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M. Stepić, L. Hadžievski and M.M. Škorić

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Modulation instability in two-dimensional nonlinear Schrödinger lattice models with dispersion and long-range interactions

Milutin Stepić, Ljupčo Hadžievski and Miloš M. Škorić*
Vinča Institute of Nuclear Sciences, P.O.B. 522, 11001 Belgrade, Serbia, Yugoslavia
**National Institute for Fusion Science, Toki, Gifu, 509-5292, Japan*
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The problem of modulation instability of continuous wave and array soliton solutions in the framework of a two-dimensional continuum-discrete nonlinear Schrödinger lattice model which accounts for dispersion and long-range interactions between elements, is investigated. Application of the linear stability analysis based on an energetic principle and a variational approach, which were originally developed for the continuum nonlinear Schrödinger model, is proposed. Analytical expressions for the corresponding instability thresholds and the growth rate spectra are calculated.

Keywords: *nonlinear dynamics, stability, continuum-discrete nonlinear Schrödinger equation*

I. INTRODUCTION

Mathematical models describing dynamical properties of the systems with interplay between nonlinearity, dispersion and discreteness attract a growing interest due to their rich applicability in different physical problems. There are many nonlinear physical systems which are both discrete and continuous, like nonlinear fiber arrays (NFA) [1-9], arrays of coupled Josephson's junctions [10], elastic energy transfer in anharmonic crystals [11], etc.. Such systems show a complex dynamical behavior exhibiting diverse physical properties like wave instabilities, solitonlike localized structures, quasi-collapse (blow-up solutions), pattern formation and spatiotemporal chaos. These phenomena were intensively studied mainly for one-dimensional (1D) continuum-discrete systems with short range interactions using a nearest-neighbor approximation [1, 4-9]. However, some physical systems can not be described in the framework of this approximation and the effect of long-range interactions between the lattice elements must be taken into account. Examples are DNA molecule chains with long-range Coulomb interactions, excitation transfer in molecular crystals and vibron energy transport in biopolymers with dipole-dipole interactions. Mathematical modelling of these systems often leads to one of the discrete or continuum-discrete variants of the universal nonlinear evolution equations like nonlinear Schrödinger (NLS), sine-Gordon, Korteweg-de Vries, Klein-Gordon and Kadomtsev-Petviashvili equations. The simplest and also most extensively studied are NLS models described by the continuum-discrete NLS (CDNLS)equation. In the physical situation where the dispersion along the lattice elements can be neglected, the CDNLS model reduces to the discrete NLS model where dynamical properties of the system are determined by an interplay between nonlinearity and discreteness. The effects of long-range dispersive interactions in 1D discrete nonlinear Schrödinger (NLS) system was investigated in [12, 13]. The long-range interaction model with a power law dependence on the distance between interacting elements was used in reference [12]. A modified interaction model in a form of Joncqière's function convenient to cover different physical situations from nearest-neighbor interactions to ultra long-range interactions was discussed in [13] . The dynamics of the discrete two-dimensional (2D) NLS system with long-range dipole-dipole interactions was studied in [14]. However, in many physical problems the dispersion along the lattice elements can not be neglected and CDNLS equation must be used as a mathematical model.

The goal of this work is to study an important problem of modulation instability of continuous wave (CW) and array soliton (AS) solutions in continuum-discrete nonlinear Schrödinger (CDNLS) model describing dynamics in two-dimensional lattice with dispersion and long-range interactions between elements. In Sec. II we define the basic evolution equation and give the continuous wave (CW) and array soliton solutions of the model. In Sec. III we describe a linear stability analysis based on an energetic principle and a variational approach which were originally developed for the continuum NLS models [15, 16]. We obtain analytical expressions for the instability thresholds and the growth rate spectra and finally, we summarize our results in Sec. IV.

