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The Lagrangian formulation of the gyrokinetic theory is generalized in order to describe the particles' dynamics as well as the self-consistent behavior of the electromagnetic fields The gyrokinetic equation for the particle distribution function and the gyrokinetic Maxwell's equations for the electromagnetic fields are both derived from the variational principle for the Lagrangian consisting of the parts of particles, fields, and their interaction. In this generalized Lagrangian formulation, the energy conservation property for the total nonlinear gyrokinetic system of equations is directly shown from the Noether's theorem This formulation can be utilized in order to derive the nonlinear gyrokinetic system of equations and the rigorously conserved total energy for fluctuations with arbitrary frequencies.

KEYWORDS gyrokinetic field theory, Lagrangian formulation

§1. INTRODUCTION

The gyrokinetic theory[1-9] is a basic framework to describe microinstabilities, turbulence, and resultant anomalous transport observed in magnetically confined plasmas. Basic equations for the gyrokinetic theory are the gyrokinetic equations for the particle distribution functions and the Maxwell's equations for the electromagnetic fields The gyrokinetic theory treats the fluctuations with perpendicular wavelengths on the order of the gyroradius ρ and frequencies on the order of the diamagnetic frequency $\omega_* \sim (\rho/L)\Omega$, and it employs the ratio ρ/L as the perturbation expansion parameter, where L is the equilibrium gradient scale length and Ω is the gyrofrequency

Two types of methods to derive the gyrokinetic equation are known. One of them is the recursive technique,[1-5] which is also used for derivation of the drift kinetic equation.[10] The recursive method is combined with the ballooning representation,[11,12] and yields the gyrokinetic equation, in which the distribution function is separated into equilibrium and perturbed parts

Another modern derivation is based on the Hamiltonian and Lagrangian formulations.[6-9] The resultant gyrokinetic equation describes the total distribution function as an invariant along the particle motion. This formulation was first utilized by Littlejohn to derive the equation for the guiding center motion.[13-15] There, the motion equation is derived from the gyrophase-independent Hamiltonian, which automatically ensures the conservation of the phase space volume and the magnetic moment even in the approximate expressions obtained by truncating the perturbation expansion up to the finite order. Also, the Hamiltonian is regarded as the conserved energy for the particle in the static electromagnetic fields

In the gyrokinetic theory, the particle Hamiltonian (or the particle energy) is not an invariant since the fluctuating electromagnetic fields are treated Instead, the conserved quantity is the total energy of the system, which is given by the sum of the kinetic energy of the particles and the energy of the electromagnetic fields However, the proof of the total energy conservation[16] is not trivial in the conventional Hamiltonian or Lagrangian formulation, where only the particle dynamics are described by the Hamiltonian or Lagrangian. Then, it seems natural that the formulation should be extended in order to derive governing equations for both the particles and the electromagnetic fields from the first principle. The purpose of the present work is to present such an extended formulation of the gyrokinetic theory.

In this paper, the gyrokinetic equation for the particle distribution function and the gyrokinetic Maxwell's equations for the electromagnetic fields are both derived from the variational principle using the Lagrangian, which consists of the parts of the particles, fields, and their interaction. This generalized Lagrangian includes the single-particle Lagrangian as a part, which has been used for the conventional Lagrangian derivation of the gyrokinetic equation. In order to treat the variational principle for the electromagnetic fields, the technique of the classical field theory[17,18] is useful. Since all the governing equations for the system are derived from the generalized Lagrangian, we can directly show the conservation of the total energy of the system with the help of the Noether's theorem.[18] This seems to be the most natural and easiest way to prove the energy conservation.

The conventional gyrokinetic theories assume that $\omega \ll \Omega$, where ω is the characteristic fluctuation frequency and Ω is the particle gyrofrequency The linear gyrokinetic theory, which can describe fluctuations with arbitrary frequencies including $\omega \sim \Omega$, was presented by Chen and Tsai[19,20] based on the recursive method Also, recently, the Lagrangian formulation of the gyrokinetic theory for arbitrary-frequency fluctuations was given by H. Qin, *et al.*[21,22] They have used the fact that, originally, the Lie perturbation method in the Lagrangian formulation[15,23] depends only on smallness of the fluctuation amplitude with no assumption on the fluctuation frequencies. However, their work is also a linear theory. The formulation presented in this paper is useful for derivation of the nonlinear gyrokinetic system of equations with the rigorously conserved total energy for the fluctuations with arbitrary frequencies.

The rest of this work is organized as follows. In §2, as a preliminary for the generalized Lagrangian formulation of the gyrokinetic theory, the variational principle is presented based on the Lagrangian, from which the Newton's motion equations for the discrete particles, the Poisson's equation, and the Ampère's law are derived. There, the conservation of the total energy is shown from the Noether's theorem. The Lagrangian formulation for the collisionless (or Vlasov) plasma is also given. In §3, the gyrokinetic theory for $\omega \ll \Omega$ is formulated based on the generalized Lagrangian formulation. From the Lagrangian variational principle, the gyrokinetic motion equations for the discrete particles and the gyrokinetic versions of the Poisson's equation and the Ampère's law are derived. The Lagrangian formulation, which gives the gyrokinetic equation for the distribution function for the collisionless case, is also shown. The gyrokinetic version of the rigorously conserved total energy is derived. In §4, the generalized Lagrangian formulation is extended to the case of arbitrary-frequency fluctuations and the nonlinear gyrokinetic system of equations with the conserved total energy are derived, which are valid even for $\omega \sim \Omega$. In §5, several limiting cases, in which the gyrokinetic equations are simplified, are considered. The small electron gyroradius limit, the quasineutrality, and the linear polarization approximation are treated as examples. The simplified gyrokinetic system of equations are given, which can describe the high-frequency electrostatic plasma fluctuations (such as the ion Bernstein waves) in the uniform magnetic field. Finally, conclusions are given in §6. Appendix gives brief explanation of the variational principle and the Noether's theorem for systems including field variables.

§2. LAGRANGIAN FOR PARTICLES AND ELECTROMAGNETIC FIELDS

In this section, we present the Lagrangian, from which the equations for particles' motion and for self-consistent electromagnetic fields are both derived through the variational principle.

2.1 Newton-Poisson-Ampère system

The variational principle to yield the governing equations for the system considered here is written in the well-known form,

$$\delta I \equiv \delta \int_{t_1}^{t_2} L dt = 0. \tag{1}$$

Here, I is called the action integral and δ represents the variation. The end points for the integral with respect to the time t are fixed to t_1 and t_2. The Lagrangian to describe the Newton-Poisson-Ampère system is written as

$$L = L_p + L_f + L_{int}, \tag{2}$$

where the parts of the particles, electromagnetic fields, and field-particle interaction are defined by

$$L_p = \sum_a \sum_{j=1}^{N_a} m_a \left(\mathbf{v}_{aj} \cdot \dot{\mathbf{x}}_{aj} - \frac{1}{2}|\mathbf{v}_{aj}|^2 \right), \tag{3}$$

$$L_f = \frac{1}{8\pi} \int_V d^3\mathbf{x} \left(|\nabla\phi(\mathbf{x}, t)|^2 - |\nabla \times \mathbf{A}(\mathbf{x}, t)|^2 \right), \tag{4}$$

and

$$L_{int} = -\sum_a \sum_{j=1}^{N_a} e_a \left(\phi(\mathbf{x}_{aj}, t) - \frac{1}{c}\dot{\mathbf{x}}_{aj} \cdot \mathbf{A}(\mathbf{x}_{aj}, t) \right), \tag{5}$$

respectively, where $\mathbf{x}_{aj}$ and $\mathbf{v}_{aj}$ are the position and velocity of the jth particle of species a with mass m_a and charge e_a, N_a denotes the number of the particles of species a, $\phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$ are the scalar and vector potentials, respectively, at the position $\mathbf{x}$ and the time t, and $\dot{} = d/dt$ represents the time derivative. Here, following the phase-space Lagrangian formalism used in the conventional guiding-center and gyrocenter theories,[8,9,15] the particle velocities $\mathbf{v}_{aj}$ are regarded as independent variables as well as the particle positions $\mathbf{x}_{aj}$. Then, the relation $\dot{\mathbf{x}}_{aj} = \mathbf{v}_{aj}$ is obtained as a result of the variational principle as shown later. The field Lagrangian L_f defined by Eq. (4) is slightly different from the one found in standard text books[17,18] in that $\partial\mathbf{A}/\partial t$ is not contained in Eq. (4). Consequently, the variational principle yields the Ampère's law instead of the Faraday's law. It implies that, in the present work as well as in the conventional gyrokinetic theory, we do not treat the electromagnetic waves with the speed of light. Also, the particle Lagrangian in Eq. (3) is only for the nonrelativistic case.

The particle variables are contained only in $L_p + L_{int}$, and we can write

$$\begin{aligned} L_p + L_{int} &= \sum_a \sum_{j=1}^{N_a} \left[\left(m_a \mathbf{v}_{aj} + \frac{e_a}{c}\mathbf{A}(\mathbf{x}_{aj}, t) \right) \cdot \dot{\mathbf{x}}_{aj} - \left(\frac{1}{2} m_a |\mathbf{v}_{aj}|^2 + e_a \phi(\mathbf{x}_{aj}, t) \right) \right] \\ &\equiv \sum_a \sum_{j=1}^{N_a} (\mathbf{p}_{aj} \cdot \dot{\mathbf{x}}_{aj} - H_{aj}) \equiv \sum_a \sum_{j=1}^{N_a} L_{aj}. \end{aligned} \tag{6}$$

Here, $\mathbf{p}_{aj}$, L_{aj}, and H_{aj} represent the canonical momentum, Lagrangian, and Hamiltonian for a single particle, respectively. The conventional Lagrangian and Hamiltonian formulations of the guiding-center and gyrocenter theories for the particle dynamics are based on the single-particle Lagrangian L_{aj} and Hamiltonian H_{aj}. In the present work, we consider the field Lagrangian part L_f as well, in order to derive the governing equations for the fields and show the energy conservation for the total system, directly.

In order to treat the variation with respect to the fields ϕ and $\mathbf{A}$, it is convenient to use the Lagrangian densities $\mathcal{L}_f$ and $\mathcal{L}_{int}$, which are defined by

$$\mathcal{L}_f = \frac{1}{8\pi} \left(|\nabla\phi(\mathbf{x}, t)|^2 - |\nabla \times \mathbf{A}(\mathbf{x}, t)|^2 \right), \tag{7}$$

and

$$\mathcal{L}_{int} = -\sum_a \sum_{j=1}^{N_a} e_a \delta^3(\mathbf{x}-\mathbf{x}_{aj}) \left(\phi(\mathbf{x},t) - \frac{1}{c}\dot{\mathbf{x}}_{aj} \cdot \mathbf{A}(\mathbf{x},t) \right) \tag{8}$$

Then, we have

$$L_f = \int_V \mathcal{L}_f d^3\mathbf{x} \quad \text{and} \quad L_{int} = \int_V \mathcal{L}_{int} d^3\mathbf{x}, \tag{9}$$

where V denotes the volume of the system.

When we take the variation of functions of t and $\mathbf{x}$, we fix their values at the end points $t = t_1, t_2$ and on the boundary surface of the volume V according to the conventional rule for the variational principle (see Appendix A). Then, with the help of partial integral, the variation of the action integral is written as

$$\begin{aligned}
\delta I = \int_{t_1}^{t_2} dt \Bigg[& \sum_a \sum_{j=1}^{N_a} \Bigg\{ \left(\frac{\partial L_{aj}}{\partial \mathbf{x}_{aj}} - \frac{d}{dt}\frac{\partial L_{aj}}{\partial \dot{\mathbf{x}}_{aj}} \right) \cdot \delta \mathbf{x}_{aj} \\
& + \left(\frac{\partial L_{aj}}{\partial \mathbf{v}_{aj}} - \frac{d}{dt} \underbrace{\frac{\partial L_{aj}}{\partial \dot{\mathbf{v}}_{aj}}}_{0} \right) \cdot \delta \mathbf{v}_{aj} \Bigg\} \\
& + \int_V d^3\mathbf{x} \Bigg\{ \left(\frac{\partial \mathcal{L}_{fi}}{\partial \phi} - \frac{\partial}{\partial t} \underbrace{\frac{\partial \mathcal{L}_{fi}}{\partial \dot{\phi}}}_{0} - \nabla \cdot \frac{\partial \mathcal{L}_{fi}}{\partial \nabla \phi} \right) \delta\phi \\
& + \left(\frac{\partial \mathcal{L}_{fi}}{\partial \mathbf{A}} - \frac{\partial}{\partial t} \underbrace{\frac{\partial \mathcal{L}_{fi}}{\partial \dot{\mathbf{A}}}}_{0} - \nabla \cdot \frac{\partial \mathcal{L}_{fi}}{\partial \nabla \mathbf{A}} \right) \cdot \delta \mathbf{A} \Bigg\} \Bigg] \\
\equiv \int_{t_1}^{t_2} dt \Bigg[& \sum_a \sum_{j=1}^{N_a} \left(\frac{\delta I}{\delta \mathbf{x}_{aj}} \cdot \delta \mathbf{x}_{aj} + \frac{\delta I}{\delta \mathbf{v}_{aj}} \cdot \delta \mathbf{v}_{aj} \right) \\
& + \int_V d^3\mathbf{x} \left(\frac{\delta I}{\delta \phi}\delta\phi + \frac{\delta I}{\delta \mathbf{A}} \cdot \delta \mathbf{A} \right) \Bigg],
\end{aligned} \tag{10}$$

where $\delta/\delta\mathbf{x}_{aj}$, $\delta/\delta\mathbf{v}_{aj}$, $\delta/\delta\phi$, and $\delta/\delta\mathbf{A}$ represent the functional derivatives, and $\mathcal{L}_{fi} = \mathcal{L}_f + \mathcal{L}_{int}$.

The variational principle written by Eq. (1) implies that each of the functional derivatives in Eq. (10) should vanish. Then, the nonrelativistic Newton's particle motion equation is derived from the Euler-Lagrange equation $\delta I/\delta \mathbf{x}_{aj} \equiv \partial L_{aj}/\partial \mathbf{x}_{aj} - d(\partial L_{aj}/\partial \dot{\mathbf{x}}_{aj})/dt = 0$, which is rewritten as

$$m_a \dot{\mathbf{v}}_{aj} = e_a \left[\mathbf{E}(\mathbf{x}_{aj},t) + \frac{1}{c}\dot{\mathbf{x}}_{aj} \times \mathbf{B}(\mathbf{x}_{aj},t) \right]. \tag{11}$$

Here, the electromagnetic fields are written by $\mathbf{E} = -\nabla\phi - c^{-1}\partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla \times \mathbf{A}$. Next, we find from $\delta I/\delta \mathbf{v}_{aj} = 0$ that

$$\dot{\mathbf{x}}_{aj} = \mathbf{v}_{aj}. \tag{12}$$

The Poisson's equation is derived from $\delta I/\delta\phi = 0$ as

$$\begin{aligned}
\nabla^2 \phi(\mathbf{x},t) &= -4\pi \sum_a e_a \sum_{j=1}^{N_a} \delta^3(\mathbf{x}-\mathbf{x}_{aj}) \\
&\equiv -4\pi \sum_a e_a n_a^{micro},
\end{aligned} \tag{13}$$

where n_a^{micro} represents the microscopic number density for species a. Also, $\delta I/\delta\mathbf{A} = 0$ gives the Ampère's law,

$$\begin{aligned}
\nabla \times [\nabla \times \mathbf{A}(\mathbf{x},t)] &= \frac{4\pi}{c} \sum_a e_a \sum_{j=1}^{N_a} \dot{\mathbf{x}}_{aj} \delta^3(\mathbf{x}-\mathbf{x}_{aj}) \\
&\equiv \frac{4\pi}{c}\mathbf{j}^{micro},
\end{aligned} \tag{14}$$

where $\mathbf{j}^{micro}$ represents the microscopic current density.

