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Path Integral Approach for Electron Transport in Disturbed Magnetic Field Lines

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Abstract

A path integral method is developed to investigate statistical property of an electron transport described as a Langevin equation in a statically disturbed magnetic field line structure; especially a transition probability of electrons strongly tied to field lines is considered. The path integral method has advantages that 1) it does not include intrinsically a growing numerical error of an orbit, which is caused by evolution of the Langevin equation under a finite calculation accuracy in a chaotic field line structure, and 2) it gives a method of understanding the qualitative content of the Langevin equation and assists to expect statistical property of the transport. Monte Carlo calculations of the electron distributions under both effects of chaotic field lines and collisions are demonstrated to comprehend above advantages through some examples. The mathematical techniques are useful to study statistical properties of various phenomena described as Langevin equations in general. By using parallel generators of random numbers, the Monte Carlo scheme to calculate a transition probability can be suitable for a parallel computation.

Key words: plasma transport, Langevin equation, path integral, Monte Carlo calculation
PACS: 52.25.Fi

1 Introduction

From the viewpoint of development of stochastic methods, we will consider a statistical analysis of an electron transport in a statically disturbed magnetic field line structure, as was discussed in Refs. [1,2]. The transport analysis is useful to understand the plasma confinement in the destroyed magnetic surfaces by MHD activities or in the peripheral region of toroidal plasmas. A collisionless transport of particles, especially electrons strongly tied to field lines, is dominated by the transport of magnetic field lines. As is well known, the equations of a field line are expressed as Hamilton's equations with an interpretation that χ is identified with the 'Hamiltonian', ζ the 'time', ψ the 'momentum', and θ the 'coordinate';

$$\frac{d\psi}{d\zeta} = -\frac{\partial}{\partial\theta}\chi(\psi,\theta,\zeta), \tag{1}$$

$$\frac{d\theta}{d\zeta} = \frac{\partial}{\partial\psi}\chi(\psi,\theta,\zeta), \tag{2}$$

where a magnetic field is expressed as $\boldsymbol{B} = \nabla\psi \times \nabla\theta - \nabla\chi \times \nabla\zeta$ with $\nabla\psi \cdot \nabla\theta \times \nabla\zeta \neq 0$ [3]. Consider a situation that a magnetic field as an integrable Hamiltonian is perturbed statically by fluctuating magnetic fields: $\chi = \chi_0(\psi)+\delta\chi(\psi,\theta,\zeta)$. By increasing the strength of the static perturbations, magnetic islands appear and overlap each other, and finally the field lines become chaotic in the ψ-θ cross section [4]. As is assumed in Refs. [1,2], if statistical property of perturbed field lines is interpreted as the Brownian diffusion, then Eq. (1) is described as $d\psi/d\zeta = -\partial\delta\chi/\partial\theta \sim W(\zeta)$, where $W(\zeta)$ is the Gaussian white noise and it satisfies $\overline{W(\zeta)} = 0$ and $\overline{W(\zeta)\, W(\zeta')} = 2D\delta(\zeta-\zeta')$. A notation $\overline{\cdots}$ means an ensemble average. A diffusion coefficient D is given as the coefficient of field lines $D = D_{\mathrm{M}}$ by using the quasi-linear approximation [2]. The hypothesis of Brownian diffusion in Refs. [1,2] seems to be justified by numerical works of Refs. [5,6]. The hypothesis may be valid, if there exists a homogeneous and completely stochastic magnetic field line structure in infinite space. However, when a transport problem in a weakly chaotic field line structure, e.g. in the structure illustrated in Fig. 1(b), is examined, the hypothesis is not trivial [7,8], and numerical methods solving the transport are needed. For the numerical estimation of statistical property of the transport, in general it is difficult to solve directly nonlinear differential equations (1) and (2) under an appropriate calculation accuracy, because numerical errors in evolution of ψ and θ grow rapidly with the 'time' ζ, owing to the chaotic field lines [9]. On the other hand, effects of collisions on the electron transport in chaotic field lines were not so sufficiently clear in previous works. Therefore, mathematical tools resolving above problems are required, and are developed in this article. In the remainder of this section and the following sections, we will introduce the mathematical tools and demonstrate the first numerical calculations by using a Monte Carlo technique based on the tools.

If statistical property of perturbed field lines is not clear and effects of collision can be rather represented as the Gaussian white noise, a motion of electron strongly tied to a field line is expressed as the following stochastic differential equations, i.e. Langevin equations, in the Cartesian coordinates (x, y, t) [10];

$$\frac{dx}{dt} = \frac{B_x(x,y,t)}{B_t(x,y,t)} + W_x(t), \tag{3}$$

$$\frac{dy}{dt} = \frac{B_y(x,y,t)}{B_t(x,y,t)} + W_y(t), \tag{4}$$

where $\boldsymbol{B} = (B_x, B_y, B_t)$ is a magnetic field with $B_t > 0$, W_x and W_y are the Gaussian white noises and they satisfy the following conditions; for $i, j = x, y$

$$\overline{W_i(t)} = 0, \tag{5}$$

$$\overline{W_i(t)\, W_j(t')} = \begin{cases} 2D_i\delta(t-t') & \text{for } i = j \\ 0 & \text{for } i \neq j. \end{cases} \tag{6}$$

A diffusion coefficient D_i is assumed to be constant and be determined by the Coulomb collisions. Note that the diffusion coefficients D_i can be extended to be functions of space (x, y), as is shown in the following sections. For the collisionless limit, the coefficient $D_i \to 0$. We assume that effects of collisions parallel to a field line are negligibly small as compared with the parallel component of an electron velocity. The Langevin equations (3) and (4) are simple as compared with the exact equations of motion; but effects of perturbed field lines can be easily investigated and the simple equation is a good candidate for fulfilment of our aim to develop basic techniques, so that we consider these equations in this article. Of course, the methods developed here are applicable to the exact equations, if the assumption of the Gaussian white noise describing effects of collision is appropriate.