II. THE MATHEMATICAL MODEL

The basic mathematical model describing two-dimensional lattice with interacting nonlinear elements in anomalous dispersion regime has a form of continuum-discrete nonlinear Schrödinger equation

$$i\frac{\partial\psi_{\vec{r}}}{\partial t}+\frac{\partial^2\psi_{\vec{r}}}{\partial z^2}+2\psi_{\vec{r}}|\psi_{\vec{r}}|^2+\sum_{\vec{r}'\ (\vec{r}'\neq\vec{r})}J_{|\vec{r}'-\vec{r}|}(\psi_{\vec{r}'}-\psi_{\vec{r}})=0, \tag{1}$$

where $\vec{r}=(n,m,0)$ ($n=0,\pm1,\pm2...N$; $m=0,\pm1,\pm2...M$) is the discrete lattice vector in a $x-y$ plane, z is the spatial continuous coordinate along the lattice elements and $\psi_{\vec{r}}=\psi_{n,m}$ is the wave function into the (n,m)-th lattice element. The nonlocal interaction term $J_{|\vec{r}'-\vec{r}|}$ describe a long-range isotropic coupling between lattice elements and depend on the distance between interacting elements. This interaction model is quite general and enables a mathematical modelling of a variety of discrete dispersive physical systems with long-range interactions. The well known interaction model for 1D discrete NLS model with a power law dependence on the distance between interacting elements was originally proposed in [12]. In our case, for 2D CDNLS model (1) for regularly spaced 2D lattice with interelement distance equal to 1, the power law dependence can be written in a form

$$J_{|\vec{r}'-\vec{r}|} = \frac{1}{|\vec{r}'-\vec{r}|^p}. \tag{2}$$

This interaction model can conveniently describe a wide class of different discrete dispersive physical systems with a long-range isotropic interactions like DNA molecule chains with a long-range Coulomb interaction ($p = 1$), propagation of optical pulses in nonlinear fiber arrays and excitation transfer in quasi two-dimensional molecule crystals ($p = 3$). On the other hand, for the sufficiently large p, the model Eq. (1) exhibits the same qualitative features as the CDNLS equation with nearest-neighbor interactions.

The CDNLS equation (1) has a Hamiltonian structure and can be written as

$$i\frac{\partial\psi_{\vec{r}}}{\partial t} = \frac{\delta H}{\delta\psi^*_{\vec{r}}}, \tag{3}$$

where H is the Hamiltonian defined by

$$H = \sum_{\vec{r}}\int_{-\infty}^{\infty}\Big(\sum_{\vec{r}'(\vec{r}'\neq\vec{r})} J_{\vec{r}'-\vec{r}}(\psi_{\vec{r}'}-\psi_{\vec{r}})\psi^*_{\vec{r}} + |\psi_{\vec{r},z}|^2 - |\psi_{\vec{r}}|^4\Big)dz. \tag{4}$$

The number of quanta P (L^2 norm) is another conserved quantity of the Eq. (1)

$$P = \sum_{\vec{r}}\int_{-\infty}^{\infty}|\psi_{\vec{r}}|^2 dz. \tag{5}$$

For the lattice with periodic boundary conditions imposed on the discrete dimensions $\vec{r}$ we can consider a set of lattice independent stationary solutions of Eq. (1) in a form

$$\psi_{\vec{r}} = f(z)\, e^{i\lambda^2 t}, \tag{6}$$

where λ is a real parameter. There are two such solutions, the first is a uniform, continuous wave (CW) solution $f_{cw} = \lambda/\sqrt{2}$, while the second one is an array soliton (SA) solution $f_{sa} = \lambda/\cosh(\lambda z)$. In both cases the parameter λ is related to the amplitude of the wave function.

III. STABILITY ANALYSIS

In order to study the stability property of the stationary solutions (6) we introduce small modulations, in a form of square integrable, perturbations

$$\psi_{\vec{r}}(x,t) = [f(z) + \delta f_{\vec{r}}(z,t)]e^{i\lambda^2 t}, \quad |\delta f_{\vec{r}}| \ll |f|. \tag{7}$$

Substituting Eq. (7) into Eq. (1) and linearizing with respect to small perturbations $\delta f_{\vec{r}}$ we arrive at the equation

$$i\frac{\partial\delta f_{\vec{r}}}{\partial t} + \frac{\partial^2\delta f_{\vec{r}}}{\partial z^2} - \lambda^2\delta f_{\vec{r}} + 4|f|^2\delta f_{\vec{r}} + 2|f|^2\delta f^*_{\vec{r}} + \sum_{\vec{r}'(\vec{r}'\neq\vec{r})} J_{\vec{r}'-\vec{r}}(\delta f_{\vec{r}'} - \delta f_{\vec{r}}) = 0. \tag{8}$$

For the perturbations with a dependence on the discrete dimensions $\vec{r}$, such as

$$\delta f_{\vec{r}}(z,t) = (a+ib)\cos(k_n n)\cos(k_m m), \tag{9}$$

where $k_n = 2\pi/(2N+1)$ and $k_m = 2\pi/(2M+1)$ are discrete wave numbers, the following eigenvalue problem is obtained

$$\begin{aligned}\frac{\partial b(z,t)}{\partial t} &= -\hat{L}_+ a(z,t),\\ \frac{\partial a(z,t)}{\partial t} &= \hat{L}_- b(z,t).\end{aligned} \tag{10}$$