We find from Eqs. (2)–(5) that the Lagrangian L has no explicit time dependence, which means that the time dependence of L appears only through the variables $\mathbf{x}_{aj}(t)$, $\dot{\mathbf{x}}_{aj}(t)$, $\mathbf{v}_{aj}(t)$, $\phi(\mathbf{x},t)$, and $\mathbf{A}(\mathbf{x},t)$. Then, we can apply the Noether's theorem (see Appendix A) to derive the conservation of the total energy of the system,

$$\frac{d}{dt}E_{tot} = 0. \tag{15}$$

Here, the conserved total energy (or the total Hamiltonian) E_{tot} is given by the Legendre transformation of the Lagrangian L. By noting that our Lagrangian includes both the discrete particle variables and the continuous field variables, we find that E_{tot} is given by

$$\begin{aligned}
E_{tot} &= \sum_a \sum_{j=1}^{N_a} \left(\dot{\mathbf{x}}_{aj} \cdot \frac{\partial L_{aj}}{\partial \dot{\mathbf{x}}_{aj}} + \dot{\mathbf{v}}_{aj} \cdot \underbrace{\frac{\partial L_{aj}}{\partial \dot{\mathbf{v}}_{aj}}}_{0} \right) \\
&\quad + \int_V d^3\mathbf{x} \left(\dot{\phi} \underbrace{\frac{\partial \mathcal{L}_{fi}}{\partial \dot{\phi}}}_{0} + \dot{\mathbf{A}} \cdot \underbrace{\frac{\partial \mathcal{L}_{fi}}{\partial \dot{\mathbf{A}}}}_{0} \right) - L \\
&= \sum_a \sum_{j=1}^{N_a} \left[\frac{1}{2} m_a |\mathbf{v}_{aj}|^2 + e_a \phi(\mathbf{x}_{aj},t) \right] - L_f \\
&= \sum_a \sum_{j=1}^{N_a} \frac{1}{2} m_a |\mathbf{v}_{aj}|^2 \\
&\quad + \frac{1}{8\pi} \int_V d^3\mathbf{x} \left(|\nabla\phi(\mathbf{x},t)|^2 + |\nabla \times \mathbf{A}(\mathbf{x},t)|^2 \right),
\end{aligned} \tag{16}$$

where Eq. (13) is also used. The total energy E_{tot} in Eq. (16) has the well-known form, which is given by the sum of the kinetic energy of the particles and the energy of the electromagnetic fields.

2.2 Vlasov-Poisson-Ampère system

In the previous subsection, we have treated the discrete particles' motion and the microscopic electromagnetic fields. Therefore, the effect of Coulomb collisions is included in the governing equations (11)–(14) and in the total energy conservation given by Eq. (15) with Eq. (16). Then, the next natural question is how the collisionless (or Vlasov) plasma is described by the generalized Lagrangian formulation, which is considered here. For the Vlasov plasma, we neglect the discreteness of the particles and describe the particles by the distribution function $f_a(\mathbf{x}, \mathbf{v}, t)$ for species a in the phase space, from which the statistically averaged (or macroscopic) density and current of the particles are calculated. The use of $(\mathbf{x}, \mathbf{v}, t)$ as the independent variables for f_a corresponds to the Eulerian picture for representing the state of the particles' ensemble. On the other hand, in the previous subsection, the orbit of each particle labeled by the index aj is pursued, which corresponds to the Lagrangian picture. In order to bridge the gap between these two different pictures, we perform the following replacement for the particle part of the Lagrangian,

$$\begin{aligned}
&\sum_{j=1}^{N_a} \rightarrow \int d^3\mathbf{x}_0 \int d^3\mathbf{v}_0 f_a(\mathbf{x}_0, \mathbf{v}_0, t_0), \\
&\mathbf{x}_{aj}(t) \rightarrow \mathbf{r}_a(\mathbf{x}_0, \mathbf{v}_0, t_0; t), \\
&\mathbf{v}_{aj}(t) \rightarrow \mathbf{u}_a(\mathbf{x}_0, \mathbf{v}_0, t_0; t).
\end{aligned} \tag{17}$$

Here, $f_a(\mathbf{x}_0, \mathbf{v}_0, t_0)$ is the distribution function at an arbitrarily specified initial time t_0, $\mathbf{r}_a(\mathbf{x}_0, \mathbf{v}_0, t_0; t)$ and $\mathbf{u}_a(\mathbf{x}_0, \mathbf{v}_0, t_0; t)$ represents the position and velocity, respectively, of the particle, which satisfy the initial conditions,

$$\mathbf{r}_a(\mathbf{x}_0, \mathbf{v}_0, t_0; t_0), = \mathbf{x}_0, \qquad \mathbf{u}_a(\mathbf{x}_0, \mathbf{v}_0, t_0; t_0), = \mathbf{v}_0. \tag{18}$$

The functional forms of $\mathbf{r}_a(\mathbf{x}_0, \mathbf{v}_0, t_0; t)$ and $\mathbf{u}_a(\mathbf{x}_0, \mathbf{v}_0, t_0; t)$ are determined by the variational principle with the Lagrangian as shown later. Once that they are obtained, the distribution function f_a for an arbitrary time t is given by

$$f_a(\mathbf{x}, \mathbf{v}, t) = \int d^3\mathbf{x}_0 \int d^3\mathbf{v}_0 \delta^3[\mathbf{x} - \mathbf{r}_a(\mathbf{x}_0, \mathbf{v}_0, t_0; t)] \times \delta^3[\mathbf{v} - \mathbf{u}_a(\mathbf{x}_0, \mathbf{v}_0, t_0; t)] f_a(\mathbf{x}_0, \mathbf{v}_0, t_0). \tag{19}$$

Using Eqs. (6) and (17), the Lagrangian for Vlasov-Poisson-Ampère system is given by

$$\begin{aligned}
L &= L_p + L_{int} + L_f \\
&= \sum_a \int d^3\mathbf{x}_0 \int d^3\mathbf{v}_0 f_a(\mathbf{x}_0, \mathbf{v}_0, t_0) \\
&\quad \times L_a[\mathbf{r}_a(\mathbf{x}_0, \mathbf{v}_0, t_0; t), \mathbf{u}_a(\mathbf{x}_0, \mathbf{v}_0, t_0, t), \dot{\mathbf{r}}_a(\mathbf{x}_0, \mathbf{v}_0, t_0; t)] \\
&\quad + \frac{1}{8\pi} \int d^3\mathbf{x} \left(|\nabla\phi(\mathbf{x}, t)|^2 - |\nabla \times \mathbf{A}(\mathbf{x}, t)|^2 \right),
\end{aligned} \tag{20}$$

with the single-particle Lagrangian L_a for species a defined, in the same way as in Eq. (6), by

$$\begin{aligned}
L_a(\mathbf{r}_a, \mathbf{u}_a, \dot{\mathbf{r}}_a) &\equiv \left(m_a \mathbf{u}_a + \frac{e_a}{c} \mathbf{A}(\mathbf{r}_a, t) \right) \cdot \dot{\mathbf{r}}_a \\
&\quad - \left(\frac{1}{2} m_a |\mathbf{u}_a|^2 + e_a \phi(\mathbf{r}_a, t) \right) \\
&\equiv \mathbf{p}_a \cdot \dot{\mathbf{r}}_a - H_a,
\end{aligned} \tag{21}$$

where $\mathbf{p}_a$ and H_a represents the canonical momentum and Hamiltonian for a single particle, respectively. Now, the governing equations for the system are derived from the variational principle in Eq. (1) with the Lagrangian in Eq. (20) In the same way as in Eqs. (11) and (12), we obtain from $\delta I/\delta \mathbf{r}_a = \delta I/\delta \mathbf{u}_a = 0$ that

$$\dot{\mathbf{r}}_a = \mathbf{u}_a. \tag{22}$$

and

$$m_a \dot{\mathbf{u}}_a = e_a \left[\mathbf{E}(\mathbf{r}_a, t) + \frac{1}{c} \mathbf{u}_a \times \mathbf{B}(\mathbf{r}_a, t) \right]. \tag{23}$$

Then, we easily obtain the incompressibility condition in the phase space,

$$\frac{\partial}{\partial \mathbf{r}_a} \cdot \dot{\mathbf{r}}_a + \frac{\partial}{\partial \mathbf{u}_a} \cdot \dot{\mathbf{u}}_a = 0. \tag{24}$$

From Eqs. (19), (22), (23), and (24), we find that the distribution function f_a satisfies the Vlasov equation,

$$\left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \frac{e_a}{m_a} \left\{ \mathbf{E}(\mathbf{x}, t) + \frac{1}{c} \mathbf{v} \times \mathbf{B}(\mathbf{x}, t) \right\} \cdot \frac{\partial}{\partial \mathbf{v}} \right] f_a(\mathbf{x}, \mathbf{v}, t) = 0. \tag{25}$$

In the same way as in Eqs. (13) and (14), $\delta I/\delta\phi = 0$ and $\delta I/\delta \mathbf{A} = 0$ give the Poisson's equation,

$$\nabla^2 \phi(\mathbf{x}, t) = -4\pi \sum_a e_a \int f_a(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v} \equiv -4\pi \sum_a e_a n_a. \tag{26}$$

and the Ampère's law,

$$\nabla \times [\nabla \times \mathbf{A}(\mathbf{x}, t)] = \frac{4\pi}{c} \sum_a e_a \int f_a(\mathbf{x}, \mathbf{v}, t) \mathbf{v} d^3\mathbf{v} \equiv \frac{4\pi}{c} \mathbf{j}, \tag{27}$$

respectively. Here, n_a and $\mathbf{j}$ represent the macroscopic particle number density for species a and the macroscopic current density, respectively. Thus, all the governing equations (25), (26), and (27) for the Vlasov-Poisson-Ampère system are derived from the Lagrangian given by Eq. (20). Then, the Noether's theorem ensures the total energy conservation as written in Eq. (15). Here, the conserved total energy is obtained as

$$\begin{aligned}
E_{tot} &= \sum_a \int d^3\mathbf{x}_0 \int d^3\mathbf{v}_0 f_a(\mathbf{x}_0, \mathbf{v}_0, t_0) \left(\dot{\mathbf{r}}_a \cdot \frac{\partial L_a}{\partial \dot{\mathbf{r}}_a} + \dot{\mathbf{u}}_a \cdot \underbrace{\frac{\partial L_a}{\partial \dot{\mathbf{u}}_a}}_{0} \right) \\
&\quad + \int_V d^3\mathbf{x} \left(\dot{\phi} \underbrace{\frac{\partial \mathcal{L}_{fi}}{\partial \dot{\phi}}}_{0} + \dot{\mathbf{A}} \cdot \underbrace{\frac{\partial \mathcal{L}_{fi}}{\partial \dot{\mathbf{A}}}}_{0} \right) - L \\
&= \sum_a \int d^3\mathbf{x} \int d^3\mathbf{v} f_a(\mathbf{x}, \mathbf{v}, t) \left[\frac{1}{2} m_a |\mathbf{v}|^2 + e_a \phi(\mathbf{x}, t) \right] - L_f \\
&= \sum_a \int d^3\mathbf{x} \int d^3\mathbf{v} f_a(\mathbf{x}, \mathbf{v}, t) \frac{1}{2} m_a |\mathbf{v}|^2 \\
&\quad + \frac{1}{8\pi} \int d^3\mathbf{x} \left(|\nabla\phi(\mathbf{x}, t)|^2 + |\nabla \times \mathbf{A}(\mathbf{x}, t)|^2 \right),
\end{aligned} \tag{28}$$

where Eq. (26) is also used. Again, the total energy has the well-known form, although here the kinetic energy part is evaluated from the distribution functions and the field energy part is for the macroscopic electromagnetic fields.

§3. LAGRANGIAN FORMULATION FOR THE GYROKINETIC-POISSON-AMPÈRE SYSTEM

Here, we proceed to the generalized Lagrangian formulation of the gyrokinetic theory based on the framework given in the previous section. In the gyrokinetic system, the electromagnetic fields are assumed to consist of equilibrium and perturbation parts. Following Brizard's terminology,[9] we refer to the phase-space variables defined from the equilibrium and perturbed fields as the guiding-center and gyrocenter coordinates, respectively. These coordinates are relevant to the independent variables for the single-particle Lagrangian. Our generalized Lagrangian for the gyrokinetic theory is expressed in terms of the gyrocenter coordinates of all particles and the electromagnetic fields. First, let us consider the expansion of the Lagrangian with respect to the amplitude of the perturbation fields.

3.1 *Perturbation expansion of the Lagrangian*

The electromagnetic fields and the corresponding scalar and vector potentials are assumed to consist of equilibrium and perturbation parts.

$$\mathbf{E} = \mathbf{E}_0(\mathbf{x}) + \Delta\mathbf{E}_1(\mathbf{x},t), \quad \mathbf{B} = \mathbf{B}_0(\mathbf{x}) + \Delta\mathbf{B}_1(\mathbf{x},t),$$
$$\phi = \phi_0(\mathbf{x}) + \Delta\phi_1(\mathbf{x},t), \quad \mathbf{A} = \mathbf{A}_0(\mathbf{x}) + \Delta\mathbf{A}_1(\mathbf{x},t). \tag{29}$$

Here, Δ represents the order of the perturbation amplitude, which is used as an expansion parameter in the gyrokinetic theory. The canonical momentum of a single particle for species a is written as

$$\mathbf{p}_a \equiv m_a\mathbf{v}_a + \frac{e_a}{c}(\mathbf{A}_0 + \Delta\mathbf{A}_1) \equiv m_a\mathbf{v}_{a0} + \frac{e_a}{c}\mathbf{A}_0, \tag{30}$$

where $\mathbf{A}_0$ and $\mathbf{A}_1$ is evaluated at the position $\mathbf{x} = \mathbf{x}_a$, and the zeroth-order particle velocity $\mathbf{v}_{a0}$ is defined in terms of the canonical momentum $\mathbf{p}_a$ and the zeroth-order vector field $\mathbf{A}_0$ as $\mathbf{v}_{a0} \equiv m_a^{-1}(\mathbf{p}_a - e_a\mathbf{A}_0/c)$. In the present work, we assume that the equilibrium $\mathbf{E}\times\mathbf{B}$ drift velocity is $\mathcal{O}(\epsilon v_T)$, where $\epsilon \sim \rho/L$ is the drift ordering parameter and v_T is the thermal velocity. Then, we put $\mathbf{E}_0 = \phi_0 = 0$ and consider $\mathbf{E}_1$ and ϕ_1 to include the fluctuation part as well as the equilibrium part corresponding to the $\mathcal{O}(\epsilon v_T)$ $\mathbf{E}\times\mathbf{B}$ drift velocity. This causes no inconsistency in the results derived in this work. [The guiding-center and gyrocenter theories for the case of the $\mathcal{O}(v_T)$ $\mathbf{E}\times\mathbf{B}$ drift velocity are found in Refs. 14 and 24–28. The extension of the general Lagrangian formulation in the present work to this large $\mathbf{E}\times\mathbf{B}$ case is possible although it is not treated here for simplicity.] Here, we also neglect the induction field $-\partial\mathbf{A}_0/\partial t$, since it is $\mathcal{O}(\epsilon^2)$ according to the conventional transport ordering and its effect on the fluctuation dynamics is negligible.

Using Eqs. (29) and (30), the single-particle Lagrangian defined in Eq. (6) is rewritten as

$$L_a = L_{a0} + \Delta L_{a1} + \Delta^2 L_{a2}, \tag{31}$$

with

$$L_{a0} = \left(m_a\mathbf{v}_{a0} + \frac{e_a}{c}\mathbf{A}_0\right)\cdot\dot{\mathbf{x}}_a - \frac{1}{2}m_a|\mathbf{v}_{a0}|^2 \equiv \mathbf{p}_a\cdot\dot{\mathbf{x}}_a - H_{a0}, \tag{32}$$

$$L_{a1} = -e_a\left(\phi_1 - \frac{1}{c}\mathbf{v}_{a0}\cdot\mathbf{A}_1\right) \equiv -e_a\psi_a \equiv -H_{a1}, \tag{33}$$

and

$$L_{a2} = -\frac{e_a^2}{2m_ac^2}|\mathbf{A}_1|^2 \equiv -H_{a2}, \tag{34}$$

where L_{an} and H_{an} $(n = 0,1,2)$ denote the n-th order single-particle Lagrangian and Hamiltonian in Δ for species a, respectively [the subscript j for labeling each particle in Eq. (6) is omitted here]. By using $\mathbf{v}_{a0}$ as the zeroth-order variable instead of $\mathbf{v}_a$, all the perturbation parts of the Lagrangian given by Eqs. (33) and (34) are confined in the Hamiltonian part, and they do not depend on $\dot{\mathbf{x}}_a$. This enables the variable transformation from the guiding-center to gyrocenter coordinates to be symplectic, as shown later.