It is meaningful that the transport described by the Langevin equations (3) and (4) is considered from another point of view, by following discussions in Ref. [2]. The equations of a field line are given as $dx/dt = h_x(x, y, t)$ and $dy/dt = h_y(x, y, t)$, where $\boldsymbol{h} = (h_x, h_y) = (B_x/B_t, B_y/B_t)$. According to Ref. [2], the confinement of electrons strongly tied to field lines can be expressed as

$$\frac{\partial p(\boldsymbol{r},t)}{\partial t} = -\nabla \cdot \boldsymbol{\Gamma}(\boldsymbol{r},t), \tag{7}$$

where $\boldsymbol{\Gamma}(\boldsymbol{r},t)$ a probability flux, $p(\boldsymbol{r},t)$ a probability density, $\boldsymbol{r} = (x, y)$, and $\nabla = (\partial/\partial x, \partial/\partial y)$. For the collisionless limit $\boldsymbol{\Gamma}(\boldsymbol{r},t) = p(\boldsymbol{r},t)\boldsymbol{h}(\boldsymbol{r},t)$, the confinement is given as

$$\frac{\partial p(\boldsymbol{r},t)}{\partial t} = -\nabla \cdot \left[p(\boldsymbol{r},t)\boldsymbol{h}(\boldsymbol{r},t)\right]. \tag{8}$$

Let field lines be chaotic. If p and $\boldsymbol{h}$ are split into two components: slowly varying terms p_0 and $\boldsymbol{h}_0$, and fluctuating terms $\tilde{p}$ and $\tilde{\boldsymbol{h}}$, i.e. $p = p_0 + \tilde{p}$ and

$\boldsymbol{h} = \boldsymbol{h}_0 + \tilde{\boldsymbol{h}}$, then the ensemble average yields

$$\frac{\partial p_0}{\partial t} = -\nabla \cdot \overline{\tilde{p}\,\tilde{\boldsymbol{h}}} \sim D_{\mathrm{M}} \nabla^2 p_0, \tag{9}$$

where $\boldsymbol{h}_0 = 0$, $\overline{\tilde{p}} = 0$, $\overline{\tilde{\boldsymbol{h}}} = 0$, and the hypothesis of Brownian diffusion is used [2]. On the other hand, when effects of collision are considered and are supposed to cause a diffusion in space (x, y), the confinement is expressed as follows. If the term $-\nabla \cdot \boldsymbol{\Gamma}$ in Eq. (7) is split into two components $-\nabla \cdot \boldsymbol{\Gamma} = -(\nabla \cdot \boldsymbol{\Gamma}_0 + \nabla \cdot \tilde{\boldsymbol{\Gamma}})$: a drift term given by field lines $-\nabla \cdot \boldsymbol{\Gamma}_0 = -\nabla \cdot [p(\boldsymbol{r}, t)\boldsymbol{h}(\boldsymbol{r}, t)]$ and a diffusion term given by collisions $-\nabla \cdot \tilde{\boldsymbol{\Gamma}} \sim D_{\mathrm{coll}} \nabla^2 p$, then we have the following Fokker-Planck equation:

$$\frac{\partial p(\boldsymbol{r}, t)}{\partial t} = -\nabla \cdot \left[p(\boldsymbol{r}, t)\boldsymbol{h}(\boldsymbol{r}, t)\right] + D_{\mathrm{coll}} \nabla^2 p, \tag{10}$$

where $D_{\mathrm{coll}} =$ const. is a diffusion coefficient of collision. Langevin equations corresponding to the above equation are Eqs. (3) and (4). The path integral method gives a solution of the Fokker-Planck equation.

The organization of the following sections is as follows. In section 2 the path integral approach is introduced to derive a transition probability, and it is applied to the electron transport in statically disturbed field lines in section 3. Section 4 presents the conclusions.

2 Path integral approach

In order to understand statistical property of random electron motions expressed by the Langevin equations, a transition probability density is derived as follows. For simplicity, we consider the Langevin equation in one dimension: $dx/dt = h(x, t) + W(t)$, where a function $h(x, t)$ is an arbitrary function of x and t, and a noise $W(t)$ is described by the Brownian process $w(t)$ with a diffusion coefficient D: $W(t) = dw(t)/dt$. Note that most techniques developed in one dimension can be easily generalized into n dimensional space. A path integral corresponding to a stochastic differential equation is originated in Ref. [11], and in this earliest work the differential equation was limited under a condition that a noise W and a path x are linearly related. In this article, a relation between W and x is generalized to be nonlinear. By following discussion in Ref. [11], the probability density functional for the noise $W(t) = dw(t)/dt$ is given as $K_w[w(t)] = \exp\{-(1/4D) \int dt\ [\dot{w}(t)]^2\}$. The probability density functional for the path $x(t)$, $K_x[x(t)]$, is related with $K_w[w(t)]$ as follows; $K_x[x(t)]\ \mathfrak{D}x(t) = K_w[w(t)]\ \mathfrak{D}w(t)$, because there is a path $x(t)$ for each noise

$W(t)$, where an integral $\int \mathfrak{D}x(t)\cdots$ is a path integral carried out in the space of the x functions. By using a notation of difference, the nonlinear relation between $W = dw/dt$ and x, $W(t) = \dot{w}(t) = \dot{x}(t) - h[x(t), t]$, can be represented as $W_j = (w_j - w_{j-1})/\varepsilon = (x_j - x_{j-1})/\varepsilon - h(x_{j-1}, t_{j-1})$, thus a 'jacobian' $\mathcal{J}$ between the 'volume' elements of $x(t)$ and $w(t)$ is constant, where $W_j = W(t_j)$, $dw = (w_j - w_{j-1})$, $dx = (x_j - x_{j-1})$ and $dt = (t_j - t_{j-1}) = \varepsilon$. Note that we have to choose the prepoint position in $h[x(t), t]$, i.e. $h(x_{j-1}, t_{j-1})$, in order to avoid the 'jacobian' being a functional of $x(t)$. Hence the probability density functional $K_x[x(t)]$ is given by $K_w[w(t)]$ with a 'jacobian' $\mathcal{J}[x(t)] =$ const.; $K_x[x(t)] =$ const. $K_w[\dot{x}(t) - h(x, t)]$. Therefore we obtain the transition probability density:

$$F(b,a) \equiv F(x_b, t_b; x_a, t_a) = \int \mathfrak{D}x(t)\ K_x[x(t)]$$
$$= \int \mathfrak{D}x(t)\ \exp\left\{ -\frac{1}{4D} \int_{t_a}^{t_b} dt\ \left(\frac{dx(t)}{dt} - h\left[x(t), t\right] \right)^2 \right\}, \tag{11}$$

where (x_a, t_a) is a start point and (x_b, t_b) an end point. The more accurate and convenient expression of the path integral $F(b, a)$ in actual calculation is given as $F(b, a) = \lim_{N\to\infty} F_N(b, a)$, where

$$F_N(b,a) = \frac{1}{C^N} \int\int \cdots \int dx_1\ dx_2 \cdots dx_j \cdots dx_{N-1} \exp\left\{ -\frac{1}{4D} S_N[b,a] \right\}, \tag{12}$$

$$S_N[b,a] = \varepsilon \sum_{j=0}^{N-1} \left[\frac{x_{j+1} - x_j}{\varepsilon} - h(x_j, t_j) \right]^2$$
$$= 2\varepsilon \sum_{j=0}^{N-1} \left\{ \underbrace{\frac{1}{2}\left(\frac{x_{j+1} - x_j}{\varepsilon} \right)^2}_{\mathcal{K}} - \underbrace{\left(h(x_j, t_j) \frac{x_{j+1} - x_j}{\varepsilon} - \frac{1}{2}\left[h(x_j, t_j)\right]^2 \right)}_{\mathcal{V}} \right\}, \tag{13}$$

and C is a normalizing factor. Note that $\mathcal{L} = \mathcal{K} - \mathcal{V}$ in the 'action' integral (13) can be interpreted as the 'Lagrangian' $\mathcal{L}$ with the 'kinetic energy' $\mathcal{K}$ and the 'potential' $\mathcal{V}$. The time interval $(t_b - t_a)$ is divided by N steps of width $\varepsilon = (t_b - t_a)/N$, i.e. $t_j = j\varepsilon + t_a$, where $t_0 = t_a$ and $t_N = t_b$. A point x_j which is a position at a time t_j is given as $x_j = x(t_j)$, where $x_0 = x_a$ and $x_N = x_b$. Each path is constructed to connect the points x_j with straight lines.

Using the transition probability density $F(b, a)$, we can define a probability density function:

$$p(x_b, t_b) = \int_{-\infty}^{\infty} dx_a\ F(x_b, t_b; x_a, t_a)\ p(x_a, t_a). \tag{14}$$

For an infinitesimal time interval ε, the probability density function $p(x, t+\varepsilon)$ is given as

$$p(x,t+\varepsilon) = \int_{-\infty}^{\infty} dz \, \frac{1}{C} \exp\left\{ -\frac{\varepsilon}{4D} \left[\frac{(x-z)}{\varepsilon} - h(z,t) \right]^2 \right\} p(z,t). \tag{15}$$

We make the substitution $z = x + \eta$ with the expectation that appreciable contributions to the integral in Eq. (15) will occur only for small η. Expanding the l.h.s. in ε and the r.h.s. in ε and η, and integrating with respect to η, we have

$$\begin{aligned} p(x,t) &+ \varepsilon \frac{\partial p(x,t)}{\partial t} + \mathcal{O}(\varepsilon^2) + \cdots \\ =& p(x,t) + \varepsilon \left\{ -\frac{\partial h(x,t)p(x,t)}{\partial x} + D\frac{\partial^2 p(x,t)}{\partial x^2} \right\} + \mathcal{O}(\varepsilon^2) + \cdots , \end{aligned} \tag{16}$$

if $h(x,t)$ is a function of class $C^{2,1}$. This will be true to order ε if $p(x,t)$ satisfies the differential equation:

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial h(x,t)p(x,t)}{\partial x} + D\frac{\partial^2 p(x,t)}{\partial x^2}, \tag{17}$$

and we obtain the Fokker-Planck equation corresponding to the Langevin equation in one dimension [10,12].

We remark that the probability can be expressed as

$$\int_{d_b}^{u_b} dx_b \; p(x_b, t_b) = \int_{S(d_b,u_b)} d_w x \; G[x(t), t], \tag{18}$$

where an integral $\int_{S(d_b,u_b)} d_w x$ and a functional $G[x(t), t]$ are defined respectively as

$$\begin{aligned} \int_{S(d_b,u_b)} d_w x = \lim_{N\to\infty} \frac{1}{C^N} \int_{d_b}^{u_b} dx_N \int_{-\infty}^{\infty} dx_{N-1} \cdots \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_1 \\ \times \exp\left\{ -\frac{\varepsilon}{4D} \sum_{j=0}^{N-1} \left(\frac{x_{j+1} - x_j}{\varepsilon} \right)^2 \right\}, \end{aligned} \tag{19}$$

$$G[x(t), t] = G[x_0, x_1, x_2, \ldots, x_N; t_0, t_1, t_2, \ldots, t_N]$$

$$= \exp\left\{\frac{\varepsilon}{2D}\sum_{k=0}^{N-1}\left[\left(\frac{x_{k+1}-x_k}{\varepsilon}\right)h(x_k,t_k)-\frac{1}{2}\left\{h(x_k,t_k)\right\}^2\right]\right\}, \quad (20)$$