The linear second-order differential operators $\hat{L}_\pm$ are defined by

$$\begin{aligned}\hat{L}_+ &= -\frac{\partial^2}{\partial z^2} + \lambda^2 - 6f^2(z) + 4\Sigma(N,M),\\ \hat{L}_- &= -\frac{\partial^2}{\partial z^2} + \lambda^2 - 2f^2(z) + 4\Sigma(N,M).\end{aligned} \tag{11}$$

The complete discrete properties of the system described by the operators $\hat{L}_\pm$ are taken into account through the term $\Sigma(N,M)$, defined by

$$\Sigma(N,M) = \sum_{n=1}^{N} J_{n,0}sin^2(\frac{k_n n}{2}) + \sum_{m=1}^{M} J_{0,m}sin^2(\frac{k_m m}{2}) - \sum_{(n,m)=1}^{(N,M)} J_{n,m}[\cos(k_n n)\cos(k_m m) - 1], \tag{12}$$

which depends on the lattice dimension and the interaction law between lattice elements.

A. Stability of the CW solution

For the case of CW solution ($f_{cw} = \lambda/\sqrt{2}$) the differential operators (11) are homogeneous and stability analysis is straightforward. The Fourier transform ($e^{-i\omega+ikx}$) of Eqs (10) gives the following dispersion relation

$$\omega^2 = (k^2 + 4\Sigma)(k^2 - 2\lambda^2 + 4\Sigma). \qquad (13)$$

The instability will occur for $\omega^2 < 0$, which leads to the following instability threshold

$$\lambda^2 > \frac{k^2}{2} + 2\Sigma(N, M). \qquad (14)$$

As seen from (14) the lowest threshold is for an excitation of small wave-number perturbations related to the modulational instability. Dispersion relation (13) and the instability threshold (14) according to the dimensionality of the lattice (one-dimensional or two-dimensional) and type of the interaction (nearest-neighbor or long-range interactions) is represented by four particular expressions:

a) One-dimensional lattice $\vec{r} = (n, 0, 0)$ with nearest-neighbor interactions

$$\omega^2 = (k^2 + 4\sin^2(\frac{k_n}{2}))(k^2 + 4\sin^2(\frac{k_n}{2}) - 2\lambda^2), \quad (15)$$

$$\lambda^2 > \frac{k^2}{2} + 2\sin^2(\frac{\pi}{2N+1}). \qquad (16)$$

Results (15)and (16) coincide with the corresponding ones obtained in [5].

b) One-dimensional lattice $\vec{r} = (n, 0, 0)$ with long-range interactions

The instability threshold in this case reads

$$\lambda^2 > \frac{k^2}{2} + 2\sum_{n=1}^{N} J_n \sin^2(\frac{\pi}{2N+1}n). \qquad (17)$$

For the interaction model with a power law dependence on the distance between the interacting elements (2) $J_n = \frac{1}{n^p}$ the instability threshold (17) reads

$$\lambda^2 > \frac{k^2}{2} + 2\sum_{n=1}^{N} \frac{\sin^2(\frac{\pi}{2N+1}n)}{n^p}. \qquad (18)$$

The instability threshold λ_c for long-range interactions as a function of the size of the 1D lattice (N) for different values of p is plotted in Fig. 1. The curve for $p = 5$ practically corresponds to the result (16) for the nearest-neighbor interactions model. Above results show that due to the increased inertia of the system the instability threshold for the long-range interactions is higher than the corresponding threshold for the nearest-neighbor interactions.

c) Two-dimensional lattice $\vec{r} = (n, m, 0)$ with nearest-neighbor interactions

The instability threshold for this case reads

$$\lambda^2 > \frac{k^2}{2} + 2[1 - \cos(\frac{k_n - k_m}{2})\cos(\frac{k_n + k_m}{2})]. \qquad (19)$$

For highly elongated 2D lattices $N >> M$, perturbations dominantly develop along the longer dimension of the lattice and the instability threshold (19) approaches values for the instability threshold for 1D lattices (16).