3.2 *Guiding-center coordinates*

The single-particle guiding-center coordinates $\mathbf{Z}_a = (Z_a^i)_{i=1,\ldots,6} = (\mathbf{X}_a, U_a, \mu_a, \xi_a)$ for species a are defined by taking account of the equilibrium electromagnetic fields. First, we consider the preliminary transformation,

$$(\mathbf{x}_a, \mathbf{v}_{a0}) \rightarrow \mathbf{z}_a = (z_a^i)_{i=1,\ldots,6} = (\mathbf{x}_a, v_{a0\|}, \mu_{a0}, \theta_a) \tag{35}$$

Here, $v_{a0\|}$, μ_{a0}, and θ_a are defined by

$$v_{a0\|} = \mathbf{v}_{a0}\cdot\mathbf{b}, \qquad \mu_{a0} = \frac{m_a v_{a0\perp}^2}{2B_0}, \tag{36}$$

and

$$\mathbf{v}_{a0\perp} = \mathbf{v}_{a0} - v_{a0\|}\mathbf{b} = -v_{a0\perp}(\sin\theta_a\,\mathbf{e}_1 + \cos\theta_a\,\mathbf{e}_2), \tag{37}$$

respectively, where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{b} \equiv \mathbf{B}_0/B_0)$ are unit vectors which form a right-hand orthogonal system at $\mathbf{x}_a$.

In order to remove the gyrophase dependence from the equilibrium part L_{a0} of the single-particle Lagrangian given by Eq. (32), we introduce the guiding-center transformation of the phase-space coordinates,

$$\mathbf{z}_a = (\mathbf{x}_a, v_{a0\|}, \mu_{a0}, \theta_a) \rightarrow \mathbf{Z}_a = (\mathbf{X}_a, U_a, \mu_a, \xi_a), \tag{38}$$

This guiding-center transformation is the near-identity Lie transform,[14,23]

$$\mathbf{X}_a = \mathbf{x}_a - \epsilon\boldsymbol{\rho}_{a0} + \mathcal{O}(\epsilon^2), \; U_a = v_{a0\|} + \mathcal{O}(\epsilon),$$
$$\mu_a = \mu_{a0} + \mathcal{O}(\epsilon), \; \xi_a = \theta_a + \mathcal{O}(\epsilon), \tag{39}$$

where $\boldsymbol{\rho}_{a0} \equiv \mathbf{b}\times\mathbf{v}_{a0}/\Omega_a$ and $\Omega_a \equiv e_aB_0/(m_ac)$. Detailed expressions for the $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ terms are found in Ref. 14. In terms of the guiding-center coordinates $(\mathbf{X}_a, U_a, \mu_a, \xi_a)$, The Lagrangian L_{a0} is written as

$$L_{a0} = \epsilon^{-1}\frac{e_a}{c}\mathbf{A}_a^*(\mathbf{X}_a, U_a, \mu_a)\cdot\dot{\mathbf{X}}_a + \epsilon\frac{m_ac}{e_a}\mu_a\dot{\xi}_a$$
$$- \bar{H}_{a0}(\mathbf{X}_a, U_a, \mu_a). \tag{40}$$

Here, the definitions of $\mathbf{A}_a^*$, and $\bar{H}_{a0}$ are written, up to the third lowest order in ϵ, by

$$\bar{H}_{a0}(\mathbf{X}_a, U_a, \mu_a) = \frac{1}{2}m_a|\mathbf{v}_{a0}(\mathbf{Z}_a)|^2 = \frac{1}{2}m_aU_a^2 + \mu_aB_0(\mathbf{X}_a), \tag{41}$$

and

$$\mathbf{A}_a^*(\mathbf{X}_a, U_a, \mu_a) = \mathbf{A}_0(\mathbf{X}_a) + \epsilon\frac{m_a c}{e_a}U_a\mathbf{b}(\mathbf{X}_a) - \epsilon^2\frac{m_a c^2}{e_a^2}\mu_a\mathbf{W}(\mathbf{X}_a), \quad (42)$$

respectively, where

$$\mathbf{v}_{a0}(\mathbf{Z}_a) = U_a\mathbf{b}(\mathbf{X}_a) - [2\mu_a B_0(\mathbf{X}_a)/m_a]^{1/2} \times [\sin\xi_a\, \mathbf{e}_1(\mathbf{X}_a) + \cos\xi_a\, \mathbf{e}_2(\mathbf{X}_a)], \quad (43)$$

and

$$\mathbf{W}(\mathbf{X}_a) = [\nabla\mathbf{e}_1(\mathbf{X}_a)]\cdot\mathbf{e}_2(\mathbf{X}_a) + \frac{1}{2}\mathbf{b}(\mathbf{X}_a)\mathbf{b}(\mathbf{X}_a)\cdot[\nabla\times\mathbf{b}(\mathbf{X}_a)]. \quad (44)$$

The single-particle Lagrangian in Eq. (40) determines the symplectic structure, which is represented by the differential 2-form ω, and the Hamiltonian flow in the single-particle phase space.[9,23,29] Taking the inverse of the matrix (ω_{ij}) with ω_{ij} being the components of the symplectic structure ω, the Poisson brackets for pairs of the guiding-center coordinates are obtained. Consequently, the nonvanishing Poisson brackets are given by

$$\{\mathbf{X}_a, \mathbf{X}_a\} = \epsilon\frac{c}{e_a B_{a\|}^*}\mathbf{b}\times\mathbf{I}, \quad (45)$$

$$\{\mathbf{X}_a, U_a\} = \frac{\mathbf{B}_a^*}{m_a B_{a\|}^*}, \quad (46)$$

$$\{\mathbf{X}_a, \xi_a\} = \epsilon\frac{c}{e_a B_{a\|}^*}\mathbf{b}\times\mathbf{W}, \quad (47)$$

$$\{U_a, \xi_a\} = -\frac{\mathbf{B}_a^*\cdot\mathbf{W}}{m_a B_{a\|}^*}, \quad (48)$$

$$\{\xi_a, \mu_a\} = \epsilon^{-1}\frac{e_a}{m_a c}, \quad (49)$$

where

$$\mathbf{B}_a^* \equiv \nabla\times\mathbf{A}_a^*, \qquad B_{a\|}^* \equiv \mathbf{B}_a^*\cdot\mathbf{b}, \quad (50)$$

and $\mathbf{I} \equiv \mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{bb}$ represents the unit dyadic. The quantities in the right-hand sides of Eqs. (45)–(47) are evaluated at $\mathbf{Z}_a = (\mathbf{X}_a, U_a, \mu_a, \xi_a)$, and $\nabla = \partial/\partial\mathbf{X}_a$ in Eq. (50). It should be noted that the Poisson bracket $\{\cdot,\cdot\}$ written here is relevant to the symplectic structure in the particle phase space only. The notation $\{\cdot,\cdot\}$ does not represent the Poisson bracket in the phase space for the total system, which requires to treat the electromagnetic fields as part of the phase-space coordinates.

We find from Eqs. (40), (41), and (45)–(49) that, in the guiding-center coordinates $\mathbf{Z}_a = (\mathbf{X}_a, U_a, \mu_a, \xi_a)$, dependence on the gyrophase ξ_a disappears from the equilibrium part of the single-particle Lagrangian L_{a0}, Hamiltonian $\bar{H}_{a0}$, and Poisson brackets. Therefore, if there are no perturbed electromagnetic fields, the gyromotion is completely decoupled from the equations of motion, and the magnetic moment μ_a is a constant of motion. However, for the turbulent system, the gyrophase dependence appears through the perturbation part of the Lagrangian, which is removed by transformation from the guiding-center to gyrocenter coordinates, as shown in the next subsection.

3.3 Gyrocenter coordinates

As mentioned at the end of §3.1, owing to the use of $\mathbf{v}_{a0}$, the perturbation parts L_{a1} and L_{a2} of the single-particle Lagrangian given by Eqs. (33) and (34) change only the Hamiltonian part, although the other part (or the symplectic part) of the equilibrium Lagrangian L_{a0} are not perturbed. As shown in Eq. (40), the symplectic part of L_{a0} has already taken a desired form in the guiding-center coordinates, which gives the gyrophase-independent Poisson brackets in Eqs. (45)–(49). Then, by the gyrocenter transformation from the guiding-center to gyrocenter coordinates,

$$\mathbf{Z}_a = (\mathbf{X}_a, U_a, \mu_a, \xi_a) \to \bar{\mathbf{Z}}_a = (\bar{\mathbf{X}}_a, \bar{U}_a, \bar{\mu}_a, \bar{\xi}_a), \quad (51)$$

we remove the gyrophase dependence of the perturbed Hamiltonian without changing the symplectic structure or the form of the Poisson brackets for the guiding-center coordinates. This is done by the symplectic Lie (or canonical) transform,[23] which is associated with appropriate generating functions [see Eq. (57)]. The resultant expression for the single-particle Lagrangian in terms of the gyrocenter coordinates $\bar{\mathbf{Z}}_a = (\bar{Z}_a^i)_{i=1,\cdots,6} = (\bar{\mathbf{X}}_a, \bar{U}_a, \bar{\mu}_a, \bar{\xi}_a)$ is given by

$$L_a = L_{a0} + \Delta L_{a1} + \Delta^2 L_{a2} = \epsilon^{-1}\frac{e_a}{c}\mathbf{A}_a^*(\bar{\mathbf{X}}_a, \bar{U}_a, \bar{\mu}_a)\cdot\dot{\bar{\mathbf{X}}}_a + \epsilon\frac{m_a c}{e_a}\bar{\mu}_a\dot{\bar{\xi}}_a - \bar{H}_a(\bar{\mathbf{X}}_a, \bar{U}_a, \bar{\mu}_a, t), \quad (52)$$

where the gyrophase-independent Hamiltonian is written up to $\mathcal{O}(\Delta^2)$ as

$$\bar{H}_a(\bar{\mathbf{X}}_a, \bar{U}_a, \bar{\mu}_a, t) = \bar{H}_{a0} + \Delta\bar{H}_{a1} + \Delta^2\bar{H}_{a2}. \quad (53)$$

The zeroth-order Hamiltonian $\bar{H}_{a0}(\bar{\mathbf{X}}_a, \bar{U}_a, \bar{\mu}_a, t)$ is given by Eq. (41) with $(\mathbf{X}_a, U_a, \mu_a)$ replaced by $(\bar{\mathbf{X}}_a, \bar{U}_a, \bar{\mu}_a)$, and the first and second-order Hamiltonians are written as

$$\bar{H}_{a1}(\bar{\mathbf{X}}_a, \bar{U}_a, \bar{\mu}_a, t) = e_a\langle\psi_a(\bar{\mathbf{Z}}, t)\rangle_{\bar{\xi}_a} = e_a\left\langle\phi_1(\bar{\mathbf{X}}_a + \epsilon\bar{\boldsymbol{\rho}}_a, t) - \frac{1}{c}\mathbf{v}_{a0}(\bar{\mathbf{Z}}_a)\cdot\mathbf{A}_1(\bar{\mathbf{X}}_a + \epsilon\bar{\boldsymbol{\rho}}_a, t)\right\rangle_{\bar{\xi}_a}, \quad (54)$$

and

$$\bar{H}_{a2}(\bar{\mathbf{X}}_a, \bar{U}_a, \bar{\mu}_a, t) = \frac{e_a^2}{2m_a c^2}\langle|\mathbf{A}_1(\bar{\mathbf{X}}_a + \epsilon\bar{\boldsymbol{\rho}}_a, t)|^2\rangle_{\bar{\xi}_a} - \frac{e_a}{2}\left\langle\left\{\tilde{S}_{a1}(\bar{\mathbf{Z}}_a, t), \tilde{\psi}_a(\bar{\mathbf{Z}}_a, t)\right\}\right\rangle_{\bar{\xi}_a}, \quad (55)$$

respectively, where $\bar{\boldsymbol{\rho}}_a = \boldsymbol{\rho}_{a0}(\bar{\mathbf{Z}}_a) = \mathbf{b}(\bar{\mathbf{X}}_a)\times\mathbf{v}_{a0}(\bar{\mathbf{Z}}_a)/\Omega_a(\bar{\mathbf{X}}_a)$. Here, the gyrophase-average and gyrophase-dependent parts of an arbitrary periodic gyrophase function $Q(\bar{\xi}_a)$ are defined by

$$\langle Q\rangle_{\bar{\xi}_a} \equiv \oint\frac{d\bar{\xi}_a}{2\pi}Q(\bar{\xi}_a) \quad \text{and} \quad \tilde{Q} \equiv Q - \langle Q\rangle_{\bar{\xi}_a}, \quad (56)$$

respectively. The Poisson brackets $\{Z^i, Z^j\}$ for the gyrocenter coordinates have the same forms as those for the guiding-center coordinates, which are given by Eqs. (45)–(49) with $\mathbf{Z}_a = (\mathbf{X}_a, U_a, \mu_a, \xi_a)$ replaced by $\bar{\mathbf{Z}}_a = (\bar{\mathbf{X}}_a, \bar{U}_a, \bar{\mu}_a, \bar{\xi}_a)$.

The relations of the gyrocenter coordinates $\bar{\mathbf{Z}}_a = (\bar{\mathbf{X}}_a, \bar{U}_a, \bar{\mu}_a, \bar{\xi}_a)$ to the guiding-center coordinates $\mathbf{Z}_a = (\mathbf{X}_a, U_a, \mu_a, \xi_a)$ are written as

$$\bar{\mathbf{Z}}_a = \mathbf{Z}_a + \Delta\{\widetilde{S}_{a1}(\mathbf{Z}_a, t), \mathbf{Z}_a\} + \mathcal{O}(\Delta^2) \qquad (57)$$

Here, the first-order generating function $\widetilde{S}_{a1}$ is determined as the solution of

$$\frac{\partial \widetilde{S}_{a1}(\bar{\mathbf{Z}}_a, t)}{\partial t} + \left\{\widetilde{S}_{a1}(\bar{\mathbf{Z}}_a, t), \bar{H}_{a0}(\bar{\mathbf{X}}_a, \bar{U}_a, \bar{\mu}_a, t)\right\} = e_a \widetilde{\psi}_a(\bar{\mathbf{Z}}_a, t) \qquad (58)$$

Using the conventional gyrokinetic assumption that $\Omega_a^{-1}\partial/\partial t = \mathcal{O}(\epsilon)$, and neglecting higher order terms in ϵ, Eq. (58) reduces to $\epsilon^{-1}\Omega_a \partial \widetilde{S}_{a1}/\partial \bar{\xi}_a = e_a \widetilde{\psi}_a$, the solution of which is given by

$$\widetilde{S}_{a1}(\bar{\mathbf{Z}}_a, t) = \epsilon \frac{e_a}{\Omega_a(\bar{\mathbf{X}}_a)} \int \widetilde{\psi}_a(\bar{\mathbf{Z}}_a, t) d\bar{\xi}_a, \qquad (59)$$

where the integral constant is determined from the condition $\langle \widetilde{S}_{a1}\rangle_{\bar{\xi}_a} = 0$. [The case, in which the fluctuation frequencies are allowed to be on the order of the gyrofrequency, is considered in the next section.] Then, we find from Eqs. (57) and (59) that $\bar{\mathbf{X}}_a = \mathbf{X}_a + \mathcal{O}(\Delta\epsilon, \Delta^2)$. Following Brizard,[9] for the particle position $\mathbf{x}_a$ as the argument of the perturbation fields, we put $\mathbf{x}_a = \bar{\mathbf{X}}_a + \epsilon\bar{\boldsymbol{\rho}}_a$ by neglecting $\mathcal{O}(\Delta\epsilon, \Delta^2, \epsilon^2)$ terms. This approximation has already been used to evaluate the fluctuations ϕ_1 and $\mathbf{A}_1$ at the position $\mathbf{x}_a$ in Eqs. (54) and (55).

