and integrating it under initial condition (A.13). Finally, in order to show (3.8) we note the identity that

$$\begin{aligned}
\overline{\tilde{u}_i^{(0)}(\mathbf{k}, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{u}_j^{(0)}(-\mathbf{k}, t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)} &= \\
\tilde{P}_{ia}(\mathbf{k}) \frac{(2\pi)^6}{L^3} \sum_{\mathbf{k}'} \overline{\tilde{v}_a^{(0)}(t | \mathbf{k}', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{\psi}^{(0)}(\mathbf{k}, t | -\mathbf{k}', t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{u}_j^{(0)}(-\mathbf{k}, t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)}, \tag{C.4}
\end{aligned}$$

which follows from continuity equation (2.21) and relation (2.19) between  $\tilde{u}_i$  and  $\tilde{v}_j$ . Here,  $\tilde{v}_a^{(0)}$  denotes DI-field of Lagrangian velocity. By substituting (3.6) and replacing  $[\tilde{u}_i^{(0)}, \tilde{v}_i^{(0)}]$  by  $[\tilde{u}_i, \tilde{v}_i]$  (assumption-1), we obtain

$$\overline{\tilde{u}_i^{(0)}(\mathbf{k}, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{u}_j^{(0)}(-\mathbf{k}, t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)} = \tilde{P}_{ia}(\mathbf{k}) \overline{\tilde{v}_a(t | \mathbf{k}, t')} \tilde{u}_j(-\mathbf{k}, t'). \tag{C.5}$$

Then, (3.8) follows from (C.5), (2.8) and (2.29).

## Appendix D

A deduction of (3.11) from (3.10) is described here. The first term in the RHS of (3.10) vanishes because of assumption-2. On substitution of

$$\tilde{u}_n^{(1)}(-\mathbf{q}, t | \mathbf{k}, \mathbf{p}, \mathbf{q}) = i \frac{(2\pi)^9}{L^6} \tilde{P}_{abc}(\mathbf{q}) \int_{t_0}^t dt' \overline{\tilde{G}_{na}^{(E0)}(-\mathbf{q}, t | \mathbf{q}, t' | \mathbf{k}, \mathbf{p}, \mathbf{q})} \tilde{u}_b^{(0)}(\mathbf{p}, t' | \mathbf{k}, \mathbf{p}, \mathbf{q}) \tilde{u}_c^{(0)}(\mathbf{k}, t' | \mathbf{k}, \mathbf{p}, \mathbf{q}), \quad (\text{D.1})$$

which is derived from (3.4), in the second term of (3.10), we obtain

$$\begin{aligned} & (\text{The second term in RHS of (3.10)}) \\ &= \frac{(2\pi)^{15}}{L^{12}} \tilde{P}_{imn}(\mathbf{k}) \sum_{\mathbf{p}} \sum_{\substack{\mathbf{q} \\ (\mathbf{k}+\mathbf{p}+\mathbf{q}=\mathbf{o})}} \tilde{P}_{abc}(\mathbf{q}) \int_{t_0}^t dt' \overline{\tilde{G}_{na}^{(E0)}(-\mathbf{q}, t | \mathbf{q}, t' | \mathbf{k}, \mathbf{p}, \mathbf{q})} \times \\ & \quad \overline{\tilde{u}_m^{(0)}(-\mathbf{p}, t | \mathbf{k}, \mathbf{p}, \mathbf{q}) \tilde{u}_b^{(0)}(\mathbf{p}, t' | \mathbf{k}, \mathbf{p}, \mathbf{q}) \tilde{u}_j^{(0)}(-\mathbf{k}, t | \mathbf{k}, \mathbf{p}, \mathbf{q}) \tilde{u}_c^{(0)}(\mathbf{k}, t' | \mathbf{k}, \mathbf{p}, \mathbf{q})}. \end{aligned} \quad (\text{D.2})$$

Assumptions-2 and 3, (3.5) and (3.8) then convert (D.2) into

$$\begin{aligned} & (\text{The second term in RHS of (3.10)}) \\ &= \frac{(2\pi)^{15}}{L^{12}} \tilde{P}_{imn}(\mathbf{k}) \sum_{\mathbf{p}} \sum_{\substack{\mathbf{q} \\ (\mathbf{k}+\mathbf{p}+\mathbf{q}=\mathbf{o})}} \tilde{P}_{abc}(\mathbf{q}) \int_{t_0}^t dt' \overline{\tilde{G}_{na}^{(E0)}(-\mathbf{q}, t | \mathbf{q}, t' | \mathbf{k}, \mathbf{p}, \mathbf{q})} \\ & \quad \times \overline{\tilde{u}_m^{(0)}(-\mathbf{p}, t | \mathbf{k}, \mathbf{p}, \mathbf{q}) \tilde{u}_b^{(0)}(\mathbf{p}, t' | \mathbf{k}, \mathbf{p}, \mathbf{q}) \tilde{u}_j^{(0)}(-\mathbf{k}, t | \mathbf{k}, \mathbf{p}, \mathbf{q}) \tilde{u}_c^{(0)}(\mathbf{k}, t' | \mathbf{k}, \mathbf{p}, \mathbf{q})} \\ &= \frac{(2\pi)^9}{L^6} \tilde{P}_{imn}(\mathbf{k}) \sum_{\mathbf{p}} \sum_{\substack{\mathbf{q} \\ (\mathbf{k}+\mathbf{p}+\mathbf{q}=\mathbf{o})}} \tilde{P}_{abc}(\mathbf{q}) \int_{t_0}^t dt' \overline{\tilde{G}_{na}^{(L0)}(t | -\mathbf{q}, \mathbf{q}, t' | \mathbf{k}, \mathbf{p}, \mathbf{q})} \tilde{Q}_{mb}(-\mathbf{p}, t, t') \tilde{Q}_{jc}(-\mathbf{k}, t, t'). \end{aligned} \quad (\text{D.3})$$

Similarly, the third term reduces to

$$\begin{aligned} & (\text{The third term in RHS of (3.10)}) \\ &= \frac{1}{2} \frac{(2\pi)^9}{L^6} \tilde{P}_{imn}(\mathbf{k}) \sum_{\mathbf{p}} \sum_{\substack{\mathbf{q} \\ (\mathbf{k}+\mathbf{p}+\mathbf{q}=\mathbf{o})}} \tilde{P}_{abc}(\mathbf{k}) \int_{t_0}^t dt' \overline{\tilde{G}_{ja}^{(L0)}(t | -\mathbf{k}, \mathbf{k}, t' | \mathbf{k}, \mathbf{p}, \mathbf{q})} \tilde{Q}_{mb}(-\mathbf{p}, t, t') \tilde{Q}_{nc}(-\mathbf{q}, t, t'). \end{aligned} \quad (\text{D.4})$$

Combination of (D.3) and (D.4) leads to

$$\begin{aligned} & \left[ \frac{\partial}{\partial t} + \nu k^2 \right] \tilde{V}_{ij}(\mathbf{k}, t, t) = \frac{1}{2} \frac{(2\pi)^9}{L^6} \tilde{P}_{imn}(\mathbf{k}) \sum_{\mathbf{p}} \sum_{\substack{\mathbf{q} \\ (\mathbf{k}+\mathbf{p}+\mathbf{q}=\mathbf{o})}} \int_{t_0}^t dt' \tilde{Q}_{mb}(-\mathbf{p}, t, t') \\ & \times \left\{ 2\tilde{P}_{abc}(\mathbf{q}) \overline{\tilde{G}_{na}^{(L0)}(t | -\mathbf{q}, \mathbf{q}, t' | \mathbf{k}, \mathbf{p}, \mathbf{q})} \tilde{Q}_{jc}(-\mathbf{k}, t, t') + \tilde{P}_{abc}(\mathbf{k}) \overline{\tilde{G}_{ja}^{(L0)}(t | -\mathbf{k}, \mathbf{k}, t' | \mathbf{k}, \mathbf{p}, \mathbf{q})} \tilde{Q}_{nc}(-\mathbf{q}, t, t') \right\} \\ & + (i \leftrightarrow j, \mathbf{k} \rightarrow -\mathbf{k}). \end{aligned} \quad (\text{D.5})$$

Multiplying the above equation by  $\tilde{P}_{\alpha i}(\mathbf{k})$ , replacing suffixes appropriately and using (2.29) and (3.9), we obtain (3.11).

## Appendix E

We derive here the LDIA equations (3.12) and (3.13) for the two-point-two-time Lagrangian velocity correlation and the response functions.

