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On transition in plasma turbulence with multiple scale lengths

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Abstract
A statistical theory of plasma turbulence which is composed of multiple-scale fluctuations is examined. Influences of statistical noise and variance of rapidly-changing variable in an adiabatic approximation are investigated. It is confirmed that the contributions of noise and variance remain higher order corrections. Transition rate of the turbulence with multiple scale lengths is obtained under the refined adiabatic approximation.

Keywords: plasma turbulence, transition, multiple scale lengths, stochastic equation, adiabatic approximation

I. Introduction

Recently, statistical theory for turbulent plasmas has shown progress. (See, e.g., [1,2] for a review.) In particular, the transition in turbulent states has attracted attentions. This is because the finding of the H-mode [3] has motivated the study of transition in confined plasmas [4-6]. (Detailed reviews are found in [1, 7, 8].) A statistical theory of plasma turbulence has been explored, in which the statistical nature of subcritical excitation has been investigated [9]. The turbulent noise induces the stochastic process to the transition, therefore the analysis from the probabilistic view is needed in addition to that from the deterministic view. A stochastic equation has been formulated as a Langevin equation, and statistical analyses [9] together with its solution have been investigated [10]. The probability density function (PDF) has been also found from the Fokker-Planck equation, which is a counter-part of the Langevin equation. The ensemble averages, transition probability and so on have been obtained. An extension of the analyses to the case where plasma turbulence is composed of multiple fluctuations with different scale lengths [11, 12] was made. (For instance, the fluctuations in the range of ion gyroradius ρ_i and those in the range of collisionless skin depth $\delta = c/\omega_p$ can be simultaneously excited in plasmas. These two kind of fluctuations with different scale lengths have nonlinear mutual interactions as is explained in [11].) In this case, statistical characteristics were examined by use of the adiabatic approximation. The time scale hierarchy was employed, and the rapidly varying variable was first solved by treating the slowly-varying element being a constant [13]. Then the most-probable value of the rapidly-varying element is substituted into the equation of the slowly-changing variable. In this procedure, the stochastic fluctuations of the rapidly-changing variable are neglected in calculating the distribution of the slowly-changing variable.

In this paper, the influence of the stochastic fluctuations of rapidly-changing variable is investigated. This contribution is found to be a higher order correction with respect to the ordering parameter which is a basis for the time-scale hierarchy. The statistical results, such as the probability density function of a stationary state or the transition probability between different states, are obtained for turbulent fluctuations which have two components with different scales.

II. Basic equations and assumptions

In order to examine the effect of the stochastic noise and the variance of rapidly-changing variable on the developments of the slowly-varying variables under the adiabatic approximation, the smallness parameter ε is introduced according to the previous analyses.

In the preceding articles, the turbulence which is composed of two kinds of fluctuations has been discussed. We considered the case that the fluctuations are composed of the semi-micro modes (such as ion temperature gradient mode, being denoted by the suffix l) and micro modes (such as the current-diffusive ballooning mode, being denoted by the suffix h). Stochastic equations for the fluctuation amplitude $\langle \phi_l^* \phi_l \rangle$ and $\langle \phi_h^* \phi_h \rangle$ have been discussed based on the hierarchical ordering

$$\varepsilon << 1 , \qquad \varepsilon \equiv k_l^2 / k_h^2 , \tag{1}$$

where ϕ is the fluctuating part of electrostatic potential, and k_l and k_h are typical values of wavenumber for semi-micro modes and micro modes, respectively. Variables are introduced as $X = k_l^2 D_l^{-2} \langle \phi_l^* \phi_l \rangle$ and $Y = k_h^2 D_h^{-2} \langle \phi_h^* \phi_h \rangle$. See [9, 12] for normalization of variables. This normalization is introduced such that $X = 1$ or $Y = 1$ is obtained as a nonlinear stationary state when each mode is analyzed independently [13]. (Here, D_l and D_h are the renormalized transport coefficients by the semi-micro fluctuations and micro fluctuations, respectively, in the case where they are independently driven without nonlinear interactions between them.) A set of stochastic equations for X and Y was derived as

$$\frac{\partial}{\partial t} X + \Lambda_X X = g_X w(t) , \tag{2a}$$

$$\frac{\partial}{\partial t} Y + \Lambda_Y Y = g_Y w(t) , \tag{2b}$$

where $w(t)$ represents a Gaussian white noise term. The damping terms Λ_X and Λ_Y satisfy the relation

$$\Lambda_X \sim O(\gamma_l) \quad \text{and} \quad \Lambda_Y \sim O(\gamma_h) .$$