and the initial distribution is the delta function $p(x_0, t_0) = \delta(x_0 - x_a)$. As a result, the probability $\int_{d_b}^{u_b} dx_b\ p(x_b, t_b)$ can be interpreted as the integral of functional $G[x(t), t]$ over the Wiener measure [13]. Then a start point x_0 is defined by the initial condition $p(x_0, t_0) = \delta(x_0 - x_a)$ and an end point x_N is given as $x_N = x_a + \xi\sigma\sqrt{t_N - t_0} = x_b$, where ξ is a Gaussianly distributed random variable with mean 0 and variance 1, and σ is a deviation $\sigma = \sqrt{2D}$. The elements of an Wiener path, $x_1, x_2, \ldots, x_{N-1}$, can be given by repeated application of the so-called 'interpolation formula' [14]: for $j = 1, 2, \ldots, N-1$

$$x_j = x(t_j) = \frac{(N-j)x_{j-1} + x_N}{N-j+1} + \xi_j\sigma\sqrt{\frac{(N-j)\varepsilon}{N-j+1}}, \quad (21)$$

where $\xi_j = \xi(t_j)$ means the random variable ξ at $t = t_j$. Therefore, the probability density $p(x_b, t_b)$ is allowed to be written as the sum of the weights $G[x^{(\ell)}(t), t]$ on the Wiener paths $x^{(\ell)}(t)$:

$$p(x_b, t_b) = \frac{1}{C}\exp\left\{-\frac{(x_b - x_a)^2}{4D(t_b - t_a)}\right\}$$
$$\times \lim_{\substack{N\to\infty \\ N_p\to\infty}} \frac{1}{N_p}\sum_{\ell=1}^{N_p} G\left[x_0, x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_k^{(\ell)}, \ldots, x_N; t_0, t_1, \ldots, t_N\right], \quad (22)$$

where N_p is the number of paths $\{x^{(\ell)}(t)\}$ starting from $x_0 = x_a$ and arriving at $x_N = x_b$, $x_k^{(\ell)}$ the k-th element of the ℓ-th Wiener path $x^{(\ell)}(t)$, and C a normalizing factor. For example, the Ornstein-Uhlenbeck process [10,15], $dx/dt = -\gamma x + dw(t)/dt$, is considered and its distribution is calculated by the path integral (22), then the distribution converges to the stationary Gaussian distribution: $p(x_b) = [\gamma/(2\pi D)]^{1/2}\exp\{-\gamma x_b{}^2/(2D)\}$, and numerical estimations of cumulants characterizing the distribution are shown in Table 1, where γ is a positive constant.

If a more generalized noise, $W = W(x, t) = u(x)dw(t)/dt$, is considered and a Langevin equation is described as $dx/dt = h(x, t) + u(x)dw(t)/dt$, then a transition probability density is given as $F(x_b, t_b; x_a, t_a) = \bar{F}(\bar{x}_b, t_b; \bar{x}_a, t_a)/[\bar{C}_0 u(x_b)]$ with $\bar{x}_a = \int_0^{x_a} dx/u(x)$ and $\bar{x}_b = \int_0^{x_b} dx/u(x)$:

$$\bar{F}(b, a) \equiv \bar{F}(\bar{x}_b, t_b; \bar{x}_a, t_a)$$
$$= \int \mathfrak{D}\bar{x}(t)\ \exp\left\{-\frac{1}{4D}\int_{t_a}^{t_b} dt\ \left(\frac{d\bar{x}(t)}{dt} - \bar{h}\left[\bar{x}(t), t\right]\right)^2\right\}$$

$$= \lim_{N\to\infty} \frac{1}{C^N} \int \int \cdots \int d\bar{x}_1 d\bar{x}_2 \cdots d\bar{x}_{N-1} \times \exp\left\{ \varepsilon \sum_{j=0}^{N-1} \left[\frac{\bar{x}_{j+1} - \bar{x}_j}{\varepsilon} - \bar{h}(\bar{x}_j, t_j) \right]^2 \right\}, \quad (23)$$

where $dw(t)/dt$ is the Gaussian white noise with the diffusion coefficient $D =$ const., a function $u(x)$ is an arbitrary continuous function satisfying $0 < u(x) < \infty$, a new path $\bar{x}(t)$ is a function of $x(t)$, $\bar{x} = \bar{x}(x)$, satisfying $d\bar{x} = \bar{x}_{j+1} - \bar{x}_j = (x_{j+1} - x_j)/u(x_j)$ for $j = 0, 1, 2, \ldots, N-1$: i.e. $x_{j+1} = x_j + u(x_j)(\bar{x}_{j+1} - \bar{x}_j)$, a function $\bar{h}$ is defined as $\bar{h}(\bar{x}, t) = h(x,t)/u(x) - D\partial u(x)/\partial x$, the transition probability density $\bar{F}(\bar{x}_b, t_b; \bar{x}_a, t_a) = u(x_b)F(x_b, t_b; x_a, t_a)/C_0$ satisfies $\int d\bar{x}_b \; \bar{F} = 1$, and $\bar{C}_0$ and C_0 are normalizing constants. Note that the Langevin equation for a new path $\bar{x}(t)$ is expressed as $d\bar{x}/dt = \bar{h}(\bar{x}, t) + dw(t)/dt$, and the same mathematical techniques (18)-(22) can be used to calculate the probability density $p(x_b, t_b)$. See also Refs. [10,16].