As for the two-point-two-time Lagrangian velocity correlation function, the viscous term of (2.27) is expressed, under assumption-1, as

$$\begin{aligned} & \text{(The viscous term of (2.27))} = \\ & -\frac{(2\pi)^9}{L^6} \nu \sum_{\mathbf{p}} p^2 \overline{\tilde{u}_i^{(0)}(\mathbf{p}, t \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{\psi}^{(0)}(-\mathbf{p}, t \| \mathbf{k}, t' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \tilde{u}_j^{(0)}(-\mathbf{k}, t' \| \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)} , \end{aligned} \quad (\text{E.1})$$

which is rewritten, using assumption-3, (3.6) and (3.8), as

$$\text{(The viscous term of (2.27))} = -\nu k^2 \tilde{Q}_{ij}(\mathbf{k}, t, t') . \quad (\text{E.2})$$

The nonlinear term of (2.27) is approximated, under assumption-1, by

$$\begin{aligned} & \text{(The nonlinear term of (2.27))} = -i \frac{(2\pi)^{12}}{L^9} \sum_{\mathbf{p}} \sum_{\mathbf{q}} \sum_{\mathbf{r}} \frac{r_i r_m r_n}{r^2} \\ & \times \left[ \overline{\tilde{u}_m^{(0)}(\mathbf{p}, t \| \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{u}_n^{(0)}(\mathbf{q}, t \| \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{\psi}^{(0)}(\mathbf{r}, t \| \mathbf{k}, t' \| \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{u}_j^{(0)}(-\mathbf{k}, t' \| \mathbf{p}, \mathbf{q}, \mathbf{r})} \right. \\ & + 2 \overline{\tilde{u}_m^{(1)}(\mathbf{p}, t \| \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{u}_n^{(0)}(\mathbf{q}, t \| \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{\psi}^{(0)}(\mathbf{r}, t \| \mathbf{k}, t' \| \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{u}_j^{(0)}(-\mathbf{k}, t' \| \mathbf{p}, \mathbf{q}, \mathbf{r})} \\ & + \overline{\tilde{u}_m^{(0)}(\mathbf{p}, t \| \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{u}_n^{(0)}(\mathbf{q}, t \| \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{\psi}^{(0)}(\mathbf{r}, t \| \mathbf{k}, t' \| \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{u}_j^{(1)}(-\mathbf{k}, t' \| \mathbf{p}, \mathbf{q}, \mathbf{r})} \\ & \left. + \overline{\tilde{u}_m^{(0)}(\mathbf{p}, t \| \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{u}_n^{(0)}(\mathbf{q}, t \| \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{\psi}^{(1)}(\mathbf{r}, t \| \mathbf{k}, t' \| \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{u}_j^{(0)}(-\mathbf{k}, t' \| \mathbf{p}, \mathbf{q}, \mathbf{r})} \right] . \end{aligned} \quad (\text{E.3})$$

The first term of the above equation vanishes because of (3.6) and assumptions-2 and 3. It is easily shown that both of the second and third terms are proportional to  $k_i$ . By substituting (B.2) in the fourth term, and using assumptions-2, 3, (3.6) and (3.8), we obtain

$$\text{(The fourth term of (E.3))} = -2 \left( \frac{2\pi}{L} \right)^3 \sum_{\mathbf{p}} \sum_{\mathbf{q}} \frac{q_a q_m q_n q_i}{q^2} \int_{t'}^t dt'' \tilde{Q}_{ma}(\mathbf{p}, t, t'') \tilde{Q}_{nj}(\mathbf{k}, t, t') . \quad (\text{E.4})$$

Therefore, (2.27) reduces to

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{V}_{ij}(\mathbf{k}, t, t') &= -\nu k^2 \tilde{Q}_{ij}(\mathbf{k}, t, t') - 2 \left( \frac{2\pi}{L} \right)^3 \sum_{\mathbf{p}} \sum_{\mathbf{q}} \frac{q_a q_m q_n q_i}{q^2} \int_{t'}^t dt'' \tilde{Q}_{ma}(\mathbf{p}, t, t'') \tilde{Q}_{nj}(\mathbf{k}, t, t') \\ &+ (\text{ terms which are proportional to } k_i ) . \end{aligned} \quad (\text{E.5})$$

By multiplying the above equation by  $\tilde{P}_i(\mathbf{k})$ , and noting that  $\tilde{P}_i(\mathbf{k}) k_i = 0$  and (2.29), we arrive at (3.12).

For the Lagrangian velocity response function we have only to deal with an ensemble average of (2.35) for  $\mathbf{k}' = -\mathbf{k}$ , i.e.

$$\frac{\partial}{\partial t} \overline{\tilde{G}_{ij}^{(L)}(t | \mathbf{k}, -\mathbf{k}, t')} =$$

$$\begin{aligned}
& -\nu \frac{(2\pi)^6}{L^3} \sum_{\mathbf{k}''} k''^2 \left[ \overline{\tilde{G}_{ij}^{(E)}(\mathbf{k}'', t | -\mathbf{k}, t')} \tilde{\psi}(-\mathbf{k}'', t | \mathbf{k}, t') + \overline{\tilde{u}_i(\mathbf{k}'', t) \tilde{\Psi}_j(-\mathbf{k}'', t | \mathbf{k}, -\mathbf{k}, t')} \right] \\
& -i \frac{(2\pi)^9}{L^6} \sum_{\mathbf{p}} \sum_{\mathbf{q}} \sum_{\mathbf{r}} \frac{r_i r_m r_n}{r^2} \left[ \overline{2\tilde{u}_m(\mathbf{p}, t) \tilde{G}_{nj}^{(E)}(\mathbf{q}, t | -\mathbf{k}, t')} \tilde{\psi}(\mathbf{r}, t | \mathbf{k}, t') + \overline{\tilde{u}_m(\mathbf{p}, t) \tilde{u}_n(\mathbf{q}, t) \tilde{\Psi}_j(\mathbf{r}, t | \mathbf{k}, -\mathbf{k}, t')} \right],
\end{aligned} \tag{E.6}$$

because only this combination appears in the equations for the correlation function (3.11) and (3.12). The viscous term of this equation is calculated, using assumptions-1 and 3, (3.5) and (3.6), to be

$$\begin{aligned}
(\text{The viscous term of (E.6)}) &= -\nu \frac{(2\pi)^6}{L^3} \sum_{\mathbf{k}''} k''^2 \overline{\tilde{G}_{ij}^{(E0)}(\mathbf{k}'', t | -\mathbf{k}, t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)} \overline{\tilde{\psi}^{(0)}(-\mathbf{k}'', t | \mathbf{k}, t' | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)} \\
&= -\nu k^2 \overline{\tilde{G}_{ij}^{(L0)}(t | \mathbf{k}, -\mathbf{k}, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)}.
\end{aligned} \tag{E.7}$$

The first term in the second brackets of (E.6) is rewritten as

$$\begin{aligned}
(\text{The first term in the second brackets of (E.6)}) &= -2i \frac{(2\pi)^9}{L^6} \sum_{\mathbf{p}} \sum_{\mathbf{q}} \sum_{\mathbf{r}} \frac{r_i r_m r_n}{r^2} \\
&\times \left[ \overline{\tilde{u}_m^{(0)}(\mathbf{p}, t | \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{G}_{nj}^{(E0)}(\mathbf{q}, t | -\mathbf{k}, t' | \mathbf{p}, \mathbf{q}, \mathbf{r})} \tilde{\psi}^{(0)}(\mathbf{r}, t | \mathbf{k}, t' | \mathbf{p}, \mathbf{q}, \mathbf{r}) \right. \\
&+ \overline{\tilde{u}_m^{(1)}(\mathbf{p}, t | \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{G}_{nj}^{(E0)}(\mathbf{q}, t | -\mathbf{k}, t' | \mathbf{p}, \mathbf{q}, \mathbf{r})} \tilde{\psi}^{(0)}(\mathbf{r}, t | \mathbf{k}, t' | \mathbf{p}, \mathbf{q}, \mathbf{r}) \\
&+ \overline{\tilde{u}_m^{(0)}(\mathbf{p}, t | \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{G}_{nj}^{(E1)}(\mathbf{q}, t | -\mathbf{k}, t' | \mathbf{p}, \mathbf{q}, \mathbf{r})} \tilde{\psi}^{(0)}(\mathbf{r}, t | \mathbf{k}, t' | \mathbf{p}, \mathbf{q}, \mathbf{r}) \\
&\left. + \overline{\tilde{u}_m^{(0)}(\mathbf{p}, t | \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{G}_{nj}^{(E0)}(\mathbf{q}, t | -\mathbf{k}, t' | \mathbf{p}, \mathbf{q}, \mathbf{r})} \tilde{\psi}^{(1)}(\mathbf{r}, t | \mathbf{k}, t' | \mathbf{p}, \mathbf{q}, \mathbf{r}) \right],
\end{aligned} \tag{E.8}$$

where higher-order terms of DI-field have been neglected (assumption-1). Thanks to assumption-3 and  $\overline{\tilde{u}_i^{(0)}} = 0$ , the first term of (E.8) vanishes. For the other terms, we employ the procedures described in §3.2. On substitution of (3.4) in the second term to eliminate  $\tilde{u}_m^{(1)}$ , we can show that it vanishes because of assumptions-2 and 3. For the third term, we use (B.1) to eliminate  $\tilde{G}_{nj}^{(E1)}$ . Then, assumption-3, (3.5), (3.6) and (3.8) reduce it to

$$\begin{aligned}
(\text{The third term of (E.8)}) &= -2 \frac{(2\pi)^9}{L^6} \frac{k_i k_m k_n}{k^2} \sum_{\mathbf{p}} \sum_{\mathbf{q}} \tilde{P}_{abc}(\mathbf{q}) \\
&\times \int_{t'}^t dt'' \overline{\tilde{G}_{na}^{(L0)}(t | -\mathbf{q}, \mathbf{q}, t'' | \mathbf{k}, \mathbf{p}, \mathbf{q})} \overline{\tilde{G}_{cj}^{(L0)}(t'' | \mathbf{k}, -\mathbf{k}, t' | \mathbf{k}, \mathbf{p}, \mathbf{q})} \tilde{Q}_{mb}(-\mathbf{p}, t, t'').
\end{aligned} \tag{E.9}$$