From Eqs. (4) and (52), the total Lagrangian for the system is given by

$$\begin{aligned} L &= L_p + L_{int} + L_f \\ &= \sum_a \sum_{j=1}^{N_a} \left[\epsilon^{-1}\frac{e_a}{c}\mathbf{A}_a^*(\bar{\mathbf{X}}_{aj}, \bar{U}_{aj}, \bar{\mu}_{aj}) \cdot \dot{\bar{\mathbf{X}}}_{aj} + \epsilon \frac{m_a c}{e_a}\bar{\mu}_{aj}\dot{\bar{\xi}}_{aj} \right. \\ &\quad \left. - \bar{H}_a(\bar{\mathbf{X}}_{aj}, \bar{U}_{aj}, \bar{\mu}_{aj}, t)\right] + \frac{1}{8\pi}\int_V d^3\mathbf{x}\,(\Delta^2|\nabla\phi(\mathbf{x})|^2 \\ &\quad - |\nabla \times [\mathbf{A}_0(\mathbf{x}) + \Delta\mathbf{A}_1(\mathbf{x})]|^2)\,. \end{aligned} \qquad (60)$$

Using this total Lagrangian L, the variational principle, Eq. (1), yields the gyrocenter motion equations, the gyrokinetic Poisson's equation, and the gyrokinetic Ampère's law. The gyrocenter motion equations are derived from $\delta I/\delta\bar{\mathbf{Z}}_{aj} = 0$ as

$$\begin{aligned} \frac{d\bar{\mathbf{X}}_{aj}}{dt} &= \{\bar{\mathbf{X}}_{aj}, \bar{H}_a(\bar{\mathbf{X}}_{aj}, \bar{U}_{aj}, \bar{\mu}_{aj})\}_{\bar{\mathbf{Z}}_{aj}} \\ &= \frac{1}{B^*_{a\parallel}}\left[\left(\bar{U}_{aj} + \Delta\frac{e_a}{m_a}\frac{\partial\Psi_a(\bar{\mathbf{Z}}_{aj})}{\partial\bar{U}_{aj}}\right)\mathbf{B}^*_a \right. \\ &\quad \left. + \epsilon c\mathbf{b} \times \left(\frac{\bar{\mu}_{aj}}{e_a}\nabla B_0 + \Delta\nabla\Psi_a(\bar{\mathbf{Z}}_{aj})\right)\right], \end{aligned} \qquad (61)$$

$$\begin{aligned} \frac{d\bar{U}_{aj}}{dt} &= \{\bar{U}_{aj}, \bar{H}_{aj}(\bar{\mathbf{X}}_{aj}, \bar{U}_{aj}, \bar{\mu}_{aj})\}_{\bar{\mathbf{Z}}_{aj}} \\ &= -\frac{\mathbf{B}^*_a}{m_a B^*_{a\parallel}} \cdot [\bar{\mu}_{aj}\nabla B_0 + \Delta e_a \nabla\Psi_a(\bar{\mathbf{Z}}_{aj})], \end{aligned} \qquad (62)$$

$$\frac{d\bar{\mu}_{aj}}{dt} = \{\bar{\mu}_{aj}, \bar{H}_{aj}(\bar{\mathbf{X}}_{aj}, \bar{U}_{aj}, \bar{\mu}_{aj})\}_{\bar{\mathbf{Z}}_{aj}} = 0, \qquad (63)$$

and

$$\begin{aligned} \frac{d\bar{\xi}_{aj}}{dt} &= \{\bar{\xi}_{aj}, \bar{H}_{aj}(\bar{\mathbf{X}}_{aj}, \bar{U}_{aj}, \bar{\mu}_{aj})\}_{\bar{\mathbf{Z}}_{aj}} \\ &= \frac{\Omega_a}{\epsilon} + \mathbf{W}\cdot\frac{d\bar{\mathbf{X}}_{aj}}{dt} + \frac{\Delta}{\epsilon}\frac{e_a^2}{m_a c}\frac{\partial\Psi_a(\bar{\mathbf{Z}}_{aj})}{\partial\bar{\mu}_{aj}}, \end{aligned} \qquad (64)$$

where the effects of the fluctuating electromagnetic fields are included in the potential Ψ_a defined by

$$\begin{aligned} \Psi_a(\bar{\mathbf{Z}}_a) &= \langle\psi_a(\bar{\mathbf{Z}}, t)\rangle_{\bar{\xi}_a} + \Delta\left[\frac{e_a}{2m_a c^2}\langle|\mathbf{A}_1(\bar{\mathbf{X}}_a + \epsilon\bar{\boldsymbol{\rho}}_a, t)|^2\rangle_{\bar{\xi}_a} \right. \\ &\quad \left. - \frac{1}{2}\left\langle\left\{\widetilde{S}_{a1}(\bar{\mathbf{Z}}_a, t), \widetilde{\psi}_a(\bar{\mathbf{Z}}_a, t)\right\}\right\rangle_{\bar{\xi}_a}\right] \end{aligned} \qquad (65)$$

The right-hand-side terms in Eqs. (61)–(64) are all evaluated at $\bar{\mathbf{Z}}_{aj}$ in the single-particle phase space. The notation $\{f, g\}_{\bar{\mathbf{Z}}_{aj}}$ is used to clearly represent that the Poisson bracket operates on functions f and g defined on the single-particle phase space associated with the coordinates $\bar{\mathbf{Z}}_{aj}$.

From $\delta I/\delta\phi_1 = 0$ and $\delta I/\delta\mathbf{A}_1 = 0$, the gyrokinetic Poisson's equation and the gyrokinetic Ampère's law are derived as

$$\begin{aligned} \Delta\nabla^2\phi_1(\mathbf{x}, t) &= -4\pi\sum_a e_a \sum_{j=1}^{N_a}\left\langle\delta^3(\bar{\mathbf{X}}_{aj} + \epsilon\bar{\rho}_{aj} - \mathbf{x}) \right. \\ &\quad \left. - \Delta\left\{\widetilde{S}_{a1}(\bar{\mathbf{Z}}_{aj}, t), \delta^3(\bar{\mathbf{X}}_{aj} + \epsilon\bar{\rho}_{aj} - \mathbf{x})\right\}_{\bar{\mathbf{Z}}_{aj}}\right\rangle_{\bar{\xi}_a} \\ &\equiv -4\pi\sum_a e_a n_{Ga}^{micro}, \end{aligned} \qquad (66)$$

and

$$\begin{aligned} &\nabla \times [\nabla \times [\mathbf{A}_0(\mathbf{x}) + \Delta\mathbf{A}_1(\mathbf{x}, t)]] \\ &= \frac{4\pi}{c}\sum_a e_a \sum_{j=1}^{N_a}\left\langle\left[\mathbf{v}_{a0}(\bar{\mathbf{Z}}_{aj}) - \Delta\frac{e_a}{m_a c}\mathbf{A}_1(\bar{\mathbf{X}}_{aj} + \epsilon\bar{\rho}_{aj}, t)\right]\right. \\ &\quad \times \delta^3(\bar{\mathbf{X}}_{aj} + \epsilon\bar{\rho}_{aj} - \mathbf{x}) \\ &\quad \left. - \Delta\left\{\widetilde{S}_{a1}(\bar{\mathbf{Z}}_{aj}, t), \mathbf{v}_{a0}(\bar{\mathbf{Z}}_{aj})\delta^3(\bar{\mathbf{X}}_{aj} + \epsilon\bar{\rho}_{aj} - \mathbf{x})\right\}_{\bar{\mathbf{Z}}_{aj}}\right\rangle_{\bar{\xi}_a} \\ &\equiv \frac{4\pi}{c}\mathbf{j}_G^{micro}, \end{aligned} \qquad (67)$$

respectively, where $\bar{\rho}_{aj} = \rho_{a0}(\bar{\mathbf{Z}}_{aj}) = \mathbf{b}(\bar{\mathbf{X}}_{aj}) \times \mathbf{v}_{a0}(\bar{\mathbf{Z}}_{aj})/\Omega_a(\bar{\mathbf{X}}_{aj})$. Here, n_{Ga}^{micro} and $\mathbf{j}_G^{micro}$ represent the gyrokinetic expressions of the microscopic density and current, respectively.

In the same way as in Eq. (16), the conserved total energy is obtained with the help of Eq. (66) as

$$E_{Gtot} = \sum_a \sum_{j=1}^{N_a}\left(\dot{\bar{\mathbf{X}}}_{aj}\cdot\frac{\partial L}{\partial\dot{\bar{\mathbf{X}}}_{aj}} + \dot{\bar{\xi}}_{aj}\frac{\partial L}{\partial\dot{\bar{\xi}}_{aj}}\right) - L$$

$$= \sum_a \sum_{j=1}^{N_a} \bar{H}_a(\bar{\mathbf{X}}_{aj}, \bar{U}_{aj}, \bar{\mu}_{aj}, t) - L_f$$

$$= \sum_a \sum_{j=1}^{N_a} \left\langle \frac{1}{2} m_a \left[\mathbf{v}_{a0}(\bar{\mathbf{Z}}_{aj}) - \Delta \frac{e_a}{m_a c} \mathbf{A}_1(\bar{\mathbf{X}}_{aj} + \epsilon \bar{\boldsymbol{\rho}}_{aj}, t) \right]^2 + \frac{e_a^2}{2\Omega_a(\bar{\mathbf{X}}_{aj})} \Delta^2 \left[\left\{ \int (\widetilde{\phi}_1)_{aj} d\bar{\xi}_{aj}, (\widetilde{\phi}_1)_{aj} \right\}_{\bar{\mathbf{Z}}_{aj}} - \frac{1}{c^2} \left\{ \int (\widetilde{\mathbf{v}_0 \cdot \mathbf{A}_1})_{aj} d\bar{\xi}_{aj}, (\widetilde{\mathbf{v}_0 \cdot \mathbf{A}_1})_{aj} \right\}_{\bar{\mathbf{Z}}_{aj}} \right] \right\rangle_{\bar{\xi}_{aj}}$$
$$+ \frac{1}{8\pi} \int_V d^3\mathbf{x} \, (\Delta^2 |\nabla \phi_1(\mathbf{x}, t)|^2 + |\nabla \times [\mathbf{A}_0(\mathbf{x}) + \Delta \mathbf{A}_1(\mathbf{x}, t)]|^2), \quad (68)$$

where $(\phi_1)_{aj} = \phi_1(\bar{\mathbf{X}}_{aj} + \epsilon \bar{\boldsymbol{\rho}}_{aj}, t)$ and $(\mathbf{v}_0 \cdot \mathbf{A}_1)_{aj} = \mathbf{v}_0(\bar{\mathbf{Z}}_{aj}) \cdot \mathbf{A}_1(\bar{\mathbf{X}}_{aj} + \epsilon \bar{\boldsymbol{\rho}}_{aj}, t)$. Here, it should be recalled from Eq (30) that $\mathbf{v}_a = \mathbf{v}_{a0} - \Delta e_a \mathbf{A}_1/(m_a c)$ represents the true particle velocity.

3.4 *Gyrokinetic Vlasov-Poisson-Ampère system*

In the previous subsection as well as in §2.1, we have described the discrete particles' motion and the microscopic electromagnetic fields, in which the effect of Coulomb collisions is included. In this subsection, the gyrokinetic system of equations for the particle distribution function and the macroscopic electromagnetic fields in the collisionless (or Vlasov) plasma are derived by the generalized Lagrangian formulation.

In the same manner as in §2.2, we perform the following replacement for the particle part of the Lagrangian in Eq. (60),

$$\sum_{j=1}^{N_a} \to \int d^6\bar{\mathbf{Z}}_0 \, D_a(\bar{\mathbf{Z}}_0) F_a(\bar{\mathbf{Z}}_0, t_0),$$
$$\bar{\mathbf{Z}}_{aj}(t) \to \mathbf{Z}_a^*(\bar{\mathbf{Z}}_0, t_0; t), \quad (69)$$

where $\bar{\mathbf{Z}}_{aj} \equiv (\bar{\mathbf{X}}_{aj}, \bar{U}_{aj}, \bar{\mu}_{aj}, \bar{\xi}_{aj})$, $\mathbf{Z}_a^* \equiv (\mathbf{X}_a^*, U_a^*, \mu_a^*, \xi_a^*)$, $\bar{\mathbf{Z}}_0 \equiv (\bar{\mathbf{X}}_0, \bar{U}_0, \bar{\mu}_0, \bar{\xi}_0)$, and $\int d^6\bar{\mathbf{Z}}_0 \equiv \int_V d^3\bar{\mathbf{X}}_0 \int_{-\infty}^{\infty} d\bar{U}_0 \int_0^{\infty} d\bar{\mu}_0 \int_0^{2\pi} d\bar{\xi}_0$. Here, $D_a(\bar{\mathbf{Z}}_0) \equiv B_{a\|}^*(\bar{\mathbf{Z}}_0)/m_a$ is the Jacobian, $F_a(\bar{\mathbf{Z}}_0, t_0)$ denotes the distribution function for species a at an arbitrarily specified initial time t_0, and $\mathbf{Z}_a^*(\bar{\mathbf{Z}}_0, t_0; t)$ represents the gyrocenter coordinates of the particle, which satisfy the initial condition,

$$\mathbf{Z}_a^*(\bar{\mathbf{Z}}_0, t_0; t_0) = \bar{\mathbf{Z}}_0. \quad (70)$$

The functional form of $\mathbf{Z}_a^*(\bar{\mathbf{Z}}_0, t_0; t)$ is determined by the variational principle with the Lagrangian as shown later. Then, the distribution function $F_a(\bar{\mathbf{Z}}, t)$ for an arbitrary time t is determined by

$$D_a(\bar{\mathbf{Z}}) F_a(\bar{\mathbf{Z}}, t) = \int d^6\bar{\mathbf{Z}}_0 \, D_a(\bar{\mathbf{Z}}_0) F_a(\bar{\mathbf{Z}}_0, t_0) \delta^6[\bar{\mathbf{Z}} - \mathbf{Z}_a^*(\bar{\mathbf{Z}}_0, t_0; t)], \quad (71)$$

where $\delta^6(\bar{\mathbf{Z}} - \mathbf{Z}_a^*) = \delta^3(\bar{\mathbf{X}} - \mathbf{X}_a^*) \delta(\bar{U} - U_a^*) \delta(\bar{\mu} - \mu_a^*) \delta[\bar{\xi} - \xi_a^*(\mathrm{mod} 2\pi)]$.

Using Eqs. (51) and (69), the Lagrangian for the gyrokinetic Vlasov-Poisson-Ampère system is given by

$$L = L_p + L_{int} + L_f$$
$$= \sum_a \int d^6\bar{\mathbf{Z}}_0 \, D_a(\bar{\mathbf{Z}}_0) F_a(\bar{\mathbf{Z}}_0, t_0) \times L_a[\mathbf{Z}_a^*(\bar{\mathbf{Z}}_0, t_0; t), \dot{\mathbf{Z}}_a^*(\bar{\mathbf{Z}}_0, t_0; t), t] + \frac{1}{8\pi} \int_V d^3\mathbf{x} \, (\Delta^2 |\nabla \phi_1(\mathbf{x})|^2 - |\nabla \times [\mathbf{A}_0(\mathbf{x}) + \Delta \mathbf{A}_1(\mathbf{x}, t)]|^2), \quad (72)$$

with the single-particle Lagrangian L_a for species a defined by

$$L_a(\mathbf{Z}_a^*, \dot{\mathbf{Z}}_a^*, t) \equiv \epsilon^{-1} \frac{e_a}{c} \mathbf{A}_a^*(\mathbf{X}_a^*, U_a^*, \mu_a^*) \cdot \dot{\mathbf{X}}_a^* + \epsilon \frac{m_a c}{e_a} \mu_a^* \dot{\xi}_a^* - \bar{H}_a(\mathbf{X}_a^*, U_a^*, \mu_a^*, t), \quad (73)$$

where the single-particle gyrocenter Hamiltonian $\bar{H}_a$ is defined by Eqs. (53)–(55) and (41). We obtain from $\delta I/\delta \mathbf{Z}_a^* = \partial L/\partial \mathbf{Z}_a^* - d(\partial L/\partial \dot{\mathbf{Z}}_a^*)/dt = 0$ that

$$\frac{d\mathbf{Z}_a^*}{dt} = \{\mathbf{Z}_a^*, \bar{H}_a(\mathbf{Z}_a^*, t)\}_{\mathbf{Z}_a^*}, \quad (74)$$

where $\bar{H}_a(\mathbf{Z}_a^*, t)$ is a simplified notation for $\bar{H}_a(\mathbf{X}_a^*, U_a^*, \mu_a^*, t)$ and $\{\cdot, \cdot\}_{\mathbf{Z}_a^*}$ represents the Poisson bracket with the same structure in $\mathbf{Z}_a^*$ as shown by Eqs. (45)–(49). The right-hand sides of the motion equations (74) are given by those of Eqs. (61)–(64) with $\bar{\mathbf{Z}}_{aj}$ replaced by $\mathbf{Z}_a^*$. Since the right-hand side of Eq. (74) is independent of the gyrophase ξ_a^*, it is easily found that $\mathbf{X}_a^*(\bar{\mathbf{Z}}_0, t_0 : t)$, $U_a^*(\bar{\mathbf{Z}}_0, t_0 : t)$, and $\mu_a^*(\bar{\mathbf{Z}}_0, t_0 : t)$ are independent of the initial gyrophase $\bar{\xi}_0$. The Jacobian D_a is also gyrophase-independent. Then, we find from Eq. (71) that, if F_a is initially gyrophase-independent, it is gyrophase-independent at any time. Hereafter, we assume without loss of generality that F_a is gyrophase-independent, $\partial F_a(\bar{\mathbf{Z}}, t)/\partial \bar{\xi} = 0$. We also obtain the gyrocenter phase-space conservation law,

$$\frac{\partial}{\partial \bar{\mathbf{Z}}} \cdot \left[D_a(\bar{\mathbf{Z}}) \{\bar{\mathbf{Z}}, \bar{H}_a(\bar{\mathbf{Z}}, t)\}_{\bar{\mathbf{Z}}} \right] = 0. \quad (75)$$