3 Electron transport in statically disturbed field lines

To investigate effects of disturbed magnetic field lines and collisions on the electron transport, we return to the Langevin equations (3) and (4). By using results in Eqs. (11) and (14), the probability density $p(x_b, y_b, t_b)$ is given as

$$p(x_b, y_b, t_b) = \int \mathfrak{D}x(t) \int \mathfrak{D}y(t) \times \prod_{i=x,y} \exp\left\{ -\frac{1}{4D_i} \int_{t_a}^{t_b} dt \left[\frac{dr_i}{dt} - h_i(x,y,t) \right]^2 \right\}, \quad (24)$$

where $\boldsymbol{r}(t) = (r_x, r_y) = (x, y)$ means a path in two dimensional space, a function $\boldsymbol{h}$ is defined as $\boldsymbol{h}(x,y,t) = (h_x, h_y) = (B_x/B_t, B_y/B_t)$ and means effects of field lines, the initial condition is given as $p(x_0, y_0, t_0) = \delta(x_0 - x_a)\delta(y_0 - y_a)$, and D_x and D_y are diffusion coefficients for the x and y directions, respectively. It is easy to show that the probability density of Eq. (24) satisfies the Fokker-Planck equation in two dimensional space. Note that for the collisionless limit $D_i \to 0$, the equations of a field line (1) and (2) are obtained. From Eq. (22), the probability density is rewritten as

$$p(x_b, y_b, t_b) = \frac{1}{C} \lim_{\substack{N\to\infty \\ N_p\to\infty}} \frac{1}{N_p} \sum_{\ell=1}^{N_p} \prod_{i=x,y} \exp\left\{ -\frac{(r_{i,b} - r_{i,a})^2}{4D_i(t_b - t_a)} \right\} \times \exp\left\{ \frac{\varepsilon}{2D_i} \sum_{j=0}^{N-1} \left[h_{i,j}^{(\ell)} \frac{r_{i,j+1}^{(\ell)} - r_{i,j}^{(\ell)}}{\varepsilon} - \frac{1}{2}\left(h_{i,j}^{(\ell)}\right)^2 \right] \right\}, \quad (25)$$

where $h^{(\ell)}_{x,j} = h_x(x^{(\ell)}_j, y^{(\ell)}_j, t_j)$, $h^{(\ell)}_{y,j} = h_y(x^{(\ell)}_j, y^{(\ell)}_j, t_j)$, $r^{(\ell)}_{x,j} = x^{(\ell)}_j$, $r^{(\ell)}_{y,j} = y^{(\ell)}_j$, $r^{(\ell)}_{x,0} = r_{x,a} = x_a$, $r^{(\ell)}_{y,0} = r_{y,a} = y_a$, $r^{(\ell)}_{x,N} = r_{x,b} = x_b$, and $r^{(\ell)}_{y,N} = r_{y,b} = y_b$. When more generalized noises, $W_i = W_i(r_i, t) = u_i(r_i)dw_i(t)/dt$, are considered and Langevin equations are described as $dr_i/dt = h_i(x, y, t) + u_i(r_i)dw_i(t)/dt$, a transition probability density is given as

$$F(x_b, y_b, t_b; x_a, y_a, t_a) = \bar{F}(\bar{x}_b, \bar{y}_b, t_b; \bar{x}_a, \bar{y}_a, t_a)/[\bar{C}_0 u_x(x_b) u_y(y_b)], \tag{26}$$

$$\bar{F}(\bar{x}_b, \bar{y}_b, t_b; \bar{x}_a, \bar{y}_a, t_a) = \int \mathfrak{D}\bar{x}(t) \int \mathfrak{D}\bar{y}(t) \times \prod_{i=x,y} \exp\left\{ -\frac{1}{4D_i} \int_{t_a}^{t_b} dt \left[\frac{d\bar{r}_i}{dt} - \bar{h}_i(\bar{x}, \bar{y}, t) \right]^2 \right\} \tag{27}$$

with $\bar{r}_{i,a} = \int_0^{r_{i,a}} dr_i/u_i(r_i)$ and $\bar{r}_{i,b} = \int_0^{r_{i,b}} dr_i/u_i(r_i)$ for $i = x, y$; where $dw_i(t)/dt$ is the Gaussian white noise with the diffusion coefficient $D_i =$ const., a function $u_i(r_i)$ is an arbitrary continuous function satisfying $0 < u_i(r_i) < \infty$, a new path $\bar{r}_i(t)$ is a function of $r_i(t)$, $\bar{r}_i = \bar{r}_i(r_i)$, satisfying $d\bar{r}_i = \bar{r}_{i,j+1} - \bar{r}_{i,j} = (r_{i,j+1} - r_{i,j})/u_i(r_{i,j})$ for $j = 0, 1, 2, \ldots, N-1$: i.e. $r_{i,j+1} = r_{i,j} + u_i(r_{i,j})(\bar{r}_{i,j+1} - \bar{r}_{i,j})$, a function $\bar{h}_i$ is defined as $\bar{h}_i(\bar{x}, \bar{y}, t) = h_i(x, y, t)/u_i(r_i) - D_i \partial u_i(r_i)/\partial r_i$, the transition probability density $\bar{F}(\bar{x}_b, \bar{y}_b, t_b; \bar{x}_a, \bar{y}_a, t_a)$ satisfies $\int\int d\bar{x}_b d\bar{y}_b \, \bar{F} = 1$, and $\bar{C}_0$ is a normalizing constant.

Hereafter, we demonstrate a Monte Carlo calculation of the electron distribution under both effects of chaotic field lines and collisions. When evolution of Eqs. (3) and (4) is directly solved under the finite calculation accuracy or the coarse graining of space-time (x, y, t), the calculated final distribution is affected by growing numerical errors of orbits, which are caused by property of chaotic field lines [17]. On the other hand, the path integral method introduced in this article has an advantage that it does not include intrinsically the growing numerical errors, as is shown obviously in Eq. (25). According to methods of Eqs. (21) and (25), the Wiener paths $(x^{(\ell)}(t), y^{(\ell)}(t))$ are generated randomly and the weights on the paths are integrated. In actual calculation of an electron distribution in a disturbed magnetic field, we choose a magnetic field modeled as the Arnold-Beltrami-Childress (ABC) field [18]:

$$\boldsymbol{B} = \begin{pmatrix} B_x \\ B_y \\ B_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ B_{t0} \end{pmatrix} + \begin{pmatrix} a \sin \omega t + c \cos \omega y \\ b \sin \omega x + a \cos \omega t \\ c \sin \omega y + b \cos \omega x \end{pmatrix}, \tag{28}$$

where a, b and c are parameters of the ABC field, and B_{t0} is constant satisfying $B_t > 0$.