For the fourth term, (B.2) is used to eliminate  $\tilde{\psi}^{(1)}$ . Then, we can rewrite it, using assumptions-2 and 3, (3.5), (3.6) and (3.8), as

$$\begin{aligned}
(\text{The fourth term of (E.8)}) &= -2 \left( \frac{2\pi}{L} \right)^3 \sum_{\mathbf{p}} \sum_{\mathbf{q}} \frac{q_i q_m q_n q_a}{q^2} \\
&\times \int_{t'}^t dt'' \overline{\tilde{G}_{nj}^{(L0)}(t | \mathbf{k}, -\mathbf{k}, t' | \mathbf{k}, \mathbf{p}, \mathbf{q})} \tilde{Q}_{ma}(\mathbf{p}, t, t'').
\end{aligned} \tag{E.10}$$

Finally, we calculate the second term in the second brackets of (E.6). By neglecting higher-order terms of DI-field under assumption-1, we obtain

$$\begin{aligned}
& \text{(The second term in the second brackets of (E.6))} = -i \frac{(2\pi)^9}{L^6} \sum_{\mathbf{p}} \sum_{\mathbf{q}} \sum_{\mathbf{r}} \frac{r_i r_m r_n}{r^2} \\
& \times \left[ \frac{\tilde{u}_m^{(0)}(\mathbf{p}, t | \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{u}_n^{(0)}(\mathbf{q}, t | \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{\Psi}_j^{(0)}(\mathbf{r}, t | \mathbf{k}, -\mathbf{k}, t' | \mathbf{p}, \mathbf{q}, \mathbf{r})}{+ 2 \tilde{u}_m^{(1)}(\mathbf{p}, t | \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{u}_n^{(0)}(\mathbf{q}, t | \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{\Psi}_j^{(0)}(\mathbf{r}, t | \mathbf{k}, -\mathbf{k}, t' | \mathbf{p}, \mathbf{q}, \mathbf{r})} \right. \\
& \left. + \frac{\tilde{u}_m^{(0)}(\mathbf{p}, t | \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{u}_n^{(0)}(\mathbf{q}, t | \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{\Psi}_j^{(1)}(\mathbf{r}, t | \mathbf{k}, -\mathbf{k}, t' | \mathbf{p}, \mathbf{q}, \mathbf{r})}{+ \tilde{u}_m^{(0)}(\mathbf{p}, t | \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{u}_n^{(0)}(\mathbf{q}, t | \mathbf{p}, \mathbf{q}, \mathbf{r}) \tilde{\Psi}_j^{(1)}(\mathbf{r}, t | \mathbf{k}, -\mathbf{k}, t' | \mathbf{p}, \mathbf{q}, \mathbf{r})} \right]. \quad (\text{E.11})
\end{aligned}$$

The first term of this equation vanishes because  $\tilde{u}_m^{(0)}(\mathbf{p} | \mathbf{p}, \mathbf{q}, \mathbf{r})$  has no correlation with  $\tilde{u}_n^{(0)}(\mathbf{q} | \mathbf{p}, \mathbf{q}, \mathbf{r})$  (assumption-2). Next, we substitute (3.4) into the second term, and (B.3) into the third term to eliminate quantities of DI-field. Then, it is easy to show that these terms vanish under assumptions-2 and 3. The second term of the second brackets of (E.6), therefore, does not contribute at all to the governing equation of Lagrangian velocity response function. Combination of (E.7), (E.9) and (E.10) converts (E.6) into

$$\begin{aligned}
& \frac{\partial}{\partial t} \overline{\tilde{G}_{ij}^{(L)}(t | \mathbf{k}, -\mathbf{k}, t')} + \nu k^2 \overline{\tilde{G}_{ij}^{(L0)}(t | \mathbf{k}, -\mathbf{k}, t | \mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)} = \\
& -2 \frac{(2\pi)^9}{L^6} \frac{k_i k_m k_n}{k^2} \sum_{\mathbf{p}} \sum_{\mathbf{q}} \tilde{P}_{abc}(\mathbf{q}) \\
& \quad \times \int_{t'}^t dt'' \overline{\tilde{G}_{na}^{(L0)}(t | -\mathbf{q}, \mathbf{q}, t'' | \mathbf{k}, \mathbf{p}, \mathbf{q})} \overline{\tilde{G}_{cj}^{(L0)}(t'' | \mathbf{k}, -\mathbf{k}, t' | \mathbf{k}, \mathbf{p}, \mathbf{q})} \tilde{Q}_{mb}(-\mathbf{p}, t, t'') \\
& -2 \left( \frac{2\pi}{L} \right)^3 \sum_{\mathbf{p}} \sum_{\mathbf{q}} \frac{q_i q_m q_n q_a}{q^2} \int_{t'}^t dt'' \overline{\tilde{G}_{nj}^{(L0)}(t | \mathbf{k}, -\mathbf{k}, t' | \mathbf{k}, \mathbf{p}, \mathbf{q})} \tilde{Q}_{ma}(\mathbf{p}, t, t''). \quad (\text{E.12})
\end{aligned}$$

By multiplying this equation by  $\tilde{P}_{j\alpha}(\mathbf{k})$  and using (3.9), we obtain (3.13).

## Appendix F

We assume that in the universal range of decaying turbulence  $Q(k, t, t)$  is characterized by  $k$ ,  $\epsilon(t)$  and  $\nu$ , and therefore written as

$$Q(k, t, t) = \nu^{\frac{11}{4}} \epsilon(t)^{-\frac{1}{4}} \tilde{Q}(\mathcal{B} \nu^{\frac{4}{3}} \epsilon(t)^{-\frac{1}{4}} k), \quad (\text{F.1})$$

where  $\mathcal{B}$  is a non-dimensional constant. Then, it can be shown, under the assumption of a power decay law (5.10) of energy, that in (3.22) the time-derivative term is smaller in magnitude than the viscous and the nonlinear terms by  $Re^{-\frac{1}{2}}$ . Hence, in the limit of  $Re \rightarrow \infty$ ,  $Q(k, t, t)$  is independent of time in this range, which also allows such a stationary form as  $G(k, t, t') = \tilde{G}(k, t - t')$  in governing equation (3.16) of  $G(k, t, t')$ . In conclusion, the solution of decaying turbulence in this range is identical to that of stationary turbulence.



## Appendix G

Equation (5.37) is derived here. Successive changes of variables in (5.22),  $t'$  to  $tt'$ ,  $t$  to  $t^{\frac{2}{3}}$  and  $(pt, qt)$  to  $(p, q)$ , lead to

$$t^{\frac{4}{3}-2} \frac{\partial}{\partial t} \left( t^{\frac{4}{3}-\frac{4}{3}} E^\dagger(t^{\frac{2}{3}}) \right) = \frac{2}{3} t^3 \iint_{\Delta_t} dpdq pq \widehat{b}(t, p, q) \int_0^1 dt' t'^{3-\frac{2}{3}} \\ \times G^\dagger(t^{\frac{2}{3}}, t^{\frac{2}{3}}t') G^\dagger(p^{\frac{2}{3}}, p^{\frac{2}{3}}t') G^\dagger(q^{\frac{2}{3}}, q^{\frac{2}{3}}t') q^{-\frac{11}{3}} E^\dagger(q^{\frac{2}{3}}t') \left[ p^{\frac{11}{3}} E^\dagger(p^{\frac{2}{3}}t') - t^{\frac{11}{3}} E^\dagger(t^{\frac{2}{3}}t') \right]. \quad (\text{G.1})$$

By integrating the both sides of the above equation from 0 to infinity with respect to  $t$ , we obtain

$$\begin{aligned} (\text{LHS}) &= - \left( \frac{2}{b} - 3 \right) \int_0^\infty dt t^{-2} E^\dagger(t), \quad (\text{G.2}) \\ (\text{RHS}) &= - \lim_{T \rightarrow \infty} \frac{2}{3} T^{\frac{2}{3}} \int_1^\infty dt \int_0^\infty dp \int_{\max\{t-p, p\}}^{t+p} dq t^3 pq \int_0^1 dt' t'^{3-\frac{2}{3}} \\ &\quad \times G^\dagger((tT)^{\frac{2}{3}}, (tT)^{\frac{2}{3}}t') G^\dagger((pT)^{\frac{2}{3}}, (pT)^{\frac{2}{3}}t') G^\dagger((qT)^{\frac{2}{3}}, (qT)^{\frac{2}{3}}t') \\ &\quad \times \left\{ \left[ \widehat{b}(t, p, q) + \widehat{b}(t, q, p) \right] (pq)^{-\frac{11}{3}} E^\dagger((pT)^{\frac{2}{3}}t') E^\dagger((qT)^{\frac{2}{3}}t') \right. \\ &\quad \left. - \left[ \widehat{b}(t, p, q) q^{-\frac{11}{3}} E^\dagger((qT)^{\frac{2}{3}}t') + \widehat{b}(t, q, p) p^{-\frac{11}{3}} E^\dagger((pT)^{\frac{2}{3}}t') \right] t^{-\frac{11}{3}} E^\dagger((tT)^{\frac{2}{3}}t') \right\}. \quad (\text{G.3}) \end{aligned}$$

By taking account of (5.27) and (5.34), we can calculate (G.3) to be

$$\begin{aligned} (\text{RHS}) &= -\frac{2}{3} \int_1^\infty dt \int_0^\infty dp \int_{\max\{t-p, p\}}^{t+p} dq t^3 pq \int_0^\infty dt' G_\infty^\dagger(t^{\frac{2}{3}}t') G_\infty^\dagger(p^{\frac{2}{3}}t') G_\infty^\dagger(q^{\frac{2}{3}}t') \\ &\quad \times \left\{ \left[ \widehat{b}(t, p, q) + \widehat{b}(t, q, p) \right] (pq)^{-\frac{11}{3}} - \left[ \widehat{b}(t, p, q) q^{-\frac{11}{3}} + \widehat{b}(t, q, p) p^{-\frac{11}{3}} \right] t^{-\frac{11}{3}} \right\}. \quad (\text{G.4}) \end{aligned}$$

Equation (5.37) follows from (G.2) and (G.4).