Here $\gamma_l = D_l k_l^2$ and $\gamma_h = D_h k_h^2$ are decorrelation rate of semi-micro modes and micro modes, respectively. In usual circumstances, D_l and D_h are of the same order of magnitude, and the relation

$$\gamma_l / \gamma_h \sim O(\varepsilon) \tag{3a}$$

or

$$\gamma_l << \gamma_h \tag{3b}$$

is satisfied with the same hierarchical ordering, $\varepsilon << 1$. This leads to the time scale hierarchy

$$\left|\Lambda_X\right| << \left|\Lambda_Y\right| . \tag{4}$$

Explicit forms of Λ_X , Λ_Y , g_X and g_Y are given in [11, 12, 13]. For instance, an analytic estimate for g_X and g_Y has been given in the limit of strong turbulence as

$$g_X^2 = \gamma_l \left(C_l\, X + C_{hl} \frac{D_h^3}{D_l^3} \frac{k_l^4}{k_h^4}\, Y \right) X \tag{5a}$$

$$g_Y^2 = \gamma_h\, C_h\, Y^2 . \tag{5b}$$

In these expressions, coefficients C_l , C_{hl} and C_h stand for numerical constants of the order of unity. Under the normalization in this article, X and Y remain to be of the order unity. The second term in RHS of Eq.(5a) is a higher order correction of the order of ε^2 . (In the analysis of [13], an estimate of $C_l = C_h = 1$ is employed, and the term of the order of ε^2 is not kept.)

In Eq.(2), the terms Λ_X and Λ_Y denote the driving or damping rate for the evolution of X and Y . In a deterministic model, in which noise terms g_X and g_Y are ignored, the flow vector $\left(\partial X/\partial t,\, \partial Y/\partial t\right)$ is given by $\left(\Lambda_X X,\, \Lambda_Y Y\right)$. The solution of equations

$$\Lambda_X X = 0 \tag{6a}$$

and

$$\Lambda_Y Y = 0 \tag{6b}$$

denotes the stationary state. Equation (6) can have multiple solutions as has been analyzed in [9-13]. Figure 1 illustrates the schematic drawing of conditions $\Lambda_X X = 0$ (dashed line) and $\Lambda_Y Y = 0$ (solid line). In this case, two solutions (named "A" and "B") are stable, and the intermediate one ("C") is unstable. The partition between states "A" and "B", and the transition rate between them are analyzed in the presence of the noise terms.

III. Statistical results

This set of stochastic equation has been studied by use of an adiabatic approximation. In this process, Eq.(2b) is solved, treating X as a parameter which is constant in time. The statistical solution for rapidly-varying element Y is obtained in terms of the probability density function (PDF) for the equilibrium state as a function of X as [9]

$$P_{\mathrm{st}}(Y) \propto g_{Y}^{-1} \exp\left(-\int_{0}^{Y} 2\, g_{Y}^{-2}(Y')\, \Lambda_{Y}(Y') Y'\, \mathrm{d}Y' \right) . \tag{7}$$

In this integral, X is fixed constant. The statistical average $\langle Y \rangle$ and the statistical variance ΔY are given from the PDF. The most probable value $Y_{*}(X)$ under fixed value of X is given the peak of PDF, and is given by the condition

$$\Lambda_{Y} Y + g_{Y} \partial g_{Y} / \partial Y = 0 . \tag{8}$$

Equation (8) is modified in comparison with Eq.(6), and the second term appear as an influence of the width of the PDF.

In the first approximation for the slowly-varying function, the rapidly-varying variable Y in Eq.(2a) was replaced by the expected value of Y under fixed value of X. We approximate the expected value of Y by $Y_{*}(X)$. Then Eq. (2a) was reduced to the following equation in the slow time-scale as

$$\frac{\partial}{\partial t} X + \Lambda_{X}(X, Y_{*}) X = g_{X}(X, Y_{*})\, w(t) . \tag{9}$$

Equation (9) was solved in [13], and the transition between different turbulent states was studied.