From Eqs. (71) and (74), we have the gyrokinetic Vlasov equation in the conservation form,

$$\frac{\partial}{\partial t} [D_a(\bar{\mathbf{Z}}) F_a(\bar{\mathbf{Z}}, t)] + \frac{\partial}{\partial \bar{\mathbf{Z}}} \cdot \left[D_a(\bar{\mathbf{Z}}) F_a(\bar{\mathbf{Z}}, t) \{\bar{\mathbf{Z}}, \bar{H}_a(\bar{\mathbf{Z}}, t)\}_{\bar{\mathbf{Z}}} \right] = 0, \quad (76)$$

which is rewritten with the help of Eq. (75) in the convection form,

$$\left[\frac{\partial}{\partial t} + \{\bar{\mathbf{Z}}, \bar{H}_a(\bar{\mathbf{Z}}, t)\}_{\bar{\mathbf{Z}}} \cdot \frac{\partial}{\partial \bar{\mathbf{Z}}} \right] F_a(\bar{\mathbf{Z}}, t) = 0. \quad (77)$$

From $\delta I/\delta \phi_1 = \delta I/\delta \mathbf{A}_1 = 0$, the gyrokinetic Poisson's equation and the gyrokinetic Ampère's law are obtained as

$$\Delta \nabla^2 \phi_1(\mathbf{x}, t) = -4\pi \sum_a e_a \int d^6\bar{\mathbf{Z}} D_a(\bar{\mathbf{Z}}) \delta^3(\bar{\mathbf{X}} + \epsilon \bar{\boldsymbol{\rho}}_a - \mathbf{x}) \times \left[F_a(\bar{\mathbf{Z}}, t) + \Delta \left\{ \widetilde{S}_{a1}(\bar{\mathbf{Z}}, t), F_a(\bar{\mathbf{Z}}, t) \right\}_{\bar{\mathbf{Z}}} \right]$$
$$\equiv -4\pi \sum_a e_a n_{Ga}, \quad (78)$$

and

$$\nabla \times [\nabla \times [\mathbf{A}_0(\mathbf{x}) + \Delta \mathbf{A}_1(\mathbf{x},t)]]$$
$$= \frac{4\pi}{c}\sum_a e_a \int d^6\bar{\mathbf{Z}} D_a(\bar{\mathbf{Z}})\delta^3(\bar{\mathbf{X}} + \epsilon\bar{\rho}_a - \mathbf{x})$$
$$\times \left(\left[\mathbf{v}_{a0}(\bar{\mathbf{Z}}) - \Delta\frac{e_a}{m_a c}\mathbf{A}_1(\bar{\mathbf{X}} + \epsilon\bar{\rho}_a, t)\right] F_a(\bar{\mathbf{Z}},t)\right.$$
$$\left. + \Delta \mathbf{v}_{a0}(\bar{\mathbf{Z}})\left\{\widetilde{S}_{a1}(\bar{\mathbf{Z}},t), F_a(\bar{\mathbf{Z}},t)\right\}_{\bar{\mathbf{Z}}}\right)$$
$$\equiv \frac{4\pi}{c}\mathbf{j}_G, \qquad (79)$$

respectively, where $\int d^6\bar{\mathbf{Z}} \equiv \int_V d^3\bar{\mathbf{X}} \int_{-\infty}^{\infty} d\bar{U} \int_0^{\infty} d\bar{\mu} \int_0^{2\pi} d\bar{\xi}$ and $\bar{\rho}_a = \bar{\rho}_{a0}(\bar{\mathbf{Z}}) = \mathbf{b}(\bar{\mathbf{X}}) \times \mathbf{v}_{a0}(\bar{\mathbf{Z}})/\Omega_a(\bar{\mathbf{X}}_a)$. Here, n_{Ga} and $\mathbf{j}_G$ represent the gyrokinetic expressions of the macroscopic density and current respectively. It should be noted that the distribution function F_a^{gc} in the guiding-center coordinates is related to the distribution function F_a in the gyrocenter coordinates by $F_a^{gc}(\bar{\mathbf{Z}},t) = F_a(\bar{\mathbf{Z}},t) + \Delta\left\{\widetilde{S}_{a1}(\bar{\mathbf{Z}},t), F_a(\bar{\mathbf{Z}},t)\right\}_{\bar{\mathbf{Z}}} + \mathcal{O}(\Delta^2)$. Thus, the right-hand sides of Eqs. (78) and (79) represent the velocity-space integrals of $F_a^{gc}(\bar{\mathbf{Z}},t)$ and $\mathbf{v}_a F_a^{gc}(\bar{\mathbf{Z}},t)$, respectively, with $\mathcal{O}(\Delta^2)$ terms neglected. Then, we find that the motion equations (74) are accurate up to $\mathcal{O}(\Delta^2)$ [see Eq. (51)] while the gyrokinetic Poisson-Ampère equations (78)-(79) are accurate up to $\mathcal{O}(\Delta)$. This combination of unbalanced orders of accuracy is a direct result of the variational principle based on the Lagrangian (72) and is necessary for the existence of the invariant total energy. Thus, the orders of accuracy for all the governing equations are determined more systematically in the present formulation than in the conventional Lagrangian or Hamiltonian gyrokinetic theories.

The conserved total energy is written with the help of Eq. (78) as

$$E_{Gtot}$$
$$= \sum_a \int d^6\bar{\mathbf{Z}}_0\, D_a(\bar{\mathbf{Z}}_0)F_a(\bar{\mathbf{Z}}_0,t_0)\dot{\mathbf{Z}}_a^* \;\frac{\partial L_a(\mathbf{Z}_a^*, \dot{\mathbf{Z}}_a^*, t)}{\partial \dot{\mathbf{Z}}_a^*} - L$$
$$= \sum_a \int d^6\bar{\mathbf{Z}}\, D_a(\bar{\mathbf{Z}})F_a(\bar{\mathbf{Z}},t)\bar{H}_a(\bar{\mathbf{Z}},t) - L_f$$
$$= \sum_a \int d^6\bar{\mathbf{Z}}\, D_a(\bar{\mathbf{Z}})F_a(\bar{\mathbf{Z}},t)$$
$$\times \left(\frac{1}{2}m_a\left[\mathbf{v}_{a0}(\bar{\mathbf{Z}}) - \Delta\frac{e_a}{m_a c}\mathbf{A}_1(\bar{\mathbf{X}} + \epsilon\bar{\rho}_a, t)\right]^2\right.$$
$$+ \frac{e_a^2}{2\Omega_a(\bar{\mathbf{X}})}\Delta^2\left[\left\{\int(\widetilde{\phi}_1)_a d\bar{\xi}, (\widetilde{\phi}_1)_a\right\}_{\bar{\mathbf{Z}}}\right.$$
$$\left.\left. -\frac{1}{c^2}\left\{\int(\widetilde{\mathbf{v}_0 \cdot \mathbf{A}_1})_a d\bar{\xi}, (\widetilde{\mathbf{v}_0 \cdot \mathbf{A}_1})_a\right\}_{\bar{\mathbf{Z}}}\right]\right)$$
$$+ \frac{1}{8\pi}\int_V d^3\mathbf{x}\,(\Delta^2|\nabla\phi(\mathbf{x},t)|^2$$
$$+ |\nabla \times [\mathbf{A}_0(\mathbf{x}) + \Delta\mathbf{A}_1(\mathbf{x},t)]|^2), \qquad (80)$$

where $(\phi_1)_a = \phi_1(\bar{\mathbf{X}} + \epsilon\bar{\rho}_a, t)$ and $(\mathbf{v}_0 \cdot \mathbf{A}_1)_a = \mathbf{v}_0(\bar{\mathbf{Z}}) \cdot \mathbf{A}_1(\bar{\mathbf{X}} + \epsilon\bar{\rho}_a, t)$. The total energy E_{Gtot} in Eq. (80) [see also Eq. (68)] contains the $\mathcal{O}(\Delta^2)$ terms rewritten by

$$\frac{e_a^2}{2\Omega_a(\bar{\mathbf{X}})}\Delta^2\left[\left\{\int(\widetilde{\phi}_1)_a d\bar{\xi}, (\widetilde{\phi}_1)_a\right\}_{\bar{\mathbf{Z}}}\right.$$
$$\left. -\frac{1}{c^2}\left\{\int(\widetilde{\mathbf{v}_0 \cdot \mathbf{A}_1})_a d\bar{\xi}, (\widetilde{\mathbf{v}_0 \cdot \mathbf{A}_1})_a\right\}_{\bar{\mathbf{Z}}}\right]$$
$$= \frac{e_a}{2}\Delta^2\left\{\widetilde{S}_{a1}(\bar{\mathbf{Z}},t), (\widetilde{\phi}_1)_a + \frac{1}{c}(\widetilde{\mathbf{v}_0 \cdot \mathbf{A}_1})_a\right\}_{\bar{\mathbf{Z}}}$$
$$= \Delta^2\left\{\widetilde{S}_{a1}(\bar{\mathbf{Z}},t), \frac{\Omega_a(\bar{\mathbf{X}})}{2\epsilon}\frac{\partial\widetilde{S}_{a1}(\bar{\mathbf{Z}},t)}{\partial\bar{\xi}} + \frac{e_a}{c}(\widetilde{\mathbf{v}_0 \cdot \mathbf{A}_1})_a\right\}_{\bar{\mathbf{Z}}}$$
$$\simeq \Delta^2\left\{\widetilde{S}_{a1}(\bar{\mathbf{Z}},t), \frac{e_a}{c}(\widetilde{\mathbf{v}_0 \cdot \mathbf{A}_1})_a\right.$$
$$\left. + \frac{1}{2}\left\{\widetilde{S}_{a1}(\bar{\mathbf{Z}},t), \frac{1}{2}m_a|\mathbf{v}_{a0}(\bar{\mathbf{Z}})|^2\right\}_{\bar{\mathbf{Z}}}\right\}_{\bar{\mathbf{Z}}}, \qquad (81)$$

which coincide with the residual terms occurring in representing the particle kinetic energy in the gyrocenter coordinates $\bar{\mathbf{Z}}$ associated with the generating function $\widetilde{S}_{a1}$.[6,7] The energy related to the ion polarization is shown to be included in these terms.[6] The conservation of the total energy (80) is a direct result of the Noether's theorem applied to the Lagrangian (72) while, in the conventional Lagrangian or Hamiltonian gyrokinetic theories, it is more troublesome to prove the energy conservation from the gyrokinetic Vlasov equation and the Poisson-Ampère equations.

§4. GYROKINETIC THEORY FOR ARBITRARY FREQUENCIES

In the previous section, the characteristic frequencies ω of the fluctuations are assumed to be much smaller than the gyrofrequency Ω_a. In this section, the general Lagrangian formulation of the gyrokinetic theory is extended to the case of arbitrary fluctuation frequencies. The gyrokinetic system of equations derived here are applicable even for studying high-frequency fluctuations in the gyrofrequency range.

In the following subsections, we find arbitrary-frequency gyrokinetic descriptions for the discrete-particle system and for the continuous Vlasov system, which reduce to the results in §3.3 and §3.4, respectively, in the low-frequency limit.

4.1 *Discrete-particle system*

For the case of arbitrary fluctuation frequencies, Eq. (59) is no longer valid, since the time derivative term in Eq. (58) can not be neglected. Then, the generating function $\widetilde{S}_{a1}$ is not determined by the fluctuation field $\widetilde{\psi}_a$ at the instant time t but takes the form of the time integral of $\widetilde{\psi}_a$. If $\widetilde{S}_{a1}$ in the Lagrangian L is regarded as the time integral of the fluctuation field, the action integral I contains the double time integral and the conventional variational principle is not applicable directly. Instead, we regard $\widetilde{S}_{a1}$ as an independent variational field and utilize the method of Lagrange undetermined multipliers to derive Eq. (58) as a result of the variational principle.

Now, let us write the total Lagrangian for the arbitrary-frequency gyrokinetic system consisting of the discrete particles and the electromagnetic fields as

$$L = L_p + L_{int} + L_f + L_c$$
$$\equiv \sum_a \sum_{j=1}^{N_a} \left[\epsilon^{-1} \frac{e_a}{c} \mathbf{A}_a^*(\bar{\mathbf{X}}_{aj}, \bar{U}_{aj}, \bar{\mu}_{aj}) \cdot \dot{\bar{\mathbf{X}}}_{aj} + \epsilon \frac{m_a c}{e_a} \bar{\mu}_{aj} \dot{\bar{\xi}}_{aj} \right.$$
$$\left. - \bar{H}_a(\bar{\mathbf{X}}_{aj}, \bar{U}_{aj}, \bar{\mu}_{aj}, t) \right] + \frac{1}{8\pi} \int_V d^3\mathbf{x} \, (\Delta^2 |\nabla \phi_1(\mathbf{x}, t)|^2$$
$$- |\nabla \times [\mathbf{A}_0(\mathbf{x}) + \Delta \mathbf{A}_1(\mathbf{x}, t)]|^2)$$
$$+ \sum_a \sum_{j=1}^{N_a} \int d^6\bar{\mathbf{Z}} \, \Lambda_a(\bar{\mathbf{Z}}; \bar{\mathbf{X}}_{aj}, \bar{U}_{aj}, \bar{\mu}_{aj}; t)$$
$$\times \left[\left(\frac{\partial}{\partial t} + \frac{\Omega_a(\bar{\mathbf{X}})}{\epsilon} \frac{\partial}{\partial \bar{\xi}} \right) \tilde{S}_{a1}(\bar{\mathbf{Z}}, t) - e_a \tilde{\psi}_a(\bar{\mathbf{Z}}, t) \right]. \quad (82)$$

The Lagrangian in Eq. (82) contains the constraint part $L_c = \sum_a \sum_{j=1}^{N_a} \int d^6\bar{\mathbf{Z}} \; (\mathcal{L}_c)_{aj}$ where $(\mathcal{L}_c)_{aj} = \Lambda_a(\bar{\mathbf{Z}}; \bar{\mathbf{X}}_{aj}, \bar{U}_{aj}, \bar{\mu}_{aj}; t)[(\partial/\partial t + \epsilon^{-1}\Omega_a(\bar{\mathbf{X}})\partial/\partial\bar{\xi})\tilde{S}_{a1}(\bar{\mathbf{Z}}, t) - e_a \tilde{\psi}_a(\bar{\mathbf{Z}}, t)]$. Here, $\tilde{S}_{a1}$ and $\Lambda_a(\bar{\mathbf{Z}}; \bar{\mathbf{X}}_{aj}, \bar{U}_{aj}, \bar{\mu}_{aj}; t)$ are regarded as new independent variational fields for the variational principle $\delta I = \delta \int_{t_1}^{t_2} L dt = 0$. The field Λ_a plays the role of Lagrange multipliers Then, from $\delta I / \delta \Lambda_a = 0$, the constraint on $\tilde{S}_{a1}$ is obtained as

$$\left(\frac{\partial}{\partial t} + \frac{\Omega_a(\bar{\mathbf{X}})}{\epsilon} \frac{\partial}{\partial \bar{\xi}} \right) \tilde{S}_{a1}(\bar{\mathbf{Z}}, t) = e_a \tilde{\psi}_a(\bar{\mathbf{Z}}, t), \quad (83)$$

which corresponds to Eq. (58) with the time derivative term retained but higher ϵ order terms neglected. Equation (83) is solved with the condition $\langle \tilde{S}_{a1} \rangle_{\bar{\xi}} = 0$. The equation to determine Λ_a is derived from $\delta I / \delta \tilde{S}_{a1} = 0$ as

$$\left(\frac{\partial}{\partial t} + \frac{\Omega_a(\bar{\mathbf{X}})}{\epsilon} \frac{\partial}{\partial \bar{\xi}} \right) \Lambda_a(\bar{\mathbf{Z}}; \bar{\mathbf{X}}_{aj}, \bar{U}_{aj}, \bar{\mu}_{aj}; t)$$
$$= -\frac{e_a}{2} \left\langle \left\{ \tilde{\psi}_a(\bar{\mathbf{Z}}_{aj}, t), \delta^3(\bar{\mathbf{X}}_{aj} - \bar{\mathbf{X}}) \delta(\bar{U}_{aj} - \bar{U}) \right.\right.$$
$$\left.\left. \times \delta(\bar{\mu}_{aj} - \bar{\mu}) \delta[\bar{\xi}_{aj} - \bar{\xi} (\text{mod} 2\pi)] \right\}_{\bar{\mathbf{Z}}_{aj}} \right\rangle_{\bar{\xi}_{aj}}. \quad (84)$$