## Appendix H

For the purpose of numerical computation of (5.22)—(5.24), it is convenient to introduce  $H$  by

$$G^\dagger(t, t') = H(t - t', t'), \quad (\text{H.1})$$

because the difference between two times in  $G^\dagger$  has more important meaning of the response time. Substituting (H.1) into (5.22) and (5.23), we obtain

$$\begin{aligned} E^\dagger(t) &= -\frac{9}{4} t^{-2+\frac{2}{3}} \int_0^\infty dk \int_{2-2/3}^\infty dq \int_{|q^{\frac{2}{3}}-k^{\frac{2}{3}}|^{\frac{2}{3}}}^q dp \int_{\max\{0, \frac{t-k}{k}\}}^\infty ds (s+1)^{\frac{1}{3}-1} k^{8-\frac{2}{3}} (pq)^2 \\ &\quad \times H(ks, k) H(ps, p) H(qs, q) \\ &\quad \times \left[ E^\dagger(p) E^\dagger(q) (pq)^{-\frac{11}{2}} \left\{ \widehat{b}(k^{\frac{2}{3}}, p^{\frac{2}{3}}, q^{\frac{2}{3}}) + \widehat{b}(k^{\frac{2}{3}}, q^{\frac{2}{3}}, p^{\frac{2}{3}}) \right\} \right. \\ &\quad \left. - E^\dagger(k) k^{-\frac{11}{2}} \left\{ E^\dagger(p) p^{-\frac{11}{2}} \widehat{b}(k^{\frac{2}{3}}, q^{\frac{2}{3}}, p^{\frac{2}{3}}) + E^\dagger(q) q^{-\frac{11}{2}} \widehat{b}(k^{\frac{2}{3}}, p^{\frac{2}{3}}, q^{\frac{2}{3}}) \right\} \right] \quad (\text{H.2}) \end{aligned}$$

and

$$H(t, t') = \exp \left[ - \int_0^\infty dt'' \int_0^{\frac{t''}{t'}} ds \int_{\frac{t+t''}{t+t''}}^{\frac{t''}{t'}} dp J(p) p^{-2} \left[ \frac{s}{t''} + 1 \right]^{\frac{1}{3}-1} E^\dagger(t'') H(s, t'') \right], \quad (\text{H.3})$$

respectively. Boundary condition (5.31) is written as

$$H(0, t') = 1, \quad (\text{H.4})$$

while, asymptotic condition (5.34) at large time is represented by

$$H(t, t') \rightarrow G_{\infty}^{\dagger}(t) \quad (\text{as } t' \rightarrow \infty). \quad (\text{H.5})$$

Relation (5.24) between  $b$  and  $\zeta$  is rewritten as

$$b = \begin{cases} \frac{4}{3(\zeta + 3)} & (2 \leq \zeta < 4), \\ \left[ \frac{21}{4} - \frac{7}{10 E_0^{\dagger}} \int_0^{\infty} dt \int_0^{\infty} dk (k+t)^{\frac{3}{2} - \frac{2\zeta}{3}} t^{3 - \frac{\zeta}{3}} \left( H(k, t) E^{\dagger}(t) \right)^2 \right]^{-1} & (\zeta = 4). \end{cases} \quad (\text{H.6})$$

We solve numerically a set of integro-differential equations (H.2), (H.3) and (H.6) under boundary conditions (5.27) and (H.4), and asymptotic conditions (5.25) and (H.5). Equations (H.3) and (H.6) are solved by an iteration method, while (H.2) by the Newton-Raphson method.