d) Two-dimensional lattice $\vec{r} = (n, m, 0)$ with long-range interactions

$$\lambda^2 > \frac{k^2}{2} + 2\{\sum_{n=1}^{N} J_{n,0}\sin^2(\frac{k_n n}{2}) + \sum_{m=1}^{M} J_{0,m}\sin^2(\frac{k_m m}{2}) - \sum_{(n,m)=1}^{(N,M)} J_{n,m}[\cos(k_n n)\cos(k_m m) - 1]\}. \quad (20)$$

For the interaction model with a power law dependence on the distance between the interacting elements (2) $J_{n,m} = \frac{1}{(n^2+m^2)^{p/2}}$ the instability threshold (20) reads

$$\lambda^2 > \frac{k^2}{2} + 2\{\sum_{n=1}^{N} \frac{\sin^2(\frac{k_n n}{2})}{n^p} + \sum_{m=1}^{M} \frac{\sin^2(\frac{k_m m}{2})}{m^p} - \sum_{(n,m)=1}^{(N,M)} \frac{[\cos(k_n n)\cos(k_m m) - 1]}{(n^2+m^2)^{\frac{p}{2}}}\}. \quad (21)$$

The instability threshold λ_c for long-range interactions as a function of the size of 2D lattice (N, M) for $p = 3$ is presented in Fig. 2. The value $p = 3$ corresponds to the case of isotropic dipole-dipole interactions discussed in [14] for two-dimensional DNLS model. The instability threshold decreases with the size of the lattice and has a minimum for the square lattice $N = M$. Increasing p leads to a decrease in λ_c and for large values of p approaches the result given by Eq. (19) for the nearest-neighbor interactions model.

B. Stability of Array Soliton solutions

Stability analysis of AS solutions is more complicated due to an explicit z dependence of the differential operators $\hat{L}_\pm$. However, the fact that the discrete properties of the system are incorporated into the operators $\hat{L}_\pm$ only via $\Sigma(N, M)$ term, enables a direct application of the mathematical methods developed for stability analysis of continuum models. In order to calculate the instability threshold and to find a detailed structure of the instability growth rate we use an energetic principle introduced by Laedke and Spatschek [15] and a variational method by Rypdal and Rasmussen [16] originally applied to a stability problem of the continuum NLS equation.

For further calculations it is convenient to substitute $\lambda z -> z$ and to express operators $\hat{L}_\pm$ in a form

$$\begin{aligned}\hat{L}_+ &= \lambda^2(\hat{S}_+ + \mu - 5),\\ \hat{L}_- &= \lambda^2(\hat{S}_- + \mu - 1),\end{aligned} \tag{22}$$

where μ is the parameter containing information about the discreteness of the system, defined by

$$\mu = \frac{4\Sigma}{\lambda^2}, \tag{23}$$

and $\hat{S}_\pm$ are Sturm-Liouville- type operators

$$\begin{aligned}\hat{S}_+ &= -\frac{\partial^2}{\partial z^2} + 6\tanh^2(z),\\ \hat{S}_- &= -\frac{\partial^2}{\partial z^2} + 2\tanh^2(z).\end{aligned} \tag{24}$$

These operators possess a well-known spectra [17]. The smallest eigenvalues $\sigma_\pm^{(0)}$ and corresponding eigenfunctions $\psi_\pm^{(0)}$ in the discrete part of the spectrum are

$$\begin{aligned}\sigma_-^{(0)} &= 1; \quad \psi_-^{(0)} = 1/\cosh(z)\ ,\\ \sigma_+^{(0)} &= 2; \quad \psi_+^{(0)} = 1/\cosh^2(z)\ .\end{aligned} \tag{25}$$

The procedure of the energetic principle described in [16] shows that the system is unstable for $0 < \mu < 3$, when the operator $\hat{L}_+$ is indefinite, and stable for $\mu > 3$, when the operator $\hat{L}_+$ is positive definite. These results lead to the next instability condition

$$\lambda > \lambda_c = \frac{2\sqrt{\Sigma(N,M)}}{\sqrt{3}}. \tag{26}$$

If we compare the above instability threshold for AS solutions with the instability threshold for CW solutions given by Eq. (14) it is obvious that for $k = 0$ the difference comes only within a numerical factor $\sqrt{2/3} \approx 0.8165$. It means that all corresponding particular results for the instability thresholds of the AS solutions can be derived from the expressions given by Eqs. (14–21) for the instability thresholds of the CW solutions, by taking $k = 0$ and just multiplying by 0.8165. It also means that the shapes of the curves displayed in figures 1 and 2 are the same as in the case of AS solutions. For 1D lattice with nearest-neighbor interactions, Eq.(26) readily recovers earlier results, obtained in [5,9].