We find from comparison between Eqs. (83) and (84) that Λ_a can be given by

$$\Lambda_a(\bar{\mathbf{Z}}; \bar{\mathbf{X}}_{aj}, \bar{U}_{aj}, \bar{\mu}_{aj}; t)$$
$$= -\frac{1}{2} \left\langle \left\{ \tilde{S}_{a1}(\bar{\mathbf{Z}}_{aj}, t), \delta^3(\bar{\mathbf{X}}_{aj} - \bar{\mathbf{X}}) \delta(\bar{U}_{aj} - \bar{U}) \right.\right.$$
$$\left.\left. \times \delta(\bar{\mu}_{aj} - \bar{\mu}) \delta[\bar{\xi}_{aj} - \bar{\xi} (\text{mod} 2\pi)] \right\}_{\bar{\mathbf{Z}}_{aj}} \right\rangle_{\bar{\xi}_{aj}}. \quad (85)$$

The same form of motion equations as Eqs. (61)–(64) are obtained from $\delta I / \delta \bar{\mathbf{Z}}_{aj} = 0$ with Eq. (83). Also, the same form of gyrokinetic Poisson's equation and Ampère's law as Eqs. (66)–(67) are derived from $\delta I / \delta \phi_1 = 0$ and $\delta I / \delta \mathbf{A}_1 = 0$, respectively, with the help of Eq. (85). Thus, the gyrokinetic theory for discrete particles and arbitrary-frequency fluctuations is given by the motion equations (61)–(64), the gyrokinetic Poisson's equation (66), the gyrokinetic Ampère's law (67), and the generating-function equation (83).

The conserved total energy is derived from the total Lagrangian in Eq. (82) as

$$E_{Gtot}$$
$$= \sum_a \sum_{j=1}^{N_a} \left(\dot{\bar{\mathbf{X}}}_{aj} \cdot \frac{\partial L}{\partial \dot{\bar{\mathbf{X}}}_{aj}} + \dot{\bar{\xi}}_{aj} \cdot \frac{\partial L}{\partial \dot{\bar{\xi}}_{aj}} \right)$$
$$+ \sum_a \int d^6\bar{\mathbf{Z}} \, \dot{\tilde{S}}_{a1} \frac{\partial (\mathcal{L}_c)_{aj}}{\partial \dot{\tilde{S}}_{a1}} - L$$
$$= \sum_a \sum_{j=1}^{N_a} \left[\bar{H}_a(\bar{\mathbf{X}}_a, \bar{U}_a, \bar{\mu}_a, t) \right.$$
$$\left. - \frac{1}{2} \left\langle \left\{ \tilde{S}_{a1}(\bar{\mathbf{Z}}_{aj}, t), \frac{\partial \tilde{S}_{a1}(\bar{\mathbf{Z}}_{aj}, t)}{\partial t} \right\}_{\bar{\mathbf{Z}}_{aj}} \right\rangle_{\bar{\xi}_{aj}} \right] - L_f$$
$$= \sum_a \sum_{j=1}^{N_a} \left\langle \frac{1}{2} m_a \left[\mathbf{v}_{a0}(\bar{\mathbf{Z}}_{aj}) - \Delta \frac{e_a}{m_a c} \mathbf{A}_1(\bar{\mathbf{X}}_a + \epsilon \bar{\boldsymbol{\rho}}_{aj}, t) \right]^2 \right.$$
$$+ \Delta^2 \left\{ \tilde{S}_{a1}(\bar{\mathbf{Z}}_{aj}, t), \frac{\Omega_a(\bar{\mathbf{X}}_{aj})}{2\epsilon} \frac{\partial \tilde{S}_{a1}(\bar{\mathbf{Z}}_{aj}, t)}{\partial \bar{\xi}_{aj}} \right.$$
$$\left.\left. + \frac{e_a}{c} (\widetilde{\mathbf{v}_0 \cdot \mathbf{A}_1})_{aj} \right\}_{\bar{\mathbf{Z}}_{aj}} \right\rangle_{\bar{\xi}_{aj}} + \frac{1}{8\pi} \int_V d^3\mathbf{x} \, (\Delta^2 |\nabla \phi_1(\mathbf{x}, t)|^2$$
$$+ |\nabla \times [\mathbf{A}_0(\mathbf{x}) + \Delta \mathbf{A}_1(\mathbf{x}, t)]|^2), \quad (86)$$

where Eqs. (66) and (83) are also used

4.2 Continuous Vlasov system

The total Lagrangian for the continuous Vlasov system with arbitrary-frequency fluctuations is given by

$$L = L_p + L_{int} + L_f + L_c$$
$$= \sum_a \int d^6\bar{\mathbf{Z}}_0 \, D_a(\bar{\mathbf{Z}}_0) F_a(\bar{\mathbf{Z}}_0, t_0)$$
$$\times L_a[\mathbf{Z}_a^*(\bar{\mathbf{Z}}_0, t_0; t), \dot{\mathbf{Z}}_a^*(\bar{\mathbf{Z}}_0, t_0; t), t]$$
$$+ \frac{1}{8\pi} \int_V d^3\mathbf{x} \, (\Delta^2 |\nabla \phi_1(\mathbf{x}, t)|^2$$
$$- |\nabla \times [\mathbf{A}_0(\mathbf{x}) + \Delta \mathbf{A}_1(\mathbf{x}, t)]|^2)$$
$$+ \sum_a \int d^6\bar{\mathbf{Z}}_0 \, D_a(\bar{\mathbf{Z}}_0) F_a(\bar{\mathbf{Z}}_0, t_0) \int d^6\bar{\mathbf{Z}}$$
$$\times \Lambda_a[\bar{\mathbf{Z}}; \mathbf{X}_a^*(\bar{\mathbf{Z}}_0, t_0; t), U_a^*(\bar{\mathbf{Z}}_0, t_0; t), \mu_a^*(\bar{\mathbf{Z}}_0, t_0; t); t]$$
$$\times \left[\left(\frac{\partial}{\partial t} + \frac{\Omega_a(\bar{\mathbf{X}})}{\epsilon} \frac{\partial}{\partial \bar{\xi}} \right) \tilde{S}_{a1}(\bar{\mathbf{Z}}, t) - e_a \tilde{\psi}_a(\bar{\mathbf{Z}}, t) \right], \quad (87)$$

where $L_a[\mathbf{Z}_a^*, \dot{\mathbf{Z}}_a^*, t]$ is the single-particle Lagrangian defined by Eq. (73) and L_c is the constraint part given by $L_c = \sum_a \int d^6\bar{\mathbf{Z}}_0 \; D_a(\bar{\mathbf{Z}}_0) F_a(\bar{\mathbf{Z}}_0, t_0) \int d^6\bar{\mathbf{Z}} \; (\mathcal{L}_c)_a$ with $(\mathcal{L}_c)_a = \Lambda_a[\bar{\mathbf{Z}}; \mathbf{X}_a^*, U_a^*, \mu_a^*; t][(\partial/\partial t + \epsilon^{-1}\Omega_a(\bar{\mathbf{X}})\partial/\partial\bar{\xi}) \tilde{S}_{a1}(\bar{\mathbf{Z}}, t) - e_a \tilde{\psi}_a(\bar{\mathbf{Z}}, t)]$. The same constraint on $\tilde{S}_{a1}$ as in Eq. (83) is derived from $\delta I / \delta \Lambda_a = 0$. From $\delta I / \delta \tilde{S}_{a1} = 0$, we have

$$\left(\frac{\partial}{\partial t} + \frac{\Omega_a(\bar{\mathbf{X}})}{\epsilon} \frac{\partial}{\partial \bar{\xi}} \right) \Lambda_a(\bar{\mathbf{Z}}; \mathbf{X}_a^*, U_{aj}^*, \mu_a^*; t)$$
$$= -\frac{e_a}{2} \left\langle \left\{ \tilde{\psi}_a(\mathbf{Z}_a^*, t), \delta^3(\mathbf{X}_a^* - \bar{\mathbf{X}}) \delta(U_a^* - \bar{U}) \right.\right.$$

$$\times\delta(\mu_a^* - \mu)\delta[\xi_a^* - \xi(\mathrm{mod}2\pi)]\Big\}_{\mathbf{Z}_a^*}\Big\rangle_{\xi_a^*}, \tag{88}$$

the solution of which is given by

$$\Lambda_a(\mathbf{Z}, \mathbf{X}_a^*, U_a^*, \mu_a^*, t)$$
$$= -\frac{1}{2}\Big\langle\Big\{\tilde{S}_{a1}(\mathbf{Z}_a^*, t), \delta^3(\mathbf{X}_a^* - \mathbf{X})\delta(U_a^* - \bar{U})$$
$$\times\delta(\mu_a^* - \bar{\mu})\delta[\xi_a^* - \bar{\xi}(\mathrm{mod}2\pi)]\Big\}_{\mathbf{Z}_a^*}\Big\rangle_{\xi_a^*} \tag{89}$$

Again, the same form of motion equations as Eqs. (74) are obtained from $\delta I/\delta \mathbf{Z}_a^* = 0$, and the same form of the gyrokinetic Vlasov equation as Eq (77) is derived. Also, the same form of gyrokinetic Poisson's equation and Ampère's law as Eqs. (78)–(79) are derived from $\delta I/\delta\phi_1 = 0$ and $\delta I/\delta \mathbf{A}_1 = 0$, respectively, with the help of Eq (89) Then, the gyrokinetic theory for the continuous Vlasov system with arbitrary-frequency fluctuations is given by the gyrokinetic Vlasov equation (77), the gyrokinetic Poisson's equation (78), the gyrokinetic Ampère's law (79), and the generating-function equation (83)

The conserved total energy derived from the total Lagrangian in Eq (87) is written as

$$E_{Gtot}$$
$$= \sum_a \int d^6\bar{\mathbf{Z}}_0\ D_a(\bar{\mathbf{Z}}_0)F_a(\bar{\mathbf{Z}}_0, t_0)\left[\dot{\mathbf{Z}}_a^* \cdot \frac{\partial L_a(\mathbf{Z}_a^*, \dot{\mathbf{Z}}_a^*, t)}{\partial \dot{\mathbf{Z}}_a^*}\right.$$
$$\left. + \int d^6\mathbf{Z}\tilde{S}_{a1}\frac{\partial(\mathcal{L}_c)_a}{\partial\dot{\tilde{S}}_{a1}}\right] - L$$
$$= \sum_a \int d^6\bar{\mathbf{Z}}\ D_a(\bar{\mathbf{Z}})F_a(\bar{\mathbf{Z}}, t)\left[\bar{H}_a(\bar{\mathbf{Z}}, t)\right.$$
$$\left. -\frac{1}{2}\left\{\tilde{S}_{a1}(\bar{\mathbf{Z}}, t), \frac{\partial\tilde{S}_{a1}(\bar{\mathbf{Z}}, t)}{\partial t}\right\}_{\bar{\mathbf{Z}}}\right] - L_f$$
$$= \sum_a \int d^6\bar{\mathbf{Z}}\ D_a(\bar{\mathbf{Z}})F_a(\bar{\mathbf{Z}}, t)$$
$$\times\left(\frac{1}{2}m_a\left[\mathbf{v}_{a0}(\bar{\mathbf{Z}}) - \Delta\frac{e_a}{m_a c}\mathbf{A}_1(\bar{\mathbf{X}}_a + \epsilon\bar{\boldsymbol{\rho}}_a, t)\right]^2\right.$$
$$\left. + \Delta^2\left\{\tilde{S}_{a1}(\bar{\mathbf{Z}}, t), \frac{\Omega_a(\bar{\mathbf{X}})}{2\epsilon}\frac{\partial\tilde{S}_{a1}(\bar{\mathbf{Z}}, t)}{\partial\bar{\xi}} + \frac{e_a}{c}(\widetilde{\mathbf{v}_0\cdot\mathbf{A}_1})_a\right\}_{\bar{\mathbf{Z}}}\right)$$
$$+ \frac{1}{8\pi}\int_V d^3\mathbf{x}\ (\Delta^2|\nabla\phi_1(\mathbf{x}, t)|^2$$
$$+|\nabla\times[\mathbf{A}_0(\mathbf{x}) + \Delta\mathbf{A}_1(\mathbf{x}, t)]|^2), \tag{90}$$

where Eqs (78) and (83) are also used.

§5. LIMITING CASES

In this section, we consider several limiting cases, in which the gyrokinetic system of equations presented in the foregoing sections can be simplified. It is emphasized here that all simplifications or approximations should be done on the level of the original total Lagrangian. Once a simplified total Lagrangian is specified, a simplified gyrokinetic system of equations with the invariant total energy are straightforwardly derived from it Therefore, in this section, we mainly show the ways of simplifying the total Lagrangian rather than the resultant simplified gyrokinetic system of equations These simplified system of equations, which retain the rigorous energy conservation, are considered to be useful for numerical simulation of plasma turbulence and anomalous transport.

5.1 Neglect of W

Neglect of the $\mathcal{O}(\epsilon^2)$ term in $\mathbf{A}_a^*$ simplifies the motion equations (61)–(64) and (74) This corresponds to putting $\mathbf{W} \to 0$, and gives $\{\mathbf{X}_a, \xi_a\} = \{U_a, \xi_a\} = 0$ in Eqs (47) and (48)

In the case of uniform equilibrium magnetic fields $\mathbf{B}_0 = \mathrm{const}$, $\mathbf{W} = 0$ is rigorously obtained, and more simplifications of the motion equations are given from $\mathbf{B}_a^* = \mathbf{B}_0$ and $B_{a\|}^* = B_0$

5.2 Small electron gyroradii

When, the electron gyroradii are negligibly small compared to the fluctuation scale lengths, we can put

$$\rho_e \to 0 \tag{91}$$

Then, the particle, guiding-center. and gyrocenter variables are regarded as equivalent to each other, $\mathbf{z}_e = \mathbf{Z}_e = \bar{\mathbf{Z}}_e$. The single-electron Lagrangian is given by

$$L_e = -\frac{e}{c}\mathbf{A}_e^*(\bar{\mathbf{X}}_e, \bar{U}_e, \bar{\mu}_e)\ \dot{\bar{\mathbf{X}}}_e - \frac{m_e c}{e}\bar{\mu}_e\dot{\bar{\xi}}_e - \bar{H}_e(\bar{\mathbf{X}}_e, \bar{U}_e, \bar{\mu}_e), \tag{92}$$

where

$$\mathbf{A}_e^*(\bar{\mathbf{X}}_e, \bar{U}_e, \bar{\mu}_e) = \mathbf{A}_0(\bar{\mathbf{X}}_e) - \frac{m_e c}{e}\bar{U}_e\mathbf{b}(\bar{\mathbf{X}}_e), \tag{93}$$

$$\bar{H}_e(\bar{\mathbf{X}}_e, \bar{U}_e, \bar{\mu}_e) = \frac{1}{2}m_e\bar{U}_e^2 + \bar{\mu}_e B_0(\bar{\mathbf{X}}_e) - e\psi_e(\bar{\mathbf{X}}_e, \bar{U}_e, t)$$
$$+ \frac{e^2}{2m_e c^2}|\mathbf{A}_1(\bar{\mathbf{X}}_e, t)|^2, \tag{94}$$

and

$$\psi_e(\bar{\mathbf{X}}_e, \bar{U}_e, t) = \phi_1(\bar{\mathbf{X}}_e, t) - \bar{U}_e A_{1\|}(\bar{\mathbf{X}}_e, t) \tag{95}$$

Here, the $\mathcal{O}(\epsilon^2)$ term in $\mathbf{A}_e^*$ is neglected. Here and hereafter, the drift-ordering parameter ϵ and the perturbation expansion parameter Δ are suppressed in the equations.