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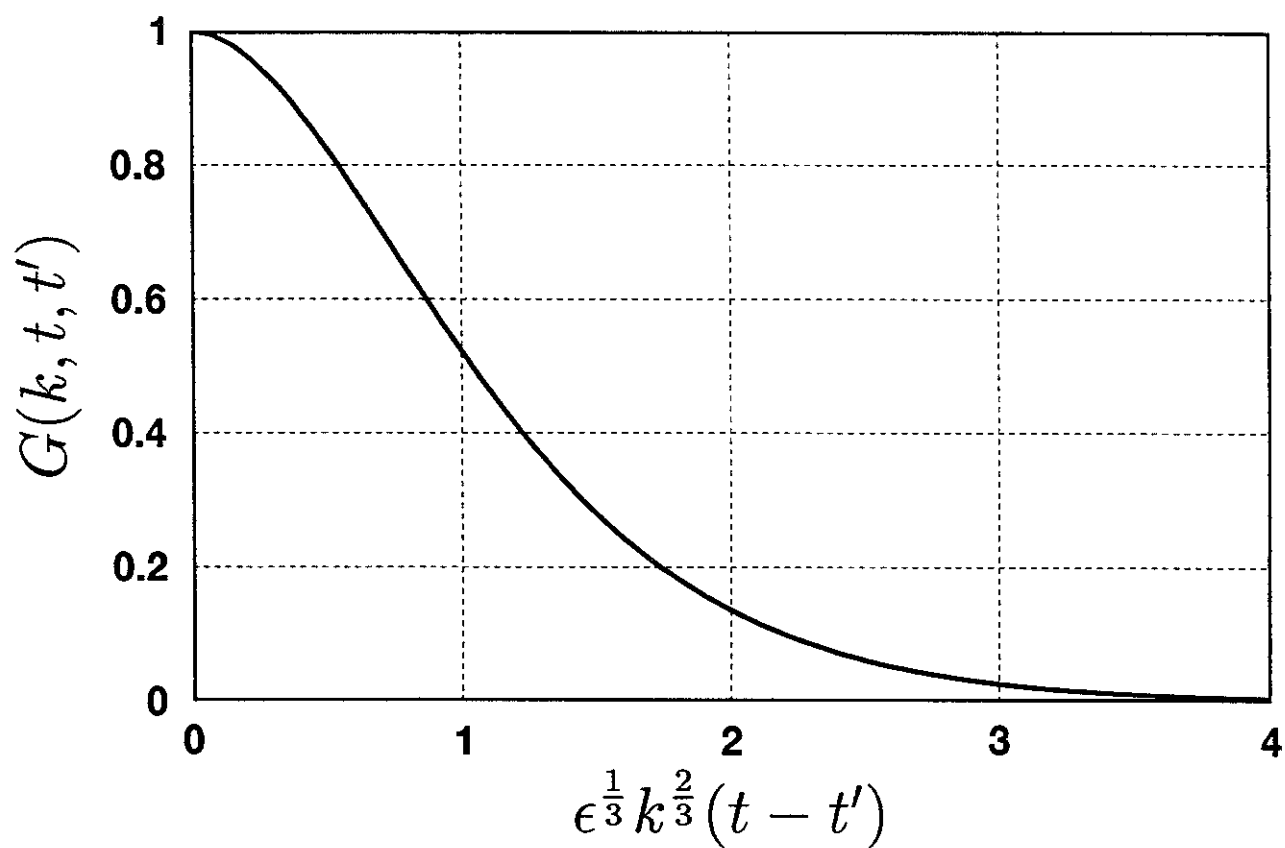
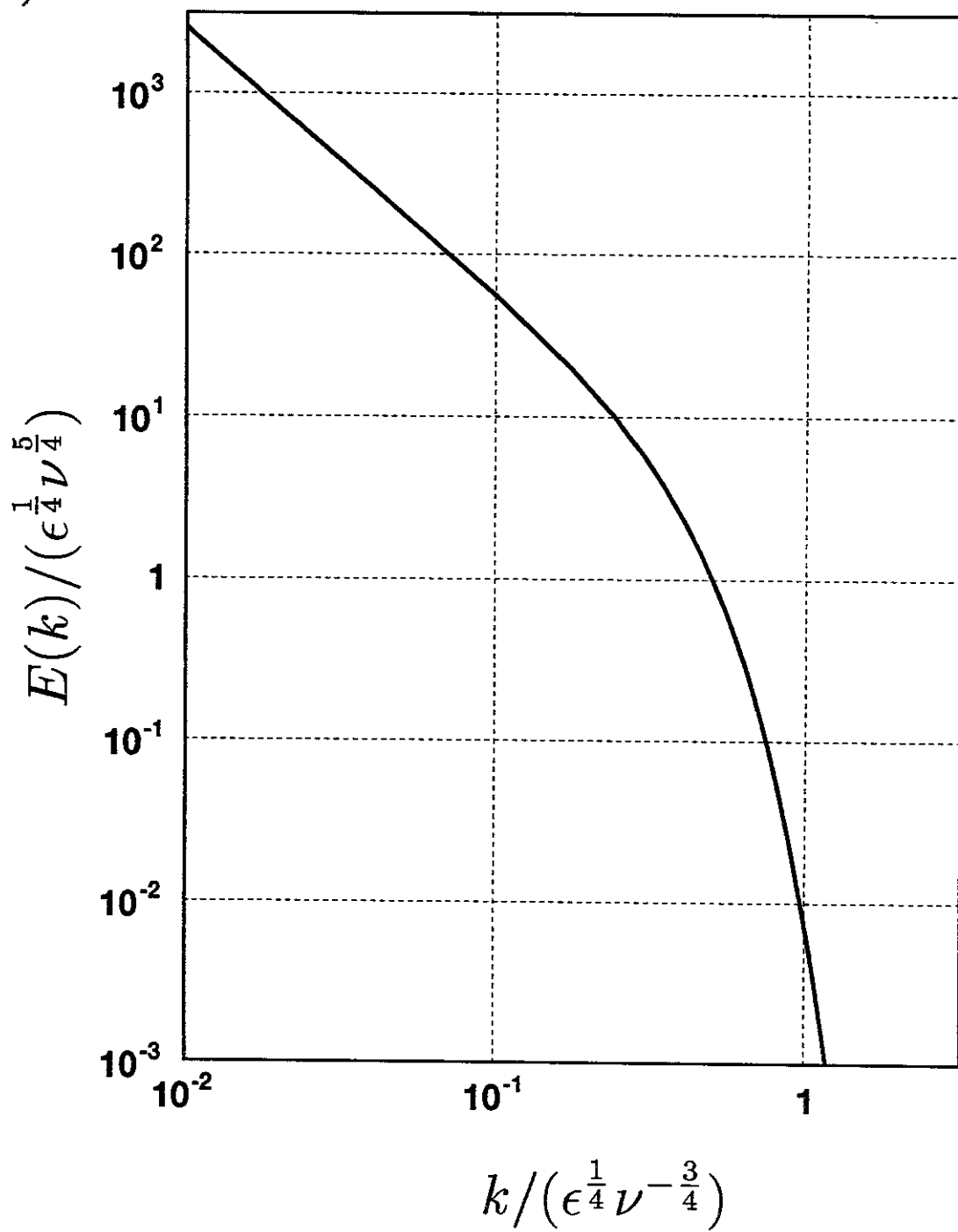
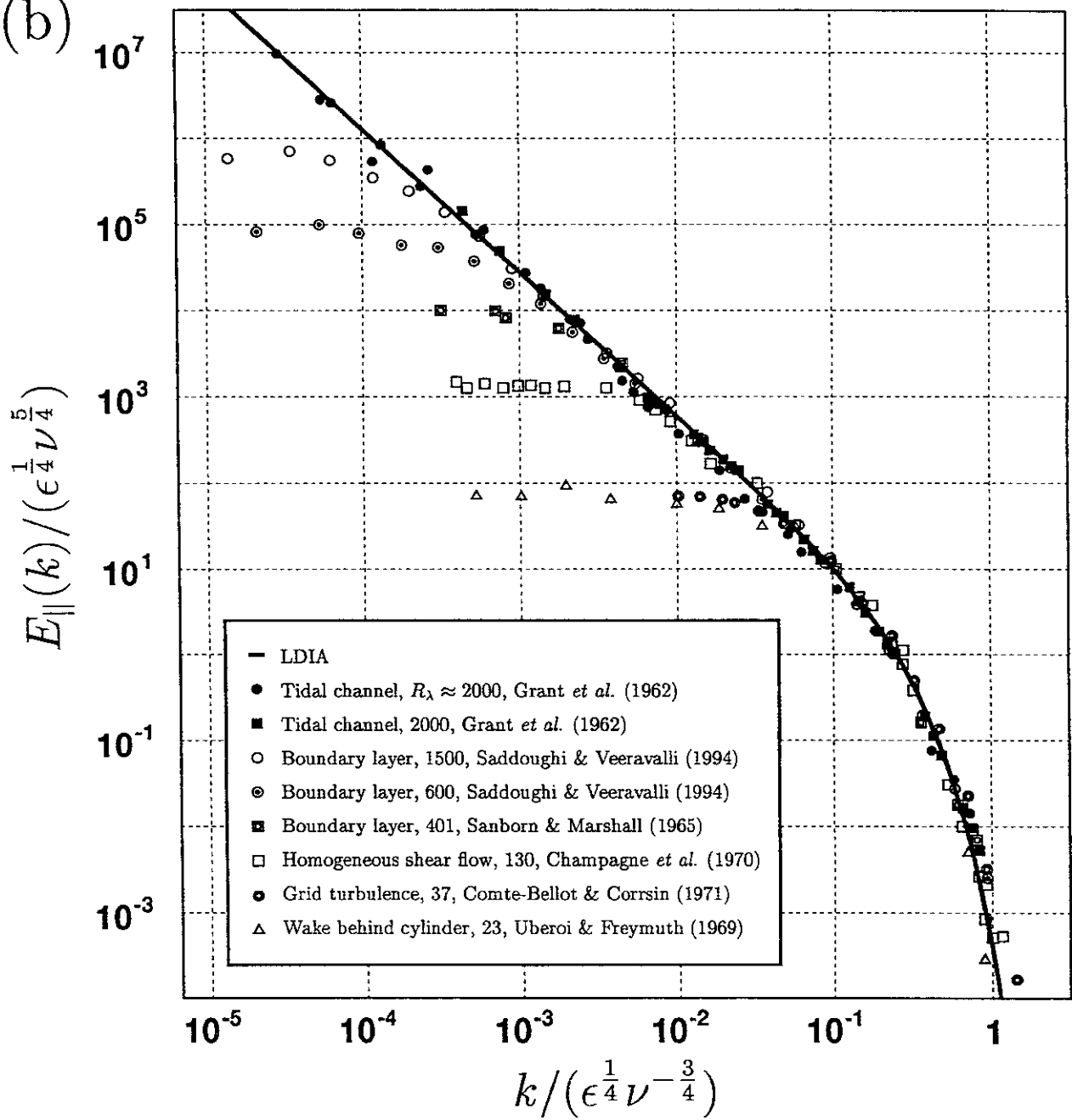


Figure 1 Lagrangian velocity response function for stationary turbulence.

(a)



(b)