The application of the energetic principle to the stability problem of AS solutions proves the existence of exponentially growing modes and gives threshold values (26) without any further detail. In order to find out more about the growth rate structure of the instability we apply a variational approach used in [16] for the continuum NLS equation and also in [9] for 1D CDNLS equation with nearest-neighbor interactions model. For the normal exponentially growing modes $a(t,z) = a(z)\exp(\gamma t)$; $b(t,z) = b(z)\exp(\gamma t)$ with the growth rate γ, the eigenvalue Eqs. (10) are transformed into

$$\begin{aligned}\hat{L}_+ a(z) &= -\Gamma b(z),\\ \hat{L}_- b(z) &= \Gamma a(z),\end{aligned} \tag{27}$$

where $\Gamma = \gamma/\lambda^2$ is the normalized growth rate. The above equations can be derived from the variation of the action

$$\delta S = \delta \int_{-\infty}^{\infty} \mathcal{L}(a, a_z, b, b_z, z)dz, \tag{28}$$

where the Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{1}{2}(a_z^2 + b_z^2) + [\frac{\mu+1}{2} - \frac{3}{\cosh^2(z)}]a^2 + [-\frac{\mu+1}{2} + \frac{1}{\cosh^2(z)}]b^2 + \Gamma ab. \tag{29}$$

The basic idea of the variational approach is to define a set of test functions $\tilde{a}(z)$ and $\tilde{b}(z)$ with some variational parameters and to calculate the action integral S. It is obvious that with this approach, obtained results will critically depend on our choice of the test functions. It was shown and also numerically confirmed in [9,16], that a good choice for the test functions are eigenfunctions of the marginally stable states, for ($\Gamma = 0$) in Eqs. (27).

$$\begin{aligned}a(z) &= 0\ ,\ b(z) = \frac{1}{\cosh(z)}\ , \quad \mu = 0\\ a(z) &= \frac{1}{\cosh^2(z)}\ ,\ b(z) = 0\ , \quad \mu = 3.\end{aligned} \tag{30}$$

Assuming test functions with two variational parameters α and β in a form

$$\tilde{a}(z) = \frac{\alpha}{\cosh^2(z)}\ , \quad \tilde{b}(z) = \frac{\beta}{\cosh(z)}\ . \tag{31}$$

we calculate the action integral

$$S = 2\alpha^2(\frac{\mu}{3} - 1) - \mu\beta^2 + \frac{\pi}{2}\Gamma\alpha\beta. \tag{32}$$

The following expression for the growth rate structure

$$\Gamma^2(\mu) = \frac{32}{\pi^2}\mu(1-\frac{\mu}{3}), \qquad (33)$$

is obtained from the conditions $\frac{\partial S}{\partial \alpha} = \frac{\partial S}{\partial \beta} = 0$.

The instability threshold λ_c corresponds to the marginally stable mode $\Gamma = 0$ of the dispersion relation (33). The expression for the instability threshold calculated from Eq. (33) coincides with the expression (26) obtained with an application of the energetic principle.

The dispersion relation (33) has the same structure as results given in [16] for the continuum NLS equation and in [9] for 1D CDNLS equation with nearest-neighbor interactions model, because complete discrete properties of the system are incorporated only via the parameter μ defined by Eq. (23). Replacing the particular expressions for μ into Eq. (33) for lattices with different dimensionality (one-dimensional or two-dimensional) and type of the interactions (nearest-neighbor or long-range interactions) we can readily obtain explicit formulae for the corresponding growth rate structure.

a) One-dimensional lattice $\vec{r} = (n, 0, 0)$ with nearest-neighbor interactions

$$\Gamma = \frac{8\sqrt{2}}{\lambda^2\pi\sqrt{3}}\sin(\frac{\pi}{2N+1})\sqrt{3\lambda^2 - 4\sin^2(\frac{\pi}{2N+1})}\,. \qquad (34)$$

b) One-dimensional lattice $\vec{r} = (n, 0, 0)$ with long-range interactions

$$\Gamma = \frac{8\sqrt{2}}{\lambda^2\pi\sqrt{3}}\sqrt{\sum_{n=1}^{N} J_n \sin^2(\frac{\pi}{2N+1}n)}\sqrt{3\lambda^2 - 4\sum_{n=1}^{N} J_n \sin^2(\frac{\pi}{2N+1}n)}. \qquad (35)$$

