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Probability of Statistical L-H Transition in Tokamaks

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Abstract

A statistical model of bifurcation of radial electric field E_r is analyzed in relation with L-H transitions of tokamaks. A noise from micro fluctuations leads to random noise for E_r. The transition of E_r occurs in a probabilistic manner. Probability density function and ensemble average of E_r are obtained, when hysteresis of E_r exists. Forward- and backward-transition probabilities are calculated. The phase boundary is shown. Due to the suppression of turbulence by E_r shear, the boundary deviates from the Maxwell's construction rule.

Keywords: L-H transition, statistical theory, probabilistic transition, transition probability, radial electric field, probability density function, random noise, hysteresis

Structural formation in inhomogeneous magnetized plasma has been one of the main issues in modern plasma physics. An important example is the H-mode transition [1] in toroidal plasmas. The key is the bifurcation of radial electric field E_r [2, 3] and its mutual interaction with turbulence which has a micro scale length (such as the ion gyroradius ρ_i or collisionless skin depth $\delta = c/\omega_p$) [4]. Theory has made progresses in explaining the existence of bifurcation in E_r of meso-scale (a hybrid between the plasma radius a and micro-scales [5, 6]) including zonal flow [7]. (See reviews, e.g., [5, 8, 9].) A further breakthrough is needed. First, the statistical and stochastic properties of L-H transition must be clarified. This is because E_r and fluctuations do not satisfy the laws of thermodynamical equilibrium. Efforts have been made to establish the far-non-equilibrium statistical law of micro turbulence,[9-15] and the role of nonlinear noise source was found important. The analyses must be extended to the L-H transition phenomena. The other is an experimental test of theories. Experiments have shown recently that the change of E_r occurs in a short time (a few times of qR/c_s, q: safety factor, R: majour radius and c_s: ion sound velocity) [16] as has been predicted.[2] This supports the model based on hard bifurcation. On the other hand, a test of observing a hysteresis by use of slow change of parameters (longer than the energy confinement time) has not shown clear hysteresis.[17]

In this article, we present a statistical model of the electric bifurcation of the L-H transitions in toroidal plasmas. Nonlinearity of micro-fluctuations statistically induces a random noise of the meso-scale E_r. Being kicked by this random noise, transitions between the L- and H-states occur in a probabilistic manner. A Langevin equation is formulated including the mechanism for hysteresis of E_r. The probability density function (PDF) of E_r is obtained, and the ensemble average is given. The flux of probability density is calculated, and the transition probability and back-transition probability are obtained. The ensemble average of E_r does not show a hysteresis although a deterministic model includes the hysteresis. The phase boundary of the statistical view is given by the condition that the H-mode and L-mode states have an equal probability. This is an extension of the Maxwell's construction rule. The phase boundary shifts to the ridge of cusp for H-to-L transition, due to the suppression of fluctuation by the E_r shear. The competition between the life time (inverse of the transition probability) and the time for the change of global parameters determines whether the hysteresis is observed experimentally or not.

We consider a thin layer near the tokamak edge. We analyze the dynamics of the radial electric field E_r (averaged over the magnetic surface) in the presence of micro fluctuations. The radial extent of E_r has the scale length ℓ . (We assume a spatial and temporal scale separation between E_r and micro-fluctuations, i.e., $\ell >> \rho_i, \delta$.) The dynamical equation for E_r has been expressed in terms of the charge-conservation equation combined with Poisson's equation as

$$\varepsilon_0 \varepsilon_\perp \frac{\partial}{\partial t} E_r = - J_r \equiv - \bar{J}_r - J_r^{\mathrm{n}} \tag{1}$$

where J_r is the radial current density, ε_0 is the vacuum susceptibility and $\varepsilon_\perp = \left(1 + 2q^2\right) c^2 v_A^{-2}$ is a dielectric constant of the magnetized toroidal plasma. The radial current has two components: a time-averaged component, $\bar{J}_r$, and the rapidly-varying part J_r^{n}. The former, i.e., the time average-part (deterministic part) $\bar{J}_r$, has various origins including the bulk viscosity of ions, ion orbit loss, and eddy damping (or zonal flow excitation) for E_r by microfluctuations.[5-9] The latter is induced by the convective nonlinearity in the vorticity equation $\tilde{V} \cdot \nabla \tilde{V}$ associated with micro fluctuations. It changes with the characteristic autocorrelation time of micro-fluctuations τ_{ac}, which is much shorter than the typical evolution time of E_r. In this article, the term J_r^{n} is considered to be a random noise. The time-average part $\bar{J}_r$ dictates the deterministic picture of bifurcations, and the noise part J_r^{n} gives a random kick for E_r and causes a probabilistic nature in transitions. The scale length ℓ is treated as a constant parameter in the statistical evolution of the magnitude of E_r, and Eq.(1) is rewritten as a Langevin equation of the magnitude of E_r as