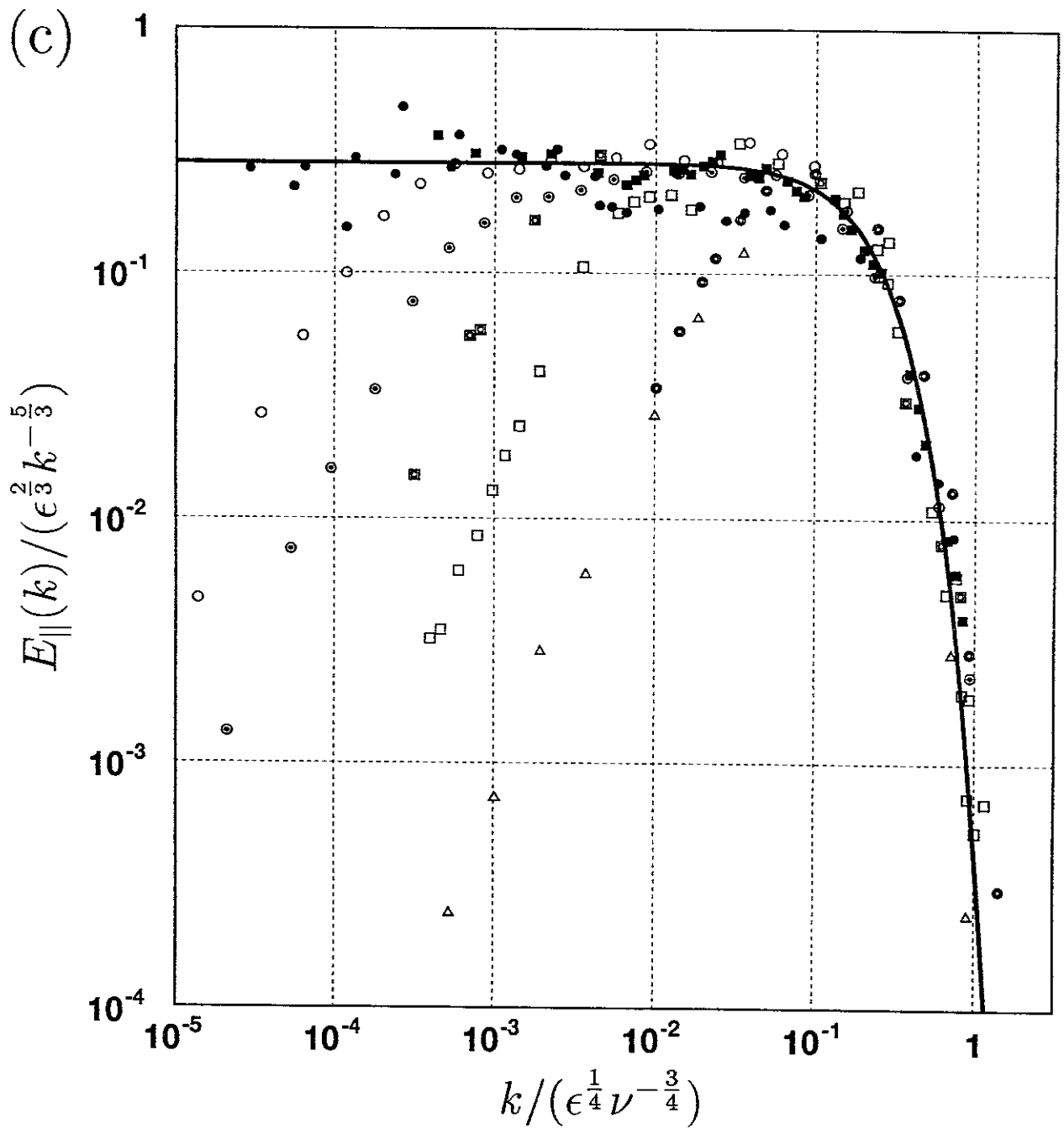
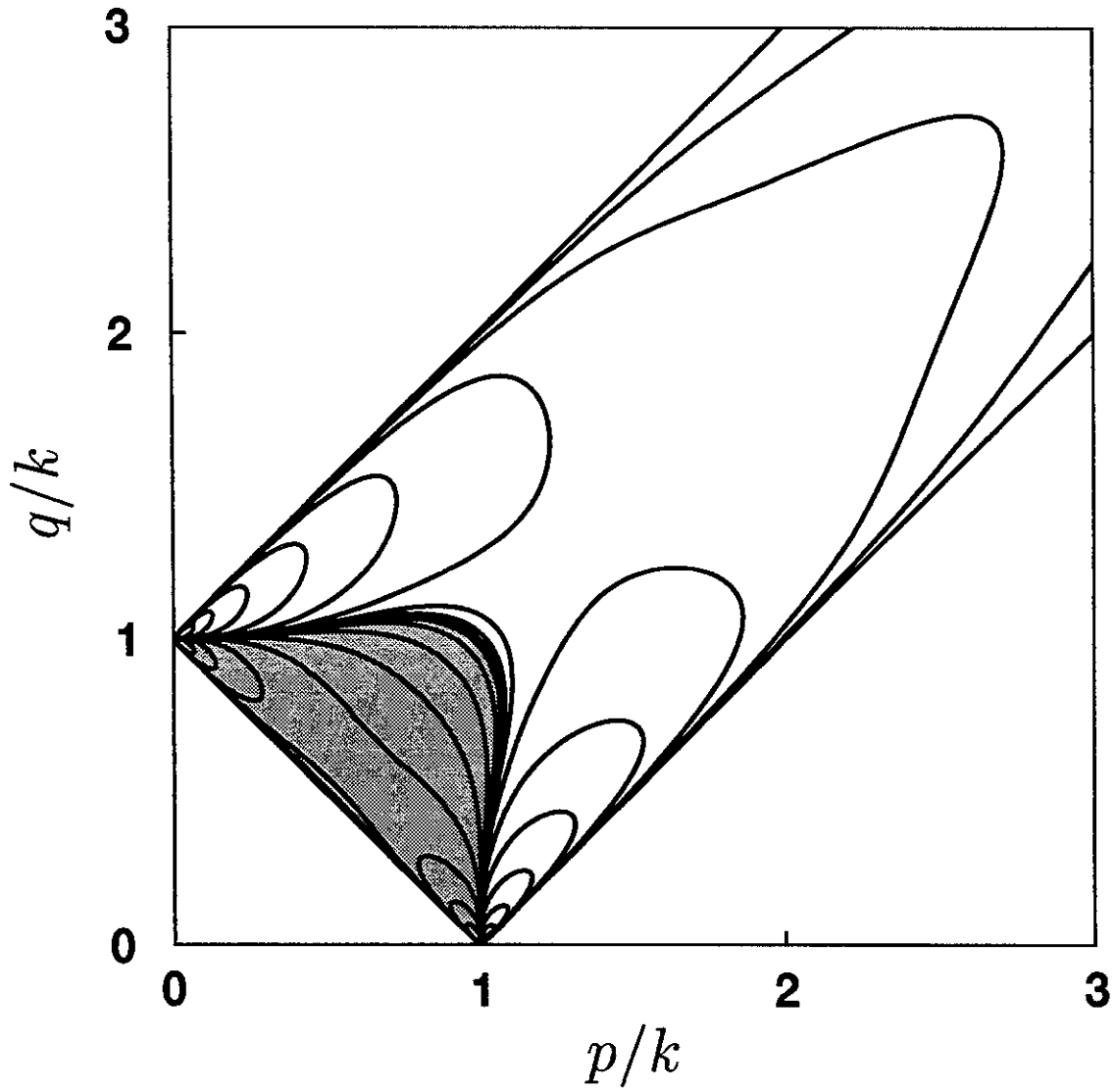
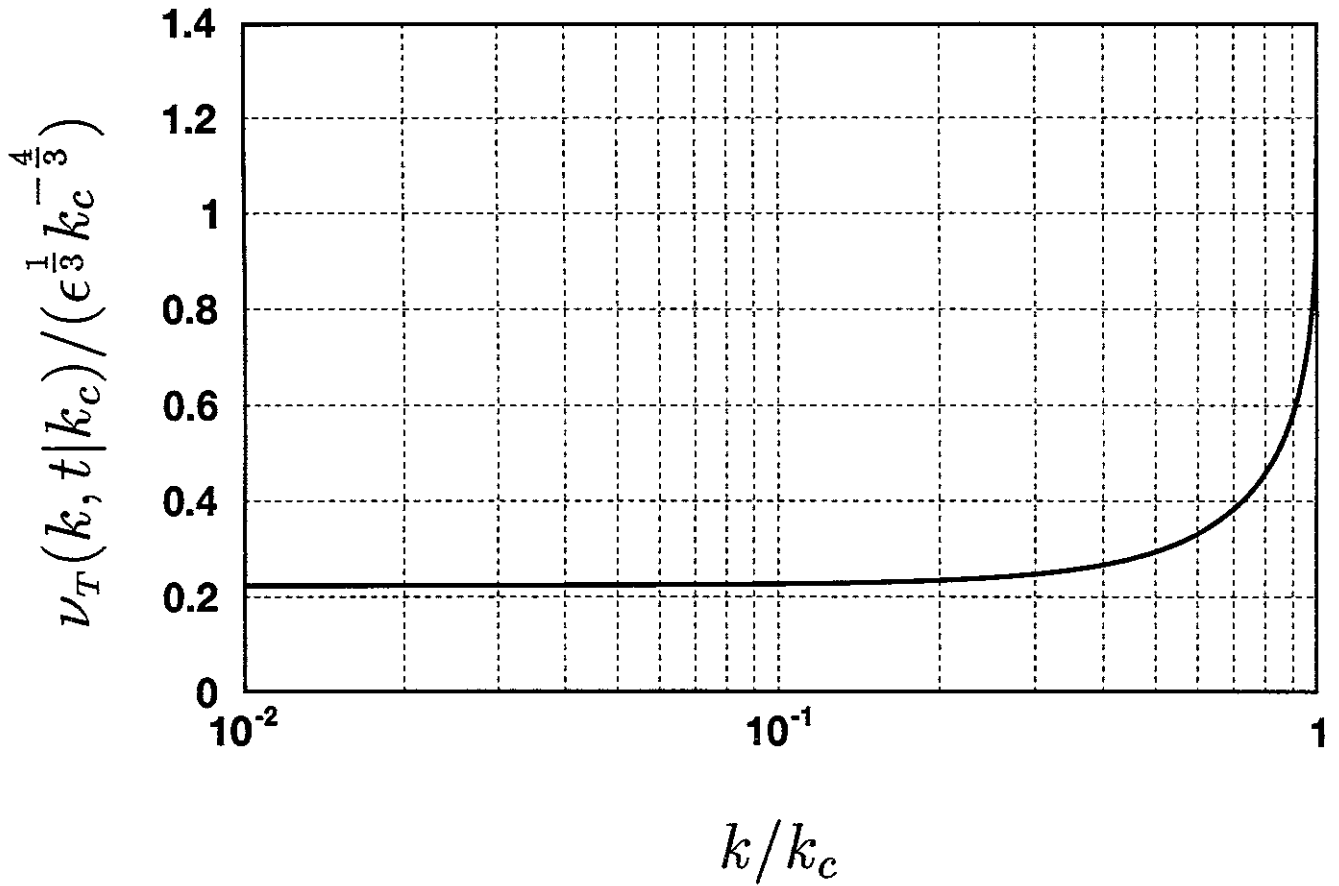


Figure 2 Energy spectra in the universal range for stationary turbulence. (a) Three-dimensional energy spectrum. (b) One-dimensional longitudinal energy spectrum. (c) One-dimensional compensated longitudinal energy spectrum. Symbols in figures (b) and (c) denote experimental data for various kinds of turbulence (taken from Chapman 1979, Grant *et al.* 1962 and Saddoughi & Veeravalli 1994).  $R_{\lambda}$  is the micro-scale Reynolds number.