For the interaction model with a power law dependence on the distance between the interacting elements (2) $J_n = \frac{1}{n^p}$ the growth rate (35) reads

$$\Gamma = \frac{8\sqrt{2}}{\lambda^2\pi\sqrt{3}}\sqrt{\sum_{i=1}^{N}\frac{\sin^2(\frac{\pi}{2N+1}n)}{n^p}}\sqrt{3\lambda^2 - 4\sum_{n=1}^{N}\frac{\sin^2(\frac{\pi}{2N+1}n)}{n^p}}. \qquad (36)$$

The figures 3a–3b show the dependence of the growth rate Γ on the soliton amplitude, for three different 1D lattices a) $N = 2$, b) $N = 8$ and c) $N = 20$. The curves for large p practically correspond to the results for the nearest-neighbor interactions model [9]. The growth rate is less sensitive on the variation of p for the lattices with lower number of elements and for $N = 1$ (one-dimensional lattice with 3 elements) all curves degenerate into a single one which corresponds to the growth rate for the nearest-neighbor interactions model.

c) Two-dimensional lattice $\vec{r} = (n, m, 0)$ with nearest-neighbor interactions

$$\Gamma = \frac{8\sqrt{2}}{\lambda^2\pi\sqrt{3}}\sqrt{1 - \cos(\pi\frac{k_n - k_m}{2})\cos(\pi\frac{k_n + k_m}{2})}$$
$$\sqrt{3\lambda^2 - 4 + 4\cos(\pi\frac{k_m - k_n}{2})\cos(\pi\frac{k_n + k_m}{2})}]. \qquad (37)$$

d) Two-dimensional lattice $\vec{r} = (n, m, 0)$ with long-range interactions

$$\Gamma = \frac{8\sqrt{2}}{\lambda^2\pi\sqrt{3}}\sqrt{\sum_{n=1}^{N} J_{n,0} sin^2(k_n n/2) + \sum_{m=1}^{M} J_{0,m} sin^2(k_m m/2) - \sum_{(n,m)=1}^{(N,M)} J_{n,m}[\cos(k_n n)\cos(k_m m) - 1]}$$
$$\sqrt{3\lambda^2 - 4\{\sum_{n=1}^{N} J_{n,0} sin^2(k_n n/2) + \sum_{m=1}^{M} J_{0,m} sin^2(k_m m/2) - \sum_{(n,m)=1}^{(N,M)} J_{n,m}[\cos(k_n n)\cos(k_m m) - 1]\}}. \qquad (38)$$

For the interaction model with a power law dependence on the distance (2) $J_{n,m} = \frac{1}{(n^2+m^2)^{p/2}}$ the growth rate (38) reads

$$\Gamma = \frac{8\sqrt{2}}{\lambda^2\pi\sqrt{3}}\sqrt{\sum_{n=1}^{N}\frac{sin^2(\frac{k_n n}{2})}{n^p} + \sum_{m=1}^{M}\frac{sin^2(\frac{k_m m}{2})}{m^p} - \sum_{(n,m)=1}^{(N,M)}\frac{\cos(k_n n)\cos(k_m m) - 1}{(n^2+m^2)^{\frac{p}{2}}}}$$

$$\sqrt{3\lambda^2 - 4[\sum_{n=1}^{N} \frac{sin^2(\frac{k_n n}{2})}{n^p} + \sum_{m=1}^{M} \frac{sin^2(\frac{k_m j}{2})}{m^p} - \sum_{(n,m)=1}^{(N,M)} \frac{\cos(k_n n)\cos(k_m m) - 1}{(n^2+m^2)^{\frac{p}{2}}}]}. \quad (39)$$

In Fig. 4a and 4b we present the growth rate Γ for the instability of the array soliton with the amplitude $\lambda = 0.5$ as a function of the size of the two-dimensional lattice (N, M) with long-range interactions for $p = 3$. Fig 4a represents the surface $\Gamma(N, M)$, while Fig. 4b is the corresponding gray scale map of the projection on the N–M plane. The black area in Fig. 4b is the region below the instability threshold.