$$\frac{\partial}{\partial \tau} X + \Lambda X = w(\tau)\, g, \tag{2}$$

where normalization is introduced for the electric field and time as $X = e\rho_{\mathrm{p}} E_r / T$ and $\tau = t\, c_s / 2qR$ and $w(t)$ is a white-noise. (ρ_{p}: ion gyroradius at poloidal magnetic field, T : plasma temperature.) The damping term, $\Lambda X = \left(1 + 2q^2\right)^{-1} \left(qR / \rho_s e c_s n_i\right) \bar{J}_r$, is the normalized current. The term g denotes the noise current J_r^{n}.

Let us consider the L-H electric bifurcation where the bulk viscosity of ions, ion orbit loss and zonal flow excitation with shear viscosity damping have the key roles.[2, 3, 7, 18] One has

$$\Lambda X = \operatorname{Im} Z\left(X + i\nu_*\right) \cdot \left(X + X_{\mathrm{NC}}\right) + \frac{\nu_b}{\left(\nu_b + \alpha X^4\right)^{1/2}} \exp\left(- \left(\nu_b + \alpha X^4\right)^{1/2}\right) - \gamma_{zonal} X \tag{3}$$

where $Z(X)$ is the plasma dispersion function, X_{NC} is the neoclassical drive and is of the order of $- \rho_p n_e^{-1} \mathrm{d}n_e / \mathrm{d}r$, $\nu_* = \nu_{ii} qR c_s^{-1}$ is the normalized ion collision frequency, $\nu_b = \varepsilon^{-3/2} \nu_*$, $\varepsilon = a/R$, α denotes the orbit squeezing [3] and γ_{zonal} is the zonal flow excitation rate combined with shear viscosity damping.[7] Zeros of Λ and relation with L-H transition have been discussed in literature.[2, 3] When $\Lambda X = 0$ has one solution, the solution describes either the L-mode state or H-mode state. When multiple solutions exist, the bifurcation has a hysteresis and the hard transition is possible to occur. A thin curve in Fig. 1(a) illustrates the solution of deterministic model, $\Lambda X = 0$, as a function of the gradient on the $X_{\rm NC}$ for a fixed value of ν_b. Bifurcation and hysteresis of the radial electric field are shown.

The magnitude of $J_r^{\rm n}$ is evaluated as follows. The nonlinearly-driven current, $J_r^{\rm n} = m_i n_i B^{-1} \langle \tilde{V} \cdot \nabla \tilde{V} \rangle$ ($\langle \ \rangle$: averaged over the magnetic surface), is given as sum of radial-Fourier components, $J_r^{\rm n} = \sum_{d_z} J_r^{\rm n}(d_z)$, where d_z is a radial wavelength of a randomly-excited current. One component is given as $\left| J_r^{\rm n}(d_z) \right| = n_i m_i B^{-3} d_z^{-1} k_0^2 \tilde{\phi}^2$ for electrostatic fluctuations, where $\tilde{\phi}$ and k_0 are the amplitude of electrostatic potential perturbation and a characteristic wave number of micro-fluctuations, respectively. (When the finite-ion-gyroradius effect is included, $\tilde{\phi}$ is screened by a numerical factor.) Time-varying current $J_r^{\rm n}(d_z)$ with various values of d_z can be simultaneously excited. Each d_z-component $J_r^{\rm n}(d_z)$ is considered to be statistically independent, so that an average of the sum of $J_r^{\rm n}(d_z)$ over the length ℓ is estimated as $\left| J_r^{\rm n} \right| = \sqrt{\ell_z / \ell} \left| J_r^{\rm n}(\ell_z) \right|$ after the law of large numbers, i.e., $\left| J_r^{\rm n} \right| = n_i m_i B^{-3} \ell_z^{-1/2} \ell^{-1/2} k_0^2 \tilde{\phi}^2$. ($\ell_z$: a characteristic value of d_z.) The fact that $J_r^{\rm n}$ changes much faster than E_r enables us to approximate it as a white noise

$$J_r^{\rm n} = n_i m_i B^{-3} \ell_z^{-1/2} \ell^{-1/2} k_0^2 \tilde{\phi}^2 \sqrt{\tau_{\rm ac}}\, w(t) , \tag{4}$$