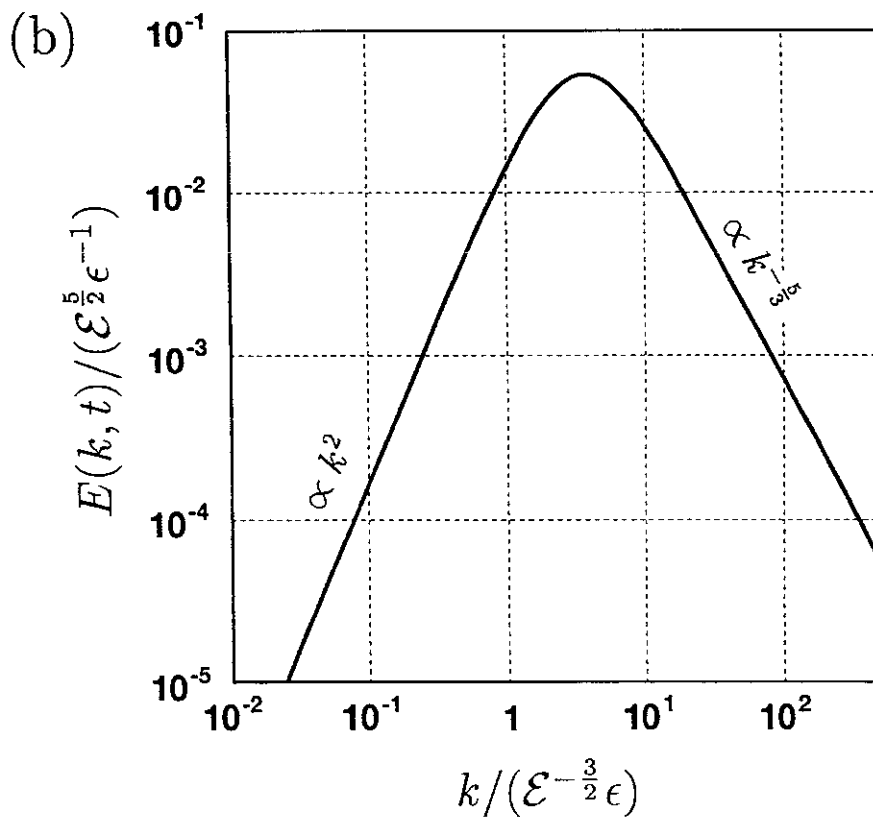
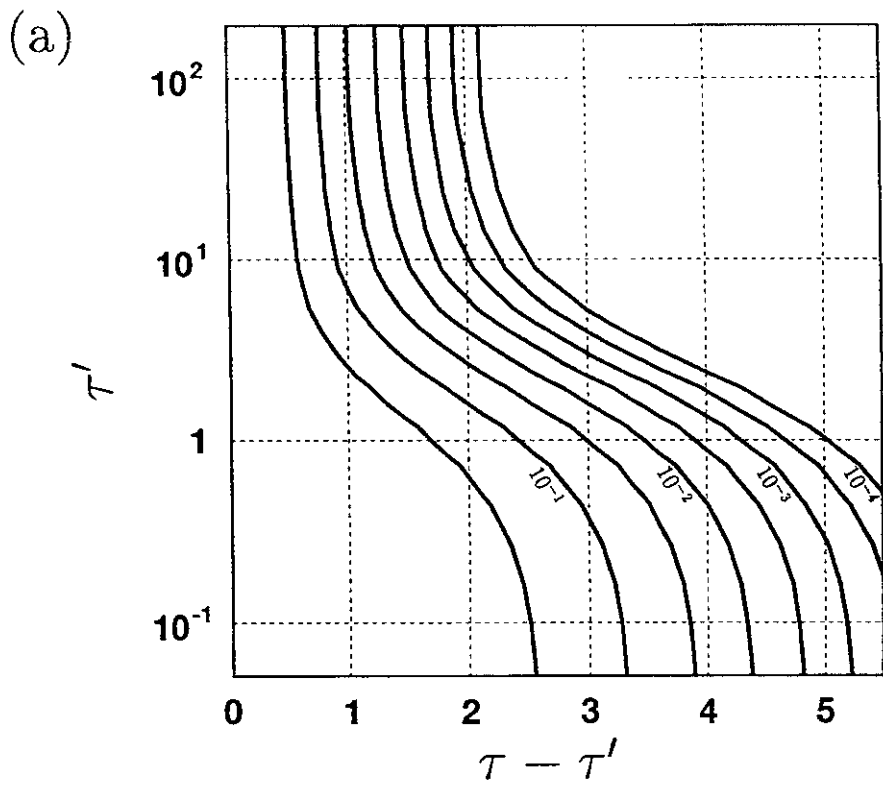


**Figure 3** Triad energy transfer function  $T^{(3)}(k, p, q)/(\epsilon k^3)$  in the inertial range for stationary turbulence. Contour levels are  $0, \pm 10^{-1.5}, \pm 10^{-1}, \pm 10^{-0.5} \dots, \pm 10^2$ . Positive regions are shaded.





**Figure 4** Wavenumber dependence of eddy-viscosity. It is nearly constant  $\nu_T \approx 0.224\epsilon^{1/3}k_c^{-4/3}$  at small wavenumbers ( $k \ll k_c$ ). The variation is less than 50% up to  $k \approx 0.6k_c$ . At cut-off wavenumber  $k_c$ , it is 5.7 times as large as that at small wavenumbers.



**Figure 5** Similar solutions in the energy containing and the inertial ranges of freely decaying turbulence for  $\zeta = 2$ . (a) Lagrangian velocity response function  $G(k, t, t')$ . Here,  $\tau = K^{\frac{1}{2}}\epsilon(t)^{\frac{1}{3}}k^{\frac{2}{3}}t$  and  $\tau' = K^{\frac{1}{2}}\epsilon(t')^{\frac{1}{3}}k^{\frac{2}{3}}t'$ . Contour levels are  $10^x$  ( $x = -0.5, -1, -1.5, \dots, -4$ ). (b) Three-dimensional energy spectrum function.