The path integral method assists to understand qualitatively complex behavior of the final distribution. An extent of contribution of the weight $G[x^{(\ell)}(t), y^{(\ell)}(t), t]$ on the Wiener path $(x^{(\ell)}(t), y^{(\ell)}(t))$ is determined by a value of its exponent:

$$\ln G[x^{(\ell)}(t), y^{(\ell)}(t), t] = \sum_{i=x,y} \frac{\varepsilon}{2D_i} \sum_{j=0}^{N-1} \left[h_{i,j}^{(\ell)} \frac{r_{i,j+1}^{(\ell)} - r_{i,j}^{(\ell)}}{\varepsilon} - \frac{1}{2}\left(h_{i,j}^{(\ell)}\right)^2 \right]. \quad (29)$$

For simplicity, the Gaussian white noise $W_i = dw_i(t)/dt$ with $D_x = D_y =$ const. is assumed hereafter. The first term of the exponent gives a drift of the distribution. The second term of the exponent, i.e. $-(1/2)\sum_{i=x,y}[h_i(x,y,t)]^2$, plays an important role, as is shown below. For a case of the ABC field with $a = 15$, $b = 1$, $c = 0.5$, $\omega = \pi/2$, and $B_{t0} = 3$ in Fig. 1(b), the second term has deep hollows at $x, y \approx \pm 2 \pm 4n$: $n = 0, 1, 2, \ldots$, as is shown in Fig. 2. Since a larger exponent gives more important contribution to the integral, the distribution is expected to be hollow around $x, y = \pm 2 \pm 4n$ and to be affected seriously by the hollows in $-(1/2)\sum_{i=x,y}[h_i(x,y,t)]^2$ for the x direction rather than the y direction. This qualitative understanding agrees with an actual calculation of the distribution in Figs. 3 and 4; the distribution for the x (or y) direction in Fig. 4 is given as the integral of $p(x_b, y_b, t_b)$ with respect to y_b (or x_b). Estimation of the cumulants is shown in Table 2.

In order to investigate effects of collisions, distributions of electrons with sufficiently large diffusion coefficients, $D_x = D_y = 10$, are calculated as solid or dash-dot lines in Fig. 5 and the cumulants are estimated in Tables 3 and 4. We find that the distributions drift to the positive or negative side with much the same absolute value of $\langle r_{i,b} - r_{i,a}\rangle_c$ and its shape for the x direction is distorted but is close to the Gaussian distribution, where $\langle\cdots\rangle_c$ means a cumulant. Distributions for the y direction seems to be Gaussian without a hollow profile, as is shown in Fig. 5(b), because Wiener paths for the y direction are not seriously affected by the hollows in $-(1/2)\sum_{i=x,y}[h_i(x,y,t)]^2$. If the parameter a decreases to zero and the other parameters b, c, ω and B_{t0} are fixed (see Fig. 1(a)), then the distribution approaches to Gaussian shown by dot lines in Fig. 5, because the hollows of the second term of the exponent are negligible in this case. Evolution of distribution is shown as solid lines in Fig. 6 and the cumulants are estimated in Table 5. The distribution for the x direction has the hollow profile with the period of $-(1/2)\sum_{i=x,y}[h_i(x,y,t)]^2$, but is still close to Gaussian. The drift of the developing distribution is much the same as at the previous time. Therefore, when the collisionality is large enough, the distribution for the x direction has a tendency to be the Gaussian-like distribution with the hollow profile and the diffusion coefficients of collisions, and the one for the y direction is not seriously affected by effects of field lines.

4 Conclusions

We have developed basic techniques of the path integral approach for the electron transport in a disturbed magnetic filed line structure. The path integral method does not include intrinsically the growing numerical errors of orbits, which are caused by evolution of a nonlinear differential equation under the finite accuracy. A method of understanding the qualitative content of Langevin equations including chaotic orbits is given by this approach. The path integral method assists to give one's intuition in bringing together physical insight and mathematical analysis. Monte Carlo calculations are demonstrated to show these advantages and the electron distributions for the ABC magnetic field are estimated. When the collisionality is sufficiently large, the distribution for the x direction has a tendency to be the Gaussian-like distribution with the hollow profile and the diffusion coefficients of collisions, and the one for the y direction is not seriously affected by effects of field lines. The Monte Carlo calculation was carried out by using 4×10^9 paths with 10^3 elements per path, i.e. 8×10^{12} random numbers were used; they were generated by the Tausworthe sequence [19]. As is suggested by Eq. (25), the Monte Carlo scheme to calculate a transition probability can be suitable for a parallel computation by using parallel generators of random numbers. The effects of chaotic field lines on the transport in various collisionality regimes after a long time will be investigated minutely by using the path integral approach in near future.

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Table 1
Numerical estimations of cumulants after a sufficient time. We assume that $\gamma = 10$, $D = 1$, and $x_a = 1$. Note that skewness is $\langle (x_b - x_a)^3 \rangle_c / {\langle (x_b - x_a)^2 \rangle_c}^{3/2}$ and kurtosis is $\langle (x_b - x_a)^4 \rangle_c / {\langle (x_b - x_a)^2 \rangle_c}^2$, where $\langle \cdots \rangle_c$ means a cumulant.

	numerical	analytical
$\langle x_b - x_a \rangle_c$	-1.001	-1 $(\langle x_b \rangle_c = 0)$
$\langle (x_b - x_a)^2 \rangle_c$	0.1031	0.1 (D/γ)
skewness	2.434×10^{-3}	0
kurtosis	3.983×10^{-3}	0