where $\sqrt{\tau_{\rm ac}}$ is explicitly written for the dimension. (A detailed argument of modelling of noise term is given in [11, 14].) When $\tau_{\rm ac}$ is much shorter than the response time of E_r, the statistical average of micro-fluctuations is calculated by treating E_r as a constant parameter. In this dc-limit, fluctuation level has been given as $\left| \tilde{\phi} \right|^2 = \left(1 + \omega_E^2 \tau_{\rm ac}^2 \right)^{-1} \left| \tilde{\phi} \right|_{\rm L}^2$, where $\left| \tilde{\phi} \right|_{\rm L}^2$ is the fluctuation level in the L-mode state, $\omega_E = B^{-1}\, {\rm d}E_r/{\rm d}r$ is the $E \times B$ shearing rate.[4, 5] Using an evaluation ${\rm d}E_r/{\rm d}r \simeq E_r/\ell$, one has $\omega_E^2 \tau_{\rm ac}^2 = \tau_{\rm ac}^2 B^{-2} \ell^{-2} E_r^2$. In the following, $\left| \tilde{\phi} \right|_L^2$ and global plasma parameters (like temperature) are treated as control parameters. The amplitude of the noise is a nonlinear function of X, and is explicitly given as

$$g = \sqrt{\hat{\tau}_{\rm ac}}\, \frac{R^2 k_0^2 \rho_i^2 \hat{\phi}^2}{a \sqrt{\ell \ell_z}} \frac{1}{1 + U X^2} , \tag{5}$$

where $\hat{\phi} = e \left| \tilde{\phi} \right|_{\rm L} / T$, $\hat{\tau}_{\rm ac} = \tau_{\rm ac} c_s / 2qR$, and $U = \left(\hat{\tau}_{\rm ac} a / \ell \right)^2$.

Statistical property of radial electric field X (the PDF of X, $P(X)$, ensemble average, transition probability between the L-and H-modes) is studied. The Fokker-Planck equation of $P(X)$ is deduced from the Langevin equation (2) as

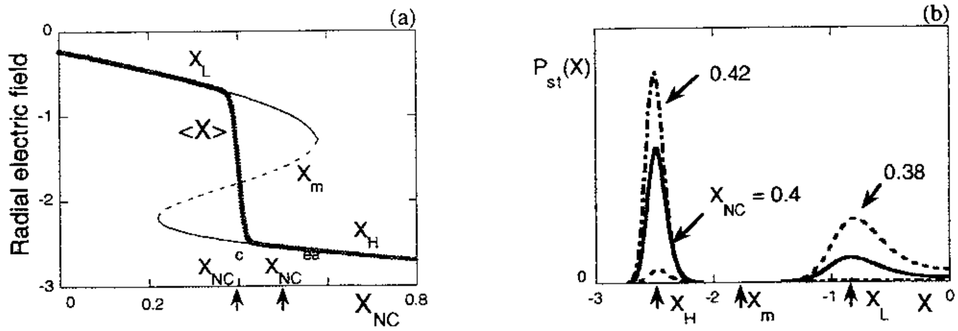

Fig.1 (a) Electric field X as a function of global gradient X_{NC} (for fixed collision frequency $\nu_* = 0.1$). The solution of $\Lambda\, X = 0$ is characterized by the cusp catastrophe (thin line). Ensemble average of the electric field $\langle X \rangle$ is shown by the thick solid line. Transition in ensemble average takes place at $X_{\mathrm{NC}} = X_{\mathrm{NC}}^{\mathrm{c}}$. $X_{\mathrm{NC}} = X_{\mathrm{NC}}^{\mathrm{ea}}$ denotes the condition of $\int J\, \mathrm{d}E = 0$. (b) PDF in a stationary state. Solid line is for $X_{\mathrm{NC}} = 0.4 \simeq X_{\mathrm{NC}}^{\mathrm{c}}$, dotted line for $X_{\mathrm{NC}} = 0.38$ (L-mode is dominant), and broken line for $X_{\mathrm{NC}} = 0.42$ (H-mode is dominant). (Parameters are: $\alpha = 0.5$, $q = 3$, $\varepsilon = a/R = 1/3$, $U = 3$ and $\Gamma = 5$.)

$$\frac{\partial}{\partial \tau} P + \frac{\partial}{\partial X}\left(\Lambda + g \frac{\partial}{\partial X} g \right) P = 0 \; . \tag{6}$$