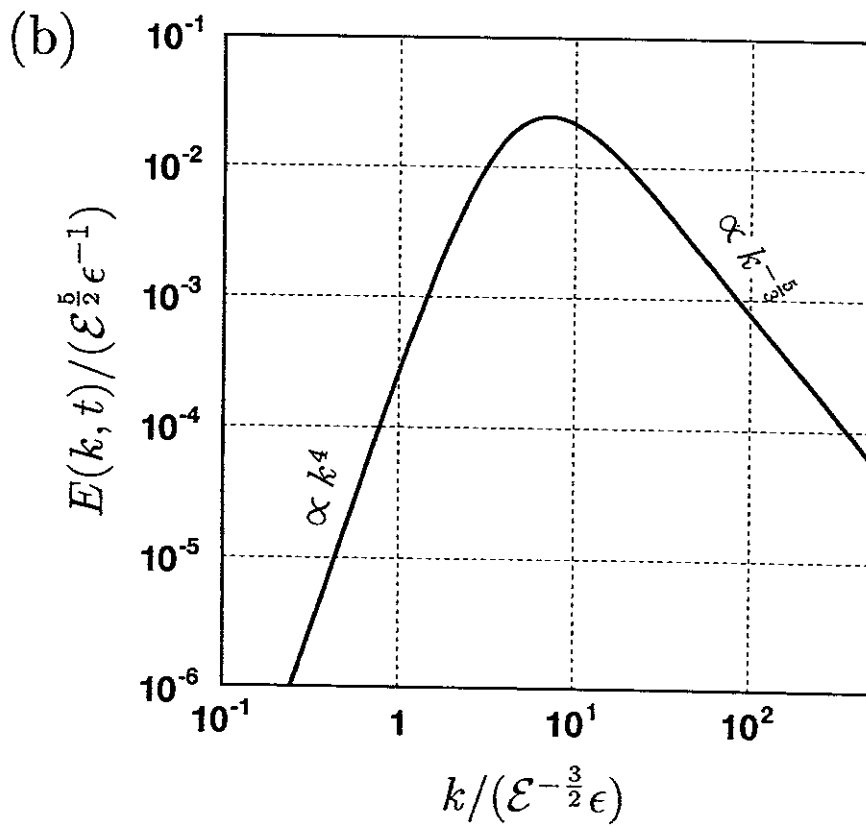
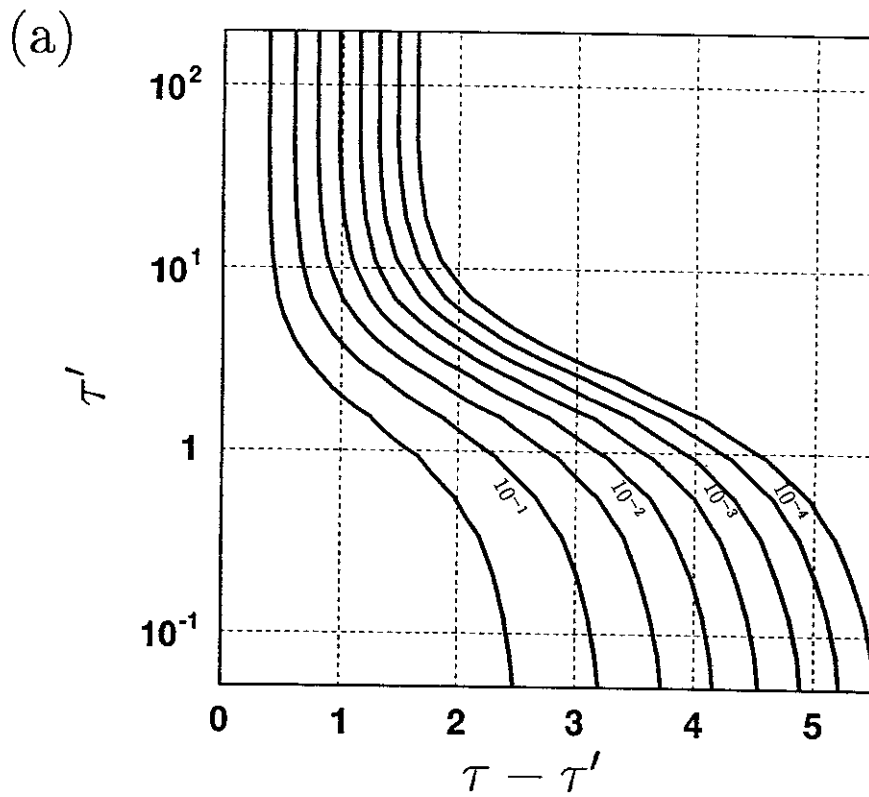
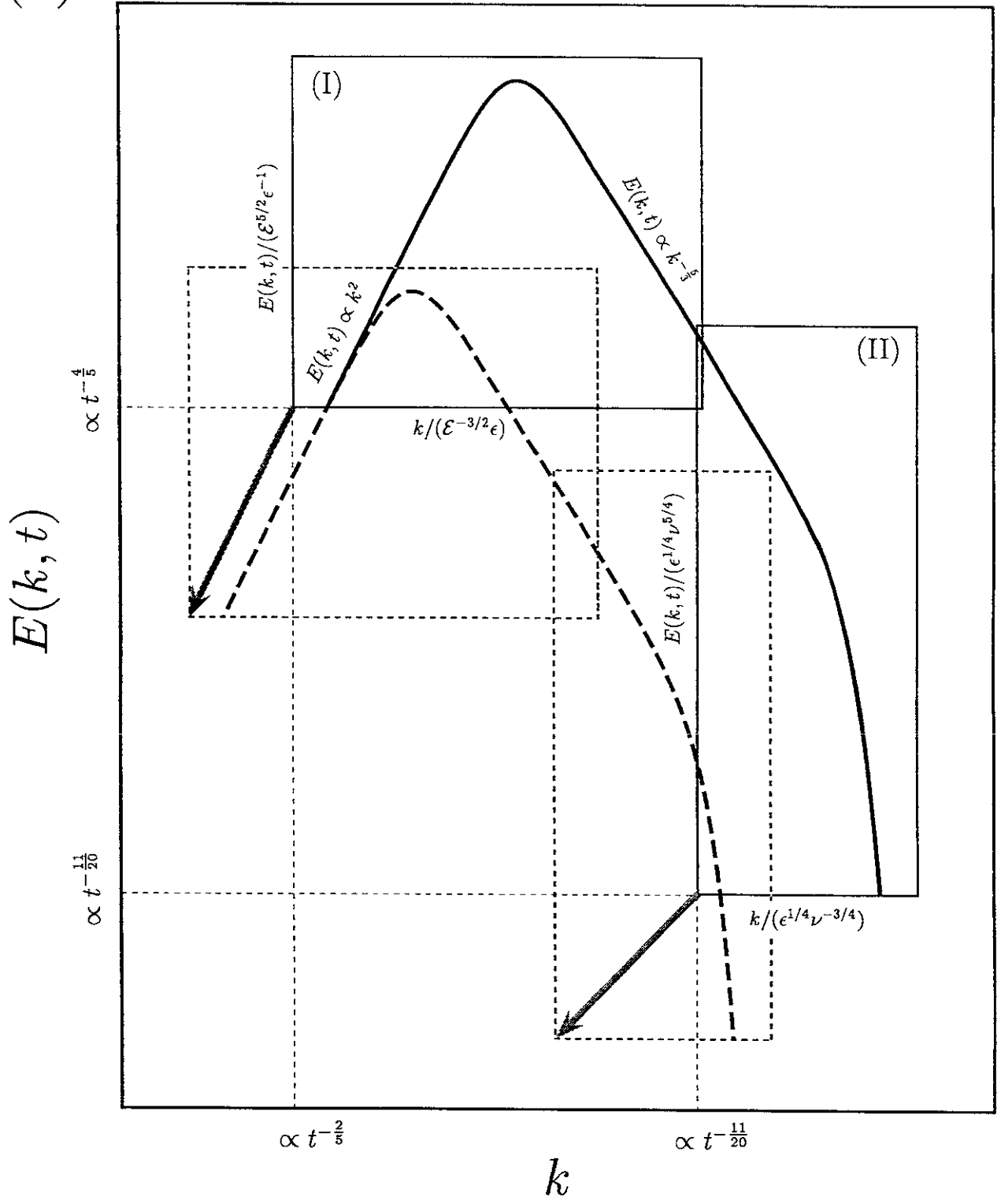
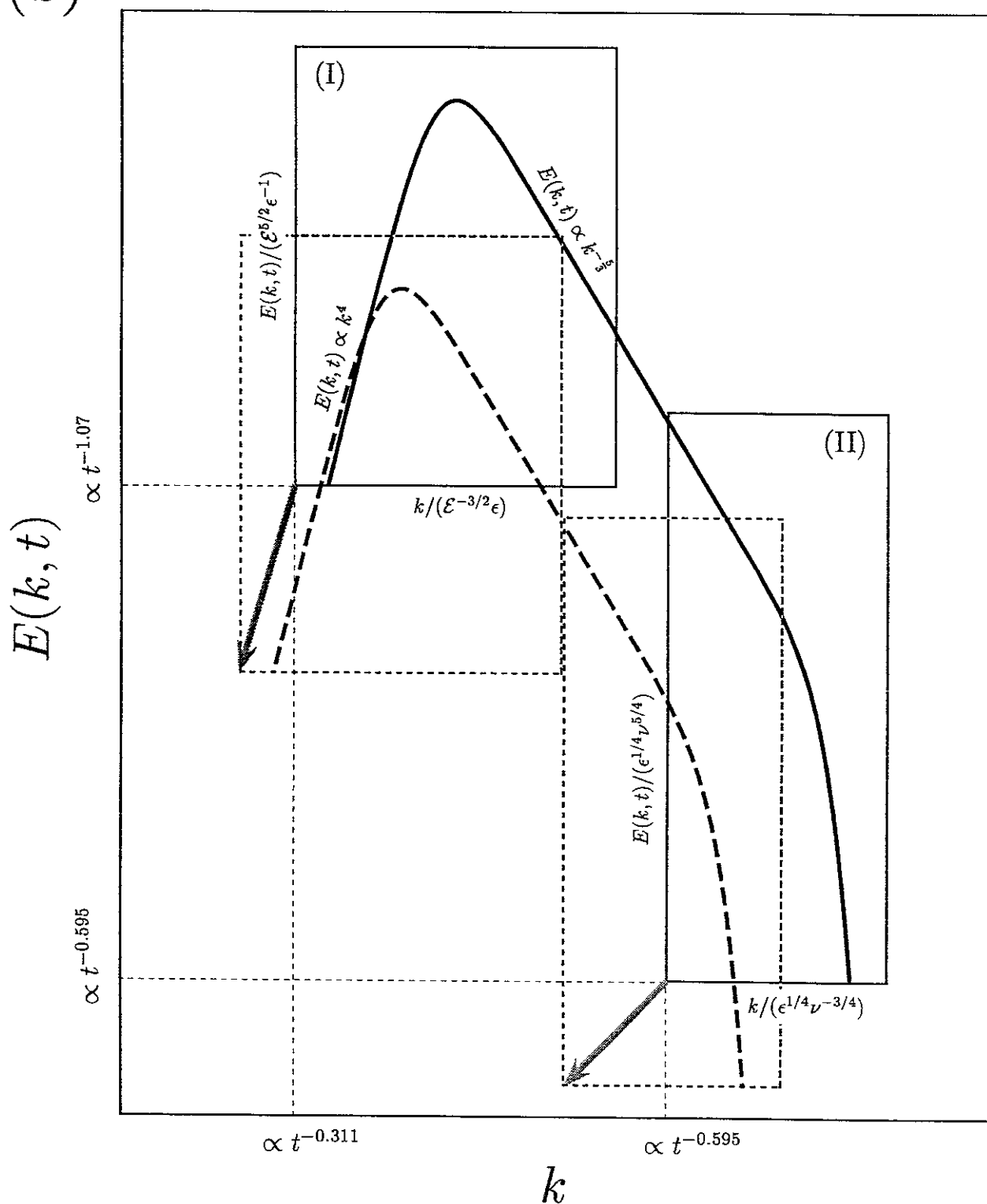


Figure 6 Same as figure 5 for  $\zeta = 4$ .

(a)



(b)



**Figure 7** Time-evolution of the three-dimensional energy spectrum of freely decaying turbulence with two similarity laws for (a)  $\zeta = 2$  and (b)  $\zeta = 4$ . Two kinds of inserted panels represent (I) the energy containing and the inertial ranges, and (II) the universal range, respectively. The energy spectra in these two ranges are connected smoothly in the inertial range between them. The two ranges move in this double logarithmic scale to the direction indicated by arrows according to the respective similarity laws.

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