5.3 Quasineutrality

The simplifications considered in the previous subsections are applicable to both the discrete system and the Vlasov continuous system given in §3 and §4. In this subsection, we consider only the Vlasov continuous system since the quasineutrality condition is valid only for macroscopic scales larger than the Debye length. The quasineutrality approximation corresponds to putting $\frac{1}{8\pi}\int_V d^3\mathbf{x}|\nabla\phi(\mathbf{x})|^2 \to 0$ in the field Lagrangian part Then, we have

$$L_f = -\frac{1}{8\pi}\int_V d^3\mathbf{x}|\nabla\times[\mathbf{A}_0(\mathbf{x}) + \Delta\mathbf{A}_1(\mathbf{x}, t)]|^2. \tag{96}$$

Using this field Lagrangian, the left-hand-side term $\nabla^2\phi_1(\mathbf{x}, t)$ in the gyrokinetic Poisson's equation (78) reduces to the quasineutrality condition $\sum_a e_a n_{Ga} = 0$. Under this approximation, the electric field energy $\frac{1}{8\pi}\int_V d^3\mathbf{x}|\phi_1(\mathbf{x},t)|^2$ disappears from the total energy (80) [or (90)].

5.4 Linear polarization

In this and next subsections, we consider the Vlasov continuous system, in which the distribution function $F_a(\bar{\mathbf{Z}},t)$ is assumed to be given by the sum of a time-independent equilibrium part $F_{a0}(\bar{\mathbf{Z}})$ and a small deviation from it. Here, $F_a(\bar{\mathbf{Z}},t)$ and $F_{a0}(\bar{\mathbf{Z}})$ are both independent of the gyrophase $\bar{\xi}$.

The right-hand sides of the gyrokinetic Poisson's equation (78) and the gyrokinetic Ampère's law (79) contain the nonlinear polarization terms, which are given by the Poisson bracket between the generating function $\widetilde{S}_{a1}$ and the distribution function F_a. These nonlinear polarization terms originate from the $\mathcal{O}(\Delta^2)$ terms in the Lagrangian,

$$\sum_a \frac{e_a}{2}\int d^6\bar{\mathbf{Z}}_0\; D_a(\bar{\mathbf{Z}}_0)F_a(\bar{\mathbf{Z}}_0,t_0)$$
$$\times\left\langle\left\{\widetilde{S}_{a1}(\mathbf{Z}_a^*(\bar{\mathbf{Z}}_0,t_0;t),t),\widetilde{\psi}_a(\mathbf{Z}_a^*(\bar{\mathbf{Z}}_0,t_0;t),t)\right\}_{\mathbf{Z}_a^*}\right\rangle_{\xi_a^*}$$
$$\text{(for nonlinear polarization).} \qquad (97)$$

The linear polarization approximation is done by replacing the above terms in the Lagrangian (72) with

$$\sum_a \frac{e_a}{2}\int d^6\bar{\mathbf{Z}}_0\; D_a(\bar{\mathbf{Z}}_0)F_{a0}(\bar{\mathbf{Z}}_0)\left\{\widetilde{S}_{a1}(\bar{\mathbf{Z}}_0,t),\widetilde{\psi}_a(\bar{\mathbf{Z}}_0,t)\right\}_{\bar{\mathbf{Z}}_0}$$
$$\text{(for linear polarization).} \qquad (98)$$

In fact, from the variational principle using the Lagrangian with this replacement for the linear polarization, the Poisson bracket terms $\{\widetilde{S}_{a1},F_{a0}\}$ appear instead of $\{\widetilde{S}_{a1},F_a\}$ as the polarization terms in the resultant gyrokinetic Poisson's equation (78) and in the gyrokinetic Ampère's law (79). Also, the $\mathcal{O}(\Delta^2)$ part $-\frac{e_a}{2}\left\langle\left\{\widetilde{S}_{a1},\widetilde{\psi}_a\right\}\right\rangle_{\bar{\xi}_a}$ of the single-particle Hamiltonian is not involved in the resultant motion equations (74). Thus, the terms associated with $\widetilde{S}_{a1}$ disappears from the gyrokinetic Vlasov equation (77). Then, the $\mathcal{O}(\Delta^2)$ terms in Eq. (81) are connected not to F_a but to F_{a0} in the total energy (80).

The results described above are still valid when the linear polarization approximation is applied to the the arbitrary-frequency case in §4.2. For that case, the linear polarization approximation is done for the Lagrangian (87) by the replacement of Eq. (97) to Eq (98) and by replacing the constraint part L_c with

$$L_c^{\rm lin-pol} = \sum_a\int d^6\bar{\mathbf{Z}}_0\; D_a(\bar{\mathbf{Z}}_0)F_{a0}(\bar{\mathbf{Z}}_0)$$
$$\times\int d^6\bar{\mathbf{Z}}\;\Lambda_a(\bar{\mathbf{Z}};\bar{\mathbf{X}}_0,\bar{U}_0,\bar{\mu}_0;t)$$
$$\times\left[\left(\frac{\partial}{\partial t}+\Omega_a\frac{\partial}{\partial\bar{\xi}}\right)\widetilde{S}_{a1}(\bar{\mathbf{Z}},t)-e_a\widetilde{\psi}_a(\bar{\mathbf{Z}},t)\right]. \quad (99)$$

A detailed example of the linear polarization approximation is given in the next subsection for the electrostatic case.

5.5 High-frequency electrostatic waves in the uniform magnetic field

The approximations given in the foregoing subsections are applicable to the gyrokinetic theory for arbitrary-frequency fluctuations shown in §4. In this subsection, we present a simplified gyrokinetic system of equations, which are valid even for high fluctuation frequencies in the ion-gyrofrequency range. Here, for simplicity, we consider only electrostatic fluctuations in the uniform magnetic field $\mathbf{B}_0 = $ const, although more general cases can be treated straightforwardly by the formulation given in §4. The resultant equations can describe the ion Bernstein waves. [In fact, the rigorous dispersion relation for the ion Bernstein waves is immediately derived from the linearized version of Eqs. (103)–(105) and (101).[22]] We also take the small ρ_e limit for electrons, and use the linear polarization approximation for multi-species ions. Then, the total Lagrangian is written as

$$L = \int d^6\bar{\mathbf{Z}}_0\; D_e F_e(\bar{\mathbf{Z}}_0,t_0)\left[\left(-\frac{e}{c}\mathbf{A}_0(\mathbf{X}_e^*)+m_eU_e^*\mathbf{b}\right)\cdot\dot{\mathbf{X}}_e^*\right.$$
$$\left.-\frac{m_ec}{e}\mu_e^*\dot{\xi}_e^*-\left(\frac{1}{2}m_e(U_e^*)^2+\mu_e^*B_0-e\phi_1(\mathbf{X}_e^*,t)\right)\right]$$
$$+\sum_{a(\rm ions)}\int d^6\bar{\mathbf{Z}}_0\; D_aF_a(\bar{\mathbf{Z}}_0,t_0)\left[\left(\frac{e_a}{c}\mathbf{A}_0(\mathbf{X}_a^*)+m_aU_a^*\mathbf{b}\right)\cdot\dot{\mathbf{X}}_a^*\right.$$
$$\left.+\frac{m_ac}{e_a}\mu_a^*\dot{\xi}_a^*-\left(\frac{1}{2}m_a(U_a^*)^2+\mu_a^*B_0+e_a\langle\phi_1(\mathbf{X}_a^*+\boldsymbol{\rho}_a^*,t)\rangle_{\xi_a^*}\right)\right]$$
$$-\sum_{a(\rm ions)}\frac{e_a^2}{2m_ac}\int d^6\bar{\mathbf{Z}}\; D_a\frac{\partial F_{a0}(\bar{\mathbf{Z}})}{\partial\mu}\frac{\partial\widetilde{S}_{a1}(\bar{\mathbf{X}},\bar{\mu},\bar{\xi},t)}{\partial\bar{\xi}}\phi_1(\bar{\mathbf{X}}+\bar{\boldsymbol{\rho}}_a,t)$$
$$+\frac{1}{8\pi}\int_V d^3\mathbf{x}|\nabla\phi_1(\mathbf{x})|^2+\sum_{a(\rm ions)}\int d^6\bar{\mathbf{Z}}_0\; D_aF_{a0}(\bar{\mathbf{Z}}_0)$$
$$\times\int d^3\bar{\mathbf{X}}d\bar{\mu}d\bar{\xi}\;\Lambda_a(\bar{\mathbf{X}},\bar{\mu},\bar{\xi};\mathbf{X}_0,\mu_0;t)$$
$$\times\left[\left(\frac{\partial}{\partial t}+\Omega_a\frac{\partial}{\partial\bar{\xi}}\right)\widetilde{S}_{a1}(\bar{\mathbf{X}},\bar{\mu},\bar{\xi},t)-e_a\widetilde{\phi}_a(\bar{\mathbf{X}}+\bar{\rho}_a,t)\right], \qquad (100)$$

where $D_e = B_0/m_e$, $D_a = B_0/m_a$, $\boldsymbol{\rho}_a^* = \boldsymbol{\rho}_{a0}(\mathbf{Z}_a^*)$, $\mathbf{Z}_a^* = (\mathbf{X}_a^*,U_a^*,\mu_a^*,\xi_a^*) = \mathbf{Z}_a^*(\bar{\mathbf{Z}}_0,t_0;t)$, and $\bar{\boldsymbol{\rho}}_a = \boldsymbol{\rho}_{a0}(\bar{\mathbf{Z}})$. Here, we have used the linear polarization part of the Lagrangian $-\sum_{a(\rm ions)}\frac{e_a^2}{2m_ac}\int d^6\bar{\mathbf{Z}}\; D_a[\partial F_{a0}(\bar{\mathbf{Z}})/\partial\mu]$ $[\partial\widetilde{S}_{a1}(\bar{\mathbf{X}},\bar{\mu},\bar{\xi},t)/\partial\bar{\xi}]\phi_1(\bar{\mathbf{X}}+\bar{\boldsymbol{\rho}}_a,t)$ which is a electrostatic version of Eq. (98) with higher ϵ-order terms neglected.

From $\delta I/\delta\Lambda_a = 0$, we obtain

$$\left(\frac{\partial}{\partial t}+\Omega_a\frac{\partial}{\partial\bar{\xi}}\right)\widetilde{S}_{a1}(\bar{\mathbf{X}},\bar{\mu},\bar{\xi},t) = e_a\widetilde{\phi}_1(\bar{\mathbf{X}}+\bar{\rho}_a,t). \quad (101)$$

The equation for Λ_a is derived from $\delta I/\delta\widetilde{S}_{a1} = 0$, which is solved with the help of Eq. (101) to give

$$\Lambda_a(\bar{\mathbf{X}},\bar{\mu},\bar{\xi};\mathbf{X}_0,\mu_0;t) = \frac{e_a}{4\pi m_ac}\delta^3(\bar{\mathbf{X}}-\bar{\mathbf{X}}_0)\frac{\partial\delta(\bar{\mu}-\bar{\mu}_0)}{\partial\mu}$$
$$\times\frac{\partial\widetilde{S}_{a1}(\bar{\mathbf{X}},\bar{\mu},\bar{\xi},t)}{\partial\bar{\xi}}. \qquad (102)$$

It should be noted that, in the electrostatic case, $\tilde{S}_{a1}$ and Λ_a are both independent of the parallel velocity. $\partial\tilde{S}_{a1}/\partial \bar{U} = \partial\Lambda_a/\partial \bar{U} = \partial\Lambda_a/\partial U'_0 = 0$

The motion equations for electrons and ions are derived from $\delta I/\delta \mathbf{Z}^*_e = \delta I/\delta \mathbf{Z}^*_a = 0$, and the kinetic equations for the electron and ion distribution functions are given by

$$\left[\frac{\partial}{\partial t} + \left(\bar{U}\mathbf{b} + \frac{c}{B_0}\mathbf{b}\times\nabla\phi_1(\bar{\mathbf{X}},t)\right)\cdot\nabla + \frac{e}{m_e}\mathbf{b}\cdot\nabla\phi_1(\bar{\mathbf{X}},t)\frac{\partial}{\partial \bar{U}}\right] F_e(\bar{\mathbf{X}},\bar{U},\bar{\mu},t) = 0, \quad (103)$$

and

$$\left[\frac{\partial}{\partial t} + \left(\bar{U}\mathbf{b} + \frac{c}{B_0}\mathbf{b}\times\nabla\langle\phi_1(\bar{\mathbf{X}}+\bar{\boldsymbol{\rho}}_a,t)\rangle_{\bar{\xi}}\right)\cdot\nabla - \frac{e_a}{m_a}\mathbf{b}\cdot\nabla\langle\phi_1(\bar{\mathbf{X}}+\bar{\boldsymbol{\rho}}_a,t)\rangle_{\bar{\xi}}\frac{\partial}{\partial \bar{U}}\right] F_a(\bar{\mathbf{X}},\bar{U},\bar{\mu},t) = 0, \quad (104)$$

respectively. From $\delta I/\delta\phi_1 = 0$, we obtain the Poisson's equation,

$$\nabla^2\phi(\mathbf{x}) = 4\pi e\int d^6\bar{\mathbf{Z}}\, D_e\delta^3(\bar{\mathbf{X}}-\mathbf{x})F_e(\bar{\mathbf{Z}},t) - 4\pi\sum_{a(\text{ions})} e_a\int d^6\bar{\mathbf{Z}}\, D_a\delta^3(\bar{\mathbf{X}}+\epsilon\bar{\boldsymbol{\rho}}_a-\mathbf{x}) \times\left[F_a(\bar{\mathbf{Z}},t) + \frac{e_a}{m_a c}\frac{\partial F_{a0}(\bar{\mathbf{Z}})}{\partial\bar{\mu}}\frac{\partial\tilde{S}_{a1}(\bar{\mathbf{X}},\bar{\mu},\bar{\xi},t)}{\partial\bar{\xi}}\right] \quad (105)$$

The closed nonlinear gyrokinetic system of equations (103)-(105) and (101) describe the high-frequency electrostatic plasma fluctuations in the uniform magnetic fields. They rigorously conserve the total energy, which is given by

$$E_{Gtot} = \int d^6\bar{\mathbf{Z}}\, D_eF_e(\bar{\mathbf{Z}},t)\left(\frac{1}{2}m_e\bar{U}^2 + \bar{\mu}B_0\right) + \sum_{a(\text{ions})}\int d^6\bar{\mathbf{Z}}\, D_a\left[F_a(\bar{\mathbf{Z}},t)\left(\frac{1}{2}m_a\bar{U}^2 + \bar{\mu}B_0\right) - \frac{\Omega_a^2}{2B_0}\frac{\partial F_{a0}(\bar{\mathbf{Z}})}{\partial\bar{\mu}}\left(\frac{\partial\tilde{S}_{a1}(\bar{\mathbf{X}},\bar{\mu},\bar{\xi},t)}{\partial\bar{\xi}}\right)^2\right] + \frac{1}{8\pi}\int_V d^3\mathbf{x}|\nabla\phi(\mathbf{x})|^2. \quad (106)$$

It should be noted that, in deriving the conservation of E_{Gtot}, the fluctuations on the boundary surface are assumed to make no contribution [see Eqs. (A·13) and (A·14) in Appendix A]. If there are any external energy sources or sinks, E_{Gtot} is not conserved. When F_{a0} is assumed to take the Maxwellian form in the velocity space perpendicular to the magnetic field, we can write $\partial F_{a0}(\bar{\mathbf{Z}})/\partial\bar{\mu} = -B_0F_{a0}/T_a$ with the perpendicular temperature T_a.