The stationary solution $P_{\mathrm{eq}}(X)$ is expressed as $P_{\mathrm{eq}}(X) \propto g^{-1} \exp\left(- S(X) \right)$ by use of the nonlinear potential as

$$S(X) = \int^{X} 4 \Lambda X g^{-2}\, \mathrm{d}X. \tag{7}$$

The minimum of $S(X)$ (apart from a correction $\ln\ g$), i.e., zero of Λ, predicts the probable state of X. The probable state agrees with the one in deterministic model, and the dominant (i.e., the most probable) state is determined by the statistical theory in which noise is kept. $S(X)$ can have two minima at $X = X_{\mathrm{L}}$ and $X = X_{\mathrm{H}}$, which are separated by the local maximum at $X = X_{\mathrm{m}}$. Figure 1(b) illustrates PDF $P_{\mathrm{eq}}(X)$ for various values of parameter X_{NC}. The PDF has two peaks, representing the hysteresis. However, the state $X = X_{\mathrm{L}}$ is dominant if $X_{\mathrm{NC}} < X_{\mathrm{NC}}^{\mathrm{c}}$ holds ($X_{\mathrm{NC}}^{\mathrm{c}} \simeq 0.4$ for the parameters or Fig.1(a)), and $X = X_{\mathrm{H}}$ is dominant if $X_{\mathrm{NC}} > X_{\mathrm{NC}}^{\mathrm{c}}$. The ensemble average $\langle X \rangle = \int X\, P_{\mathrm{eq}}(X)\, \mathrm{d}X$ changes smoothly as the global control parameter varies. The ensemble average $\langle X \rangle$ is illustrated by a thick curve in Fig.1(a). When one solution of bistable branches is chosen as an initial condition, many transitions in between $X_{\rm H}$ and $X_{\rm L}$ branches occur in a long time, and $P(X)$ reaches to $P_{\rm eq}(X)$.

The transition probability is obtained by calculating a flux of probability density from the Fokker-Planck equation (6), and is expressed by use of the potential $S(X)$.[13, 15] The probabilities of the L-to-H transition and back-transition are given as

$$r_{\rm L \to H} = \frac{\sqrt{\Lambda_{\rm L}\Lambda_{\rm m}}}{2\pi} \exp\left(S(X_{\rm L}) - S(X_{\rm m})\right), \tag{8a}$$

$$r_{\rm H \to L} = \frac{\sqrt{\Lambda_{\rm H}\Lambda_{\rm m}}}{2\pi} \exp\left(S(X_{\rm H}) - S(X_{\rm m})\right), \tag{8b}$$

respectively, where the time rates $\Lambda_{\rm L,\,m,\,H}$ are given as $\Lambda_{\rm L,\,m,\,H} = 2X \left| \partial\Lambda/\partial X \right|$ at $X = X_{\rm L,\,m,\,H}$. Note that the time rates are normalized and $\Lambda_{\rm L,\,m,\,H}$ are of the order unity. Therefore, the dominant dependence of the transition probability comes from the exponential parts in Eq.(8). By use of Eq.(5), the transition probability is explicitly evaluated as,

$$S(X_{\rm L}) - S(X_{\rm m}) = -\Gamma I_{\rm L} \equiv -\Gamma \int_{X_{\rm m}}^{X_{\rm L}} \Lambda X \left(1 + UX^2\right)^2 {\rm d}X \ , \tag{9a}$$

$$S(X_{\rm H}) - S(X_{\rm m}) = -\Gamma I_{\rm H} \equiv -\Gamma \int_{X_{\rm H}}^{X_{\rm m}} \Lambda X \left(1 + UX^2\right)^2 {\rm d}X \tag{9b}$$

with the coefficient $\Gamma = 2\,\hat{\tau}_{ac}^{-1} a^2\, \ell\, \ell_z R^{-4} k_0^{-4} \rho_i^{-4}\, \hat{\phi}^{-4}$. Substitution of Eq.(9) into Eq.(8) provides the transition probability and back-transition probability as $r_{\rm L \to H} = \sqrt{\Lambda_{\rm L}\Lambda_{\rm m}}\,(2\pi)^{-1} \exp\left(-\Gamma I_{\rm L}\right)$, and $r_{\rm H \to L} = \sqrt{\Lambda_{\rm H}\Lambda_{\rm m}}\,(2\pi)^{-1} \exp\left(-\Gamma I_{\rm H}\right)$. Integrals $I_{\rm H}$ and $I_{\rm L}$ are calculated and are of the order unity. (Details are explained in a full paper [19].)