Table 2
Numerical estimations of cumulants for the ABC magnetic field with $a = 15$, $b = 1$, $c = 0.5$, $\omega = \pi/2$, and $B_{t0} = 3$. We assume that $D_x = D_y = 2$, $(x_a, y_a, t_a) = (0, 0, 0)$, and $t_b = 1$. Note that skewness is $\langle (r_{i,b} - r_{i,a})^3 \rangle_c / {\langle (r_{i,b} - r_{i,a})^2 \rangle_c}^{3/2}$ and kurtosis is $\langle (r_{i,b} - r_{i,a})^4 \rangle_c / {\langle (r_{i,b} - r_{i,a})^2 \rangle_c}^2$.

	x direction $(i = x)$	y direction $(i = y)$
$\langle r_{i,b} - r_{i,a} \rangle_c$	3.232	3.011
$\langle (r_{i,b} - r_{i,a})^2 \rangle_c$	4.332	4.175
skewness	-3.588×10^{-2}	1.552×10^{-2}
kurtosis	-1.773×10^{-1}	-1.506×10^{-1}

Table 3
Numerical estimations of cumulants for the ABC magnetic field with $a = 15$, $b = 1$, $c = 0.5$, $\omega = \pi/2$, and $B_{t0} = 3$. We assume that $D_x = D_y = 10$, $(x_a, y_a, t_a) = (0, 0, 0)$, and $t_b = 1$.

	x direction $(i = x)$	y direction $(i = y)$
$\langle r_{i,b} - r_{i,a} \rangle_c$	3.410	3.343
$\langle (r_{i,b} - r_{i,a})^2 \rangle_c$	19.95	19.97
skewness	-1.582×10^{-3}	-1.895×10^{-3}
kurtosis	-1.659×10^{-2}	-1.472×10^{-2}

Table 4
Numerical estimations of cumulants for the ABC magnetic field with $a = -15$, $b = 1$, $c = 0.5$, $\omega = \pi/2$, and $B_{t0} = 3$. We assume that $D_x = D_y = 10$, $(x_a, y_a, t_a) = (0, 0, 0)$, and $t_b = 1$.

	x direction ($i = x$)	y direction ($i = y$)
$\langle r_{i,b} - r_{i,a} \rangle_c$	-3.402	-3.361
$\langle (r_{i,b} - r_{i,a})^2 \rangle_c$	19.95	20.09
skewness	1.632×10^{-3}	7.362×10^{-3}
kurtosis	-1.643×10^{-2}	-1.743×10^{-2}

Table 5
Numerical estimations of cumulants for the ABC magnetic field with $a = 15$, $b = 1$, $c = 0.5$, $\omega = \pi/2$, and $B_{t0} = 3$. We assume that $D_x = D_y = 10$, $(x_a, y_a, t_a) = (0, 0, 0)$, and $t_b = 5$.

	x direction ($i = x$)	y direction ($i = y$)
$\langle r_{i,b} - r_{i,a} \rangle_c$	3.400	3.363
$\langle (r_{i,b} - r_{i,a})^2 \rangle_c$	99.74	100.0
skewness	-4.897×10^{-3}	-9.915×10^{-3}
kurtosis	-2.049×10^{-2}	-2.043×10^{-2}

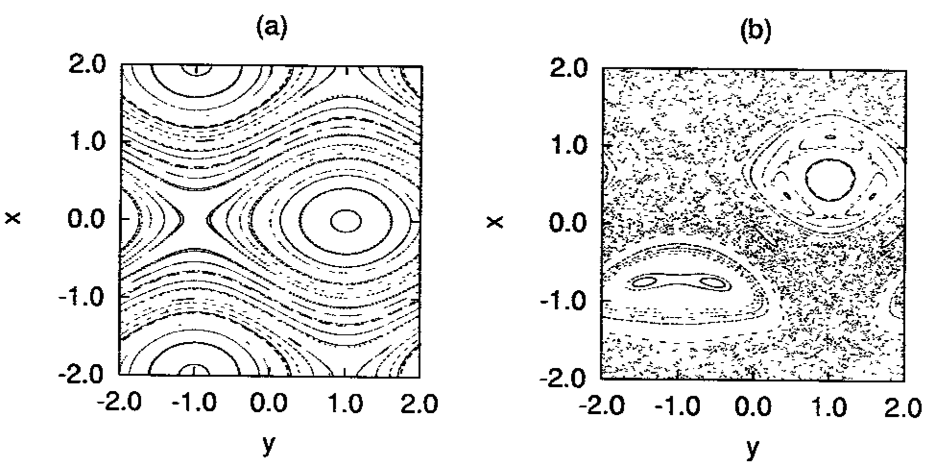

Fig. 1. Field line structures of the Arnold-Beltrami-Childress (ABC) magnetic fields with (a) $a = 0$, $b = 1$, $c = 0.5$, $\omega = \pi/2$ and $B_{t0} = 3$, and (b) $a = 15$, $b = 1$, $c = 0.5$, $\omega = \pi/2$ and $B_{t0} = 3$. See Eq. (28).

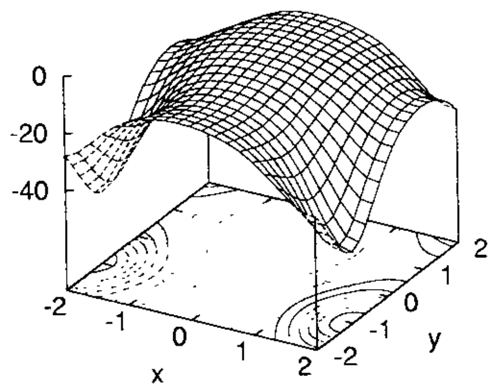

Fig. 2. A function $-(1/2)\sum_{i=x,y}[h_i(x,y,t)]^2$ with $a = 15$, $b = 1$, $c = 0.5$, $\omega = \pi/2$, $B_{t0} = 3$, and $t = 0$.