As shown in §5 3, from the variational principle using the Lagrangian (100) with the term $\frac{1}{8\pi}\int_V d^3\mathbf{x}|\nabla\phi(\mathbf{x})|^2$ neglected, the quasineutrality condition [Eq (105) with the left-hand side term vanishing] is derived Then, the electric field energy $\frac{1}{8\pi}\int_V d^3\mathbf{x}|\phi_1(\mathbf{x},t)|^2$ disappears from the total energy (106)

Further simplification is given by the adiabatic-electron approximation That corresponds to the following replacement of the electron Lagrangian part in Eq. (100),

$$\int d^6\bar{\mathbf{Z}}_0\, D_eF_e(\bar{\mathbf{Z}}_0,t_0)L_e(\mathbf{Z}^*_e,\dot{\mathbf{Z}}^*_e,t) \longrightarrow \int_V d^3\mathbf{x}\, e\phi_1(\mathbf{x},t)n_0(\mathbf{x})\left[1+\frac{e}{2T_e(\mathbf{x})}\phi_1(\mathbf{x},t)\right]$$

(for adiabatic electrons), (107)

where n_0 is the equilibrium electron density and T_e is the equilibrium electron temperature. In fact, it is easily confirmed that the variational principle for the Lagrangian using Eq. (107) makes changes in the gyrokinetic Poisson's equation (105) and in the conserved total energy (106), which are written as

$$\int d^6\bar{\mathbf{Z}}\, D_e\delta^3(\bar{\mathbf{X}}-\mathbf{x})F_e(\bar{\mathbf{Z}},t) \longrightarrow n_0(\mathbf{x})\left[1+\frac{e}{T_e(\mathbf{x})}\phi_1(\mathbf{x},t)\right]$$

(adiabatic electron density),

(108)

and

$$\int d^6\bar{\mathbf{Z}}\, D_eF_e(\bar{\mathbf{Z}},t)\left(\frac{1}{2}m_e\bar{U}^2+\bar{\mu}B_0\right) \longrightarrow \int_V d^3\mathbf{x}\, n_0(\mathbf{x})\frac{e^2}{2T_e(\mathbf{x})}|\phi_1(\mathbf{x},t)|^2, \quad (109)$$

respectively. In this approximation, no equation for the electron distribution function like Eq. (103) is derived or required for the closed system of equations.

The difference between the high-frequency gyrokinetic theory and the conventional low-frequency one is that, for the high-frequency case, the generating function $\tilde{S}_{a1}$ can not be determined instantly from the fluctuation ϕ_1 due to the time derivative term retained in Eq (101). Let us write the electrostatic potential in terms of the Fourier components with wavenumber vectors $\mathbf{k}$,

$$\phi_1(\mathbf{x},t) = \sum_{\mathbf{k}}\phi_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (110)$$

Then, the solution of Eq. (101) is explicitly written as

$$\tilde{S}_{a1}(\bar{\mathbf{X}},\bar{\mu},\bar{\xi},t) = \sum_{\mathbf{k}}\sum_{n\neq 0} e^{i\mathbf{k}\cdot\bar{\mathbf{X}}}e^{in(\bar{\xi}-\alpha_{\mathbf{k}})}J_n(k_\perp\bar{\rho}_a) \times\int_{t_0}^{t} dt' e^{-in\Omega_a(t-t')}\phi_{\mathbf{k}}(t'), \quad (111)$$

where the initial condition $\tilde{S}_{a1}(\bar{\mathbf{X}},\bar{\mu},\bar{\xi},t_0) = 0$ is used Here, $\mathbf{k} = k_\parallel\mathbf{b} - k_\perp(\sin\alpha_{\mathbf{k}}\ \mathbf{e}_1 + \cos\alpha_{\mathbf{k}}\ \mathbf{e}_2)$, $\bar{\rho}_a = (c/e_a)(2m_a\bar{\mu}/B_0)^{1/2}$, $n = \pm1,\pm2,\cdots$ $(n\neq 0)$, and J_n is the nth order Bessel function. In the low-frequency limit, $\int_{t_0}^{t} dt'e^{-in\Omega_a(t-t')}\phi_{\mathbf{k}}(t')$ in Eq. (111) is replaced by $(in\Omega_a)^{-1}\phi_{\mathbf{k}}(t)$, which reproduces the generating function given by Eq. (59)

§6. CONCLUSIONS

In this work, the generalized Lagrangian formulation of the gyrokinetic theory has been presented. The total Lagrangian, which consists of the parts of particles, electromagnetic fields, and their interaction, is shown to derive the gyrokinetic particle motion equations, Poisson's equation, and Ampère's law. Owing to the use of the total Lagrangian, the total energy, which is rigorously conserved, is directly derived from the Noether's theorem. The Lagrangian formulation is given for the discrete-particle system and for the continuous Vlasov system. In the former case, all the particles' phase-space variables and the microscopic electromagnetic fields are described, while, in the latter case, the one-body distribution functions and the macroscopic electromagnetic fields are treated.

The nonlinear gyrokinetic system of equations for the case of arbitrary fluctuation frequencies are also derived from the generalized Lagrangian formulation. The high-frequency properties of the fluctuations are included in the generating functions for the gyrocenter-variable transformation. The rigorously conserved total energy for this arbitrary-frequency nonlinear gyrokinetic system is also shown

Several limiting cases are considered, in which the gyrokinetic equations are simplified and more easily tractable for numerical simulation. The small electron gyroradius limit, the quasineutrality, and the linear polarization approximation are treated as examples. All the simplifications, which are applicable to the arbitrary fluctuation frequency case as well, are done on the level of the original Lagrangian. Then, the variational principle automatically yields the simplified gyrokinetic equations for the particles (or the distribution functions) and the fields, for which the conserved total energy is derived. The simplified gyrokinetic system of equations are written in detail to describe the high-frequency electrostatic plasma fluctuations in the uniform magnetic field. They are useful for studying the fluctuations in the ion-gyrofrequency range such as the ion Bernstein waves.

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Appendix: VARIATIONAL PRINCIPLE AND NOETHER'S THEOREM

In this appendix, the Lagrangian variational principle and the Noether's theorem are briefly explained in a partly modified way from the standard text books.[17,29)

The action integral is given by

$$I = \int_{t_1}^{t_2} L dt. \quad \text{(A·1)}$$

The total Lagrangians L considered in this work are all written in the form,

$$L = L[(\eta_\alpha), (\dot{\eta}_\alpha)], \quad \text{(A·2)}$$

where the field variables η_α are functions of $(\mathbf{x}_\alpha, t)$, α is a label to specify the field, and $\dot{} = \partial/\partial t$ is the time derivative. Here, $\mathbf{x}_\alpha$ denotes a l_α-dimensional vector variable, $\mathbf{x}_\alpha = (x_{\alpha 1}, \cdots, x_{\alpha l_\alpha})$. When $l_\alpha = 0$, η_α represents a function of the time t alone like the particle's position $\mathbf{x}_j(t)$ and velocity $\mathbf{v}_j(t)$ in §2.1. The electromagnetic potential fields $\phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$ correspond to η_α with $l_\alpha = 3$. Also, $l_\alpha = 6$ is given for $\mathbf{r}(\mathbf{x}_0, \mathbf{v}_0, t_0; t)$ (where t_0 is a fixed parameter) in §2.2, and $l_\alpha = 7$ for Λ_a in §4.2.

The Lagrangian L is a functional of the fields (or $\mathbf{x}_\alpha$-functions) η_α and $\dot{\eta}_\alpha$. We note that the part of the Lagrangian associated with η_α and $\dot{\eta}_\alpha$ for specified α are written in the form

$$L_\alpha(\eta_\alpha, \eta_\alpha)$$
$$= \int d^{l_\alpha}\mathbf{x}_\alpha \mathcal{L}_\alpha[\eta_\alpha(\mathbf{x}_\alpha, t), \dot{\eta}_\alpha(\mathbf{x}_\alpha, t), \nabla_\alpha \eta_\alpha(\mathbf{x}_\alpha, t), \cdots], \quad \text{(A·3)}$$

where $\nabla_\alpha = \partial/\partial \mathbf{x}_\alpha$, and $\cdots$ represents possible dependencies on $\mathbf{x}_\alpha$ and on the other fields η_β ($\beta \neq \alpha$). For example, in the case of §2.1, $L_{aj}(\mathbf{x}_{aj}, \dot{\mathbf{x}}_{aj}) = m_a \mathbf{v}_{aj}(t) \cdot \dot{\mathbf{x}}_{aj}(t) - e_a[\phi(\mathbf{x}_{aj}(t), t) - c^{-1}\dot{\mathbf{x}}_{aj}(t) \cdot \mathbf{A}(\mathbf{x}_{aj}(t), t)]$ and $L_\phi = \int d^3\mathbf{x} \mathcal{L}_\phi$, where $\mathcal{L}_\phi = \frac{1}{8\pi}|\nabla\phi(\mathbf{x}, t)|^2 - \sum_a \sum_{j=1}^{N_a} e_a \phi(\mathbf{x}, t) \delta^3(\mathbf{x} - \mathbf{x}_{aj}(t))$. In this case, L_{aj} and L_ϕ share part of the interaction Lagrangian L_{int} in Eq. (5). Thus, as shown by this example, we generally have $L \neq \sum_\alpha L_\alpha$.

The variational principle is written as

$$\delta I = \sum_\alpha \int_{t_1}^{t_2} dt \int d^{l_\alpha}\mathbf{x}_\alpha \left[\frac{\partial \mathcal{L}_\alpha}{\partial \eta_\alpha} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}_\alpha}{\partial \dot{\eta}_\alpha}\right) - \nabla_\alpha \cdot \left(\frac{\partial \mathcal{L}_\alpha}{\partial \nabla_\alpha \eta_\alpha}\right)\right] \delta\eta_\alpha = 0, \quad \text{(A·4)}$$

where the variation $\delta\eta_\alpha(\mathbf{x}_\alpha, t)$ is taken to be zero at the temporal endpoints t_1 and t_2 as well as on the boundary surface of the integral $\int d^{l_\alpha}\mathbf{x}_\alpha$. We obtain from Eq. (A·4) the Euler-Lagrange equations

$$\frac{\delta I}{\delta \eta_\alpha} \equiv \frac{\partial \mathcal{L}_\alpha}{\partial \eta_\alpha} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}_\alpha}{\partial \dot{\eta}_\alpha}\right) - \nabla_\alpha \cdot \left(\frac{\partial \mathcal{L}_\alpha}{\partial \nabla_\alpha \eta_\alpha}\right) = 0. \quad \text{(A·5)}$$

Next, let us consider the following infinitesimal transformations of t, $\mathbf{x}_\alpha$, and $\eta_\alpha(\mathbf{x}_\alpha, t)$ simultaneously,

$$t \to t' = t + \delta t,$$
$$\mathbf{x}_\alpha \to \mathbf{x}'_\alpha = \mathbf{x}_\alpha + \delta\mathbf{x}_\alpha,$$
$$\eta_\alpha(\mathbf{x}_\alpha, t) \to \eta'_\alpha(\mathbf{x}'_\alpha, t') = \eta_\alpha(\mathbf{x}_\alpha, t) + \delta\eta_\alpha(\mathbf{x}_\alpha, t). \quad \text{(A·6)}$$

Here, δt and $\delta\mathbf{x}_\alpha$ are generally functions of $(\mathbf{x}_\alpha, t)$, and $\delta\eta_\alpha(\mathbf{x}_\alpha, t)$ consists of the variations in the functional form of η_α and in the variables $(\mathbf{x}_\alpha, t)$,

$$\delta\eta_\alpha(\mathbf{x}_\alpha, t) = \bar{\delta}\eta_\alpha(\mathbf{x}_\alpha, t) + \delta t \; \dot{\eta}_\alpha + \delta\mathbf{x}_\alpha \cdot \nabla_\alpha \eta_\alpha, \quad \text{(A·7)}$$

where $O(\delta^2)$ terms are neglected, and $\delta\eta_\alpha(\mathbf{x}_\alpha, t) = \eta'_\alpha(\mathbf{x}_\alpha, t) - \eta_\alpha(\mathbf{x}_\alpha, t)$. The infinitesimal transformations in Eq. (A·6) also causes the variation in the action integral.

$$I - I' = \int_{t'_1}^{t'_2} L' dt', \qquad (A·8)$$

where, as in Eq. (A·3) the part of L' associated with η'_α and $\dot{\eta}'_\alpha = \partial\eta'_\alpha/\partial t'$ for specified α is given by

$$L'_\alpha = \int d^{l_\alpha}\mathbf{x}'_\alpha \mathcal{L}_\alpha[\eta'_\alpha(\mathbf{x}'_\alpha, t'), \dot{\eta}'_\alpha(\mathbf{x}'_\alpha, t'), \nabla'_\alpha\eta'_\alpha(\mathbf{x}'_\alpha, t'), \cdots] \qquad (A·9)$$

Using the Euler-Lagrange equations (A·5) the variation in the action integral under the transformations in Eq. (A·6) is written as

$$\delta I = I' - I$$
$$= -\int_{t_1}^{t_2} dt \left[\frac{dG}{dt} + \sum_\alpha \int d^{l_\alpha}\mathbf{x}_\alpha \nabla_\alpha \cdot \mathbf{J}_\alpha\right], \qquad (A·10)$$

where

$$G = \delta t \left(\sum_\alpha \int d^{l_\alpha}\mathbf{x}_\alpha \dot{\eta}_\alpha \frac{\partial \mathcal{L}_\alpha}{\partial \dot{\eta}_\alpha} - L\right)$$
$$+ \sum_\alpha \int d^{l_\alpha}\mathbf{x}_\alpha \left(\delta\mathbf{x}_\alpha \cdot \nabla_\alpha \eta_\alpha \frac{\partial \mathcal{L}_\alpha}{\partial \dot{\eta}_\alpha} - \delta\eta_\alpha \frac{\partial \mathcal{L}_\alpha}{\partial \dot{\eta}_\alpha}\right), \qquad (A·11)$$

and

$$\mathbf{J}_\alpha = \delta t\, \dot{\eta}_\alpha \frac{\partial \mathcal{L}_\alpha}{\partial \nabla_\alpha \eta_\alpha} + \delta\mathbf{x}_\alpha \cdot \nabla_\alpha \eta_\alpha \frac{\partial \mathcal{L}_\alpha}{\partial \nabla_\alpha \eta_\alpha} - \delta\mathbf{x}_\alpha \mathcal{L}_\alpha$$
$$- \delta\eta_\alpha \frac{\partial \mathcal{L}_\alpha}{\partial \nabla_\alpha \eta_\alpha}. \qquad (A·12)$$

When the action integral I is invariant under the transformations in Eq. (A·6), we obtain from Eq. (A·10) with arbitrariness of t_1 and t_2,

$$\frac{dG}{dt} + \sum_\alpha \int d^{l_\alpha}\mathbf{x}_\alpha \nabla_\alpha \cdot \mathbf{J}_\alpha = 0, \qquad (A·13)$$

which is the main conclusion of the Noether's theorem. If $\mathbf{J}_\alpha$ vanish on the boundaries of the integral regions $\int d^{l_\alpha}\mathbf{x}_\alpha$, G is conserved,

$$\frac{dG}{dt} = 0. \qquad (A·14)$$

The Noether's theorem is widely applicable to derivation of the conservation laws. For example, when $\mathcal{L}_\alpha$ is independent of η_α for $\alpha = \bar{\alpha}$, the action integral I is obviously invariant under the transform given by

$$\delta t = 0, \qquad \delta\mathbf{x}_\alpha = 0, \qquad \delta\eta_\alpha = \epsilon\delta_{\alpha\bar{\alpha}}. \qquad (A·15)$$

where ϵ is an infinitesimal constant parameter. Then, we find from Eqs. (A·11) and (A·14) that

$$\int d^{l_{\bar{\alpha}}}\mathbf{x}_{\bar{\alpha}} \frac{\partial \mathcal{L}_{\bar{\alpha}}}{\partial \dot{\eta}_{\bar{\alpha}}} = \text{const} \qquad (A·16)$$

The conservation of the magnetic moment for the gyrophase-independent Lagrangian is regarded as a special case of this example.

The total Lagrangians considered in this work have no explicit time dependence, which means that their time dependences are only through the functions $\eta_\alpha(\mathbf{x}_\alpha, t)$. Thus, the action integral I is invariant under the infinitesimal transformation given by

$$\delta t = \epsilon, \qquad \delta\mathbf{x}_\alpha = 0, \qquad \delta\eta_\alpha = 0. \qquad (A·17)$$

Then, from Eqs. (A·11) and (A·14) we immediately obtain the total energy conservation,

$$E_{tot} \equiv \sum_\alpha \int d^{l_\alpha}\mathbf{x}_\alpha \dot{\eta}_\alpha \frac{\partial \mathcal{L}_\alpha}{\partial \dot{\eta}_\alpha} - L = \text{const}. \qquad (A·18)$$

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