The phase boundary between the L-mode and H-mode (e.g., $X_{\rm NC}^{\rm c}$) is defined by the condition that both are equally observed. The probability that the state is found to stay in the L-state is given as $P_{\rm L} = r_{\rm H \to L} / \left(r_{\rm L \to H} + r_{\rm H \to L}\right)$. That for the H-state is given by $P_{\rm H} = r_{\rm L \to H} / \left(r_{\rm L \to H} + r_{\rm H \to L}\right)$. The equal-probability condition, $r_{\rm L \to H} = r_{\rm H \to L}$, is given from Eq.(8) as

$$S(X_{\rm H}) = S(X_{\rm L}) + \tfrac{1}{2} \ln\left(\Lambda_{\rm L}/\Lambda_{\rm H}\right) . \tag{10}$$

Apart from a weak logarithmic term, it is approximated as $S(X_\mathrm{H}) = S(X_\mathrm{L})$, i.e.,

$$\int_{X_\mathrm{H}}^{X_\mathrm{L}} \Lambda\, X \left(1 + UX^2\right)^2 \mathrm{d}X = 0 \ . \qquad (11)$$

This result is an extension of the Maxwell's construction rule. When the noise is independent of X, Eq.(11) gives that the condition $\int_{X_\mathrm{H}}^{X_\mathrm{L}} \Lambda\, X\, \mathrm{d}X = 0$ describes the boundary of phases. $\int \Lambda\, X\, \mathrm{d}X$ corresponds to a work function $\int J\, \mathrm{d}E$, and Maxwell's construction is deduced. The correction of UX^2 in the integrand (turbulence suppression term) is important in the H-mode $X \simeq X_\mathrm{H}$. By this effect, the phase boundary of ensemble average (X_NC^c in Fig.1(a)) deviates from the conventional criterion (denoted by $X_\mathrm{NC}^\mathrm{ea}$ in Fig.1(a)), and the region of the H-mode becomes wider. It is noted that the boundary does not depend on the magnitude of fluctuations. A phase diagram in a control parameter space $\left(\nu_b, X_\mathrm{NC}\right)$ is obtained and is explicitly given in a separate paper.[19]

We further address the problem whether a hysteresis in X is observed in experiments or not. The ensemble-averaged value $\langle X \rangle$ is reached if the averaging time is longer than the life time, $\tau_\mathrm{life}^{\mathrm{L}\rightarrow\mathrm{H}} = 1 / r_{\mathrm{L}\rightarrow\mathrm{H}}$.[15] The observation of hysteresis critically depends on the ratio between the life time of one state and the time of global parameter change, τ_global. If $\tau_\mathrm{global} >> \tau_\mathrm{life}$ holds, two states are well equilibrated via abundant transitions. The averaged observations do not depend on from where the global parameters have evolved. In the case of $\tau_\mathrm{global} \sim \tau_\mathrm{life}$, the observed results strongly depend on from which branch the parameters have evolved, i.e., from the L-mode or from the H-mode; The hysteresis in the response of X to the global parameters is observed.

In summary, the statistical theory of the E_r bifurcation in the edge of tokamaks was analyzed. Micro fluctuations induce a random noise for the transition to occur in a probabilistic manner, when a hysteresis is predicted in the deterministic model analysis. The PDF and ensemble average of E_r were obtained. The probability of L/H transition was obtained, and a life-time of each state was calculated. The phase boundary of two states was given by the equal-probability condition for the H-and L-states. Owing to the suppression effect on turbulent noise by the E_r shear, the boundary was found to deviate from the Maxwell's construction rule. Implications to experiments are as follows: First, the appearance of H-mode in plasma parameters must be judged by the ensemble averages of statistical models which have a noise source, not by a value of deterministic model. (See Fig.1(a).) Due to the noise, each transition occurs being scattered around the ensemble average. This must be noticed in the future comparison of experimental database with theories. Second, the ensemble average $\langle X \rangle$ does not show a hysteresis against global parameters X_{NC}, even though a deterministic model predicts the hysteresis. (Ensemble average of heat flux is also obtained. This is discussed in a full paper [19].) Third, the observation of hysteresis in experiments critically depends on the speed of global parameter change: this is another feature which characterizes the non-equilibrium properties. Fourth, the probabilistic onsets may change the occurrence of dithering between H-and L-states; In conjunction with it, coupled dynamics with global pressure gradient are discussed in [19]. In this article, the model of Eq.(3) was taken to show a typical example of probabilistic transition. Other mechanisms have been known to influence L-H transitions.[9] The inclusion of zonal flow excitation in statistical theory [14] or the coupling of dynamics of different scale lengths [20] must be investigated for quantitative analysis of tokamak plasmas, and are left for future studies.

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