IV. CONCLUSION

In this work, we have analytically studied detailed stability properties of the continuous wave and array soliton solutions in two-dimensional lattices with dispersion and long-range interactions, described by the CDNLS equation. The linear stability of the array soliton is solved by applying the energetic principle and the variational method which were originally developed for the continuum NLS equation [15,16]. We have obtained the instability thresholds and first results for the growth rate spectra that are valid for the two-dimensional NLS lattice with a long-range isotropic coupling between lattice elements. Explicit expressions for the long-range isotropic interactions with a power law dependence on the distance between interacting elements are also calculated. Our results for highly elongated lattices and large p, recover formulae for the one-dimensional lattice with nearest-neighbor interactions, obtained in earlier papers [5,6,9].

The results presented in this study are based on the linear stability analysis and indicate a presence of exponentially growing modes in the system giving no predictions on the subsequent nonlinear evolution stage. Based on the results for 1D and 2D CDNLS systems with nearest-neighbor interactions [4,5] and for 2D discrete NLS models (without dispersion) with long-range interactions [14], it is plausible to expect an existence of the quasi-collapse process and solitary structures localized in both continuum and discrete dimensions. A detail study of these problems in the framework of 2D CDNLS lattice model with long-range interactions as well as the problem of stability of the multi-dimensional continuum-discrete solitary waves will be given in a separate publication.

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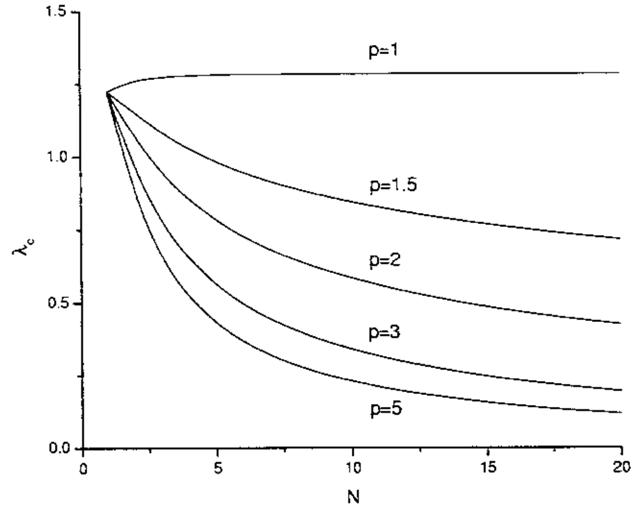

FIG. 1: Dependence of the instability threshold λ_c on the size of the one-dimensional lattice N with long-range interactions for different values of p.

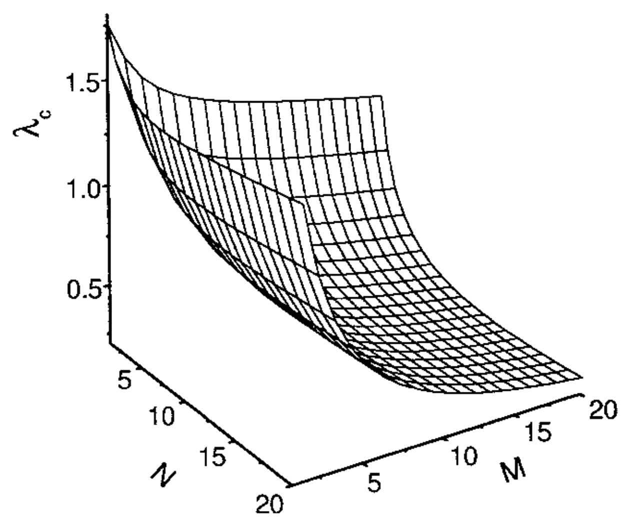

FIG. 2: The instability threshold λ_c as a function of the size of the two-dimensional lattice (N, M) with long-range interactions for $p = 3$.