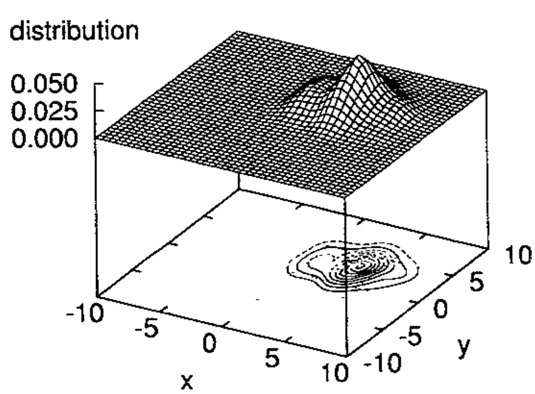

Fig. 3. A distribution $p(x,y,t)$ of random electrons at $t = t_b = 1$ for the ABC magnetic field with $a = 15$, $b = 1$, $c = 0.5$, $\omega = \pi/2$, and $B_{t0} = 3$. We assume that the start point $(x_a, y_a, t_a) = (0,0,0)$ and diffusion coefficients $D_x = D_y = 2$, where an isotropic diffusion is assumed and the diffusion coefficients are set to satisfy a condition that random electrons can be spread sufficiently over one period of the function $-(1/2)\sum_{i=x,y}[h_i(x,y,t)]^2$ on the x-y cross section, $\sqrt{2D_i t_b} = 2$.

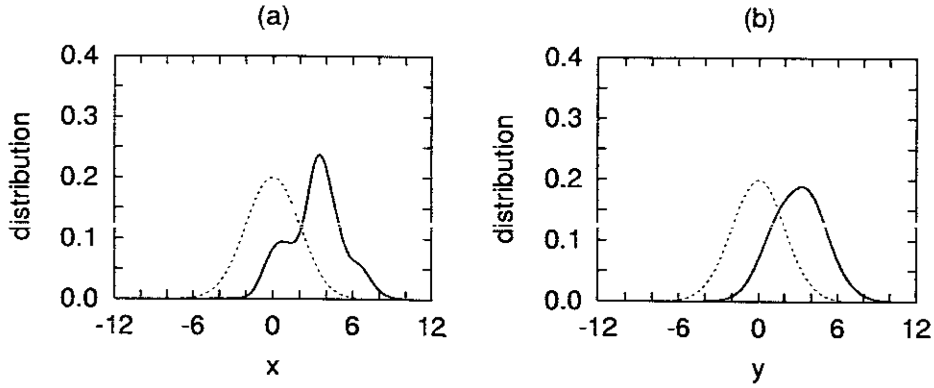

Fig. 4. Distributions at $t = t_b = 1$ for (a) the x direction and (b) the y direction. The distribution for the x (or y) direction is given as the integral of $p(x, y, t)$ in Fig. 3 with respect to y (or x). A dot line is the Gaussian distribution with mean 0 and variance 4.

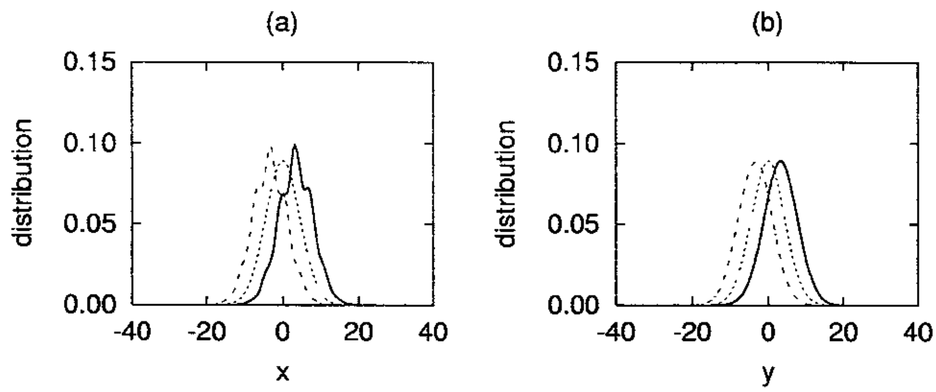

Fig. 5. Distributions of random electrons at $t_b = 1$ for (a) the x direction and (b) the y direction. A solid line is calculated by applying the path integral method to the ABC field with $a = 15$, $b = 1$, $c = 0.5$, $\omega = \pi/2$, and $B_{t0} = 3$. We assume that the start point $(x_a, y_a, t_a) = (0, 0, 0)$ and diffusion coefficients $D_x = D_y = 10$. If $a = -15$ and the other parameters are fixed, then a dash-dot line is obtained. A dot line is the Gaussian distribution with mean 0 and variance 20.

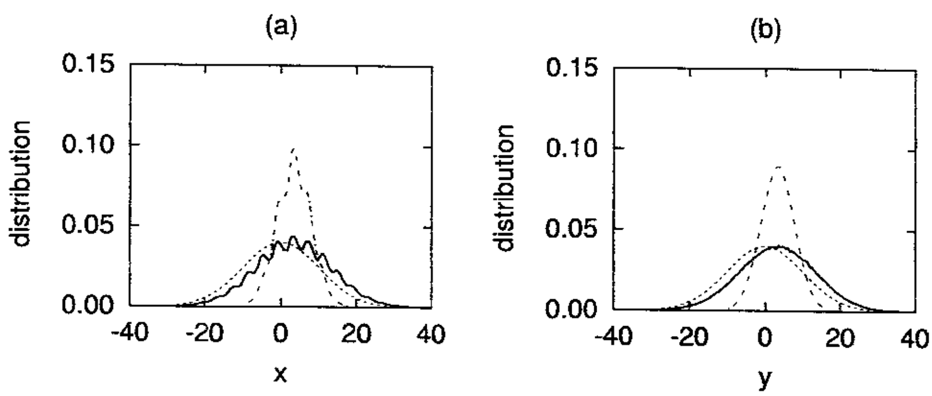

Fig. 6. Evolution of distributions for (a) the x direction and (b) the y direction. A dash-dot line is the distribution at $t_b = 1$ and a solid line is one at $t_b = 5$, where $D_x = D_y = 10$. A dot line is the Gaussian distribution with mean 0 and variance 100.

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