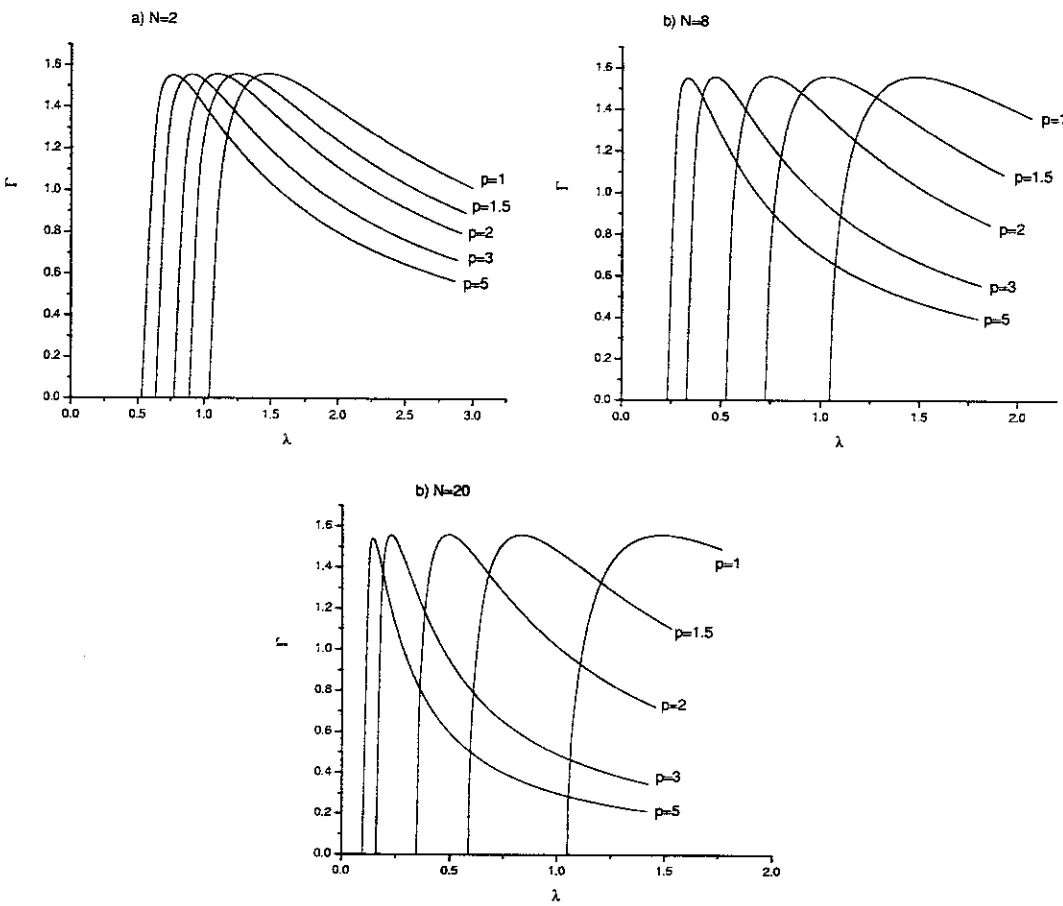

FIG. 3: Dependence of the growth rates Γ on the soliton amplitude λ for three different one-dimensional lattices with long-range interactions a) $N = 3$, b) $N = 8$ and c) $N = 20$.

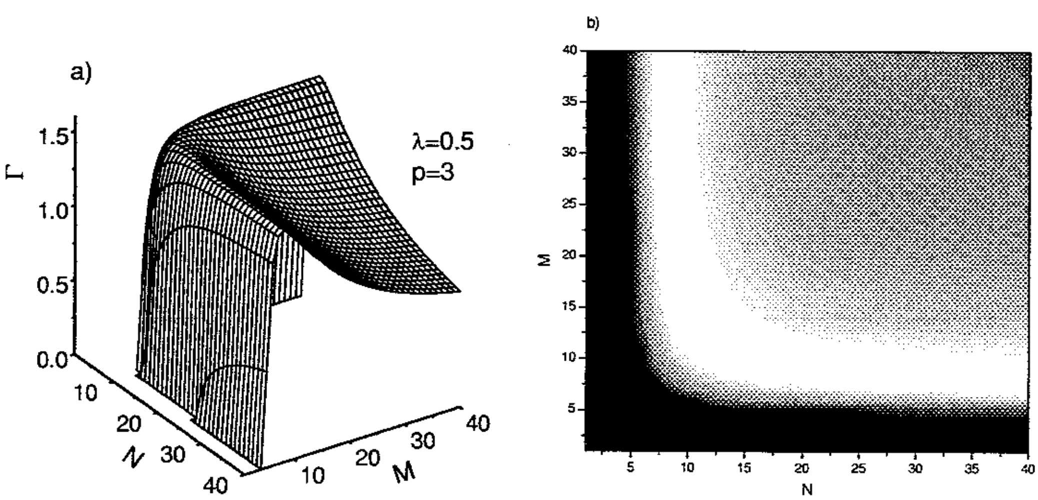

FIG. 4: Growth rates Γ of the instability of the array soliton solution with amplitude $\lambda = 0.5$ as a function of the size of the two-dimensional lattice (N, M) with long-range interactions for $p = 3$; (4a) surface $\Gamma(N, M)$; (4b) corresponding gray scale map of the projection on the $N - M$ plane.