The influences of the statistical noise g_{Y} and the variance ΔY on the response of X are investigated here. We write

$$Y = Y_{*} + \tilde{y} \tag{10}$$

where $\tilde{y}$ is a stochastic variable which changes in the time scale of Λ_{Y}. One has

$$\frac{\partial}{\partial t} \tilde{y} + \nu\, \tilde{y} = g_{Y*}\, w(t) , \tag{11}$$

where

$$\nu = \frac{\partial}{\partial Y} \left(Y \Lambda_{Y} \right) \text{ at } Y = Y_{*} \text{ , and } g_{Y*} = g_{Y}(X, Y_{*}) . \tag{12}$$

A simple estimate of the variance is given as $\langle \tilde{y}^{2} \rangle = g_{Y*}^{2} / 2\nu$, i. e.,

$$\langle \tilde{y}^2 \rangle = \frac{g_{Y*}^2}{2}\left(\frac{\partial (Y\,\Lambda_Y)}{\partial Y}\right)^{-1} . \tag{13}$$

Substitution of Eq.(10) into Λ_X and g_X^2 give

$$\Lambda_X = \Lambda_X(X,\,Y_*) + \left(\frac{\partial \Lambda_X}{\partial Y}\right)\tilde{y} + \frac{1}{2}\left(\frac{\partial^2 \Lambda_X}{\partial Y^2}\right)\tilde{y}^2 + \cdots \tag{14a}$$

and

$$g_X = g_X(X,\,Y_*) + \frac{\partial}{\partial Y}\,g_X\,\tilde{y} + \frac{1}{2}\,\frac{\partial^2}{\partial Y^2}\,g_X\,\tilde{y}^2 + \cdots . \tag{14b}$$

Since $w(t)$ in Eq.(2a) and that in Eq.(2b) are statistically independent, so that the second term in RHS of Eq.(14b) vanishes after averaging over $\tilde{y}$. The noise term $g_X w(t)$ in Eq.(2a) is approximated as

$$g_X w(t) = \left\{ g_X(X,\,Y_*) + \frac{1}{2}\,\frac{\partial^2 g_X}{\partial Y^2}\,\langle \tilde{y}^2 \rangle \right\} w(t) + \cdots . \tag{15}$$

In the slow time scale, $\tilde{y}^2$ in the third term of RHS of Eq.(14a) is replaced by the average $\langle \tilde{y}^2 \rangle$. Substituting Eqs.(14a) and (15) into Eq.(2a), the stochastic equation for the slowly-varying part is modified as

$$\frac{\partial}{\partial t}X + \left\{ \Lambda_X(X,\,Y_*) + \frac{1}{2}\left(\frac{\partial^2 \Lambda_X}{\partial Y^2}\right)\langle \tilde{y}^2 \rangle \right\} X = \left\{ g_X(X,\,Y_*) + \frac{1}{2}\,\frac{\partial^2 g_X}{\partial Y^2}\,\langle \tilde{y}^2 \rangle \right\} w(t) - \left(\frac{\partial \Lambda_X}{\partial Y}\right) X\,\tilde{y} + \cdots . \tag{16}$$

The second term in the RHS of Eq.(14a) is placed at the end of the RHS of Eq.(16). As is discussed in the appendix, the last term in the RHS of Eq.(16) is modelled by an independent Gaussian white noise term. With this procedure, Eq.(16) is rewritten as

$$\frac{\partial}{\partial t}X + \hat{\Lambda}_X\,X = \hat{g}_X\,w(t) \tag{17}$$

with

$$\hat{\Lambda}_X = \Lambda_X(X, Y_*) + \frac{1}{2}\left(\frac{\partial^2 \Lambda_X}{\partial Y^2}\right)\langle \tilde{y}^2 \rangle . \tag{18}$$

and the effective noise amplitude as

$$\hat{g}_X^2 = \left\{ g_X(X, Y_*) + \frac{1}{2}\frac{\partial^2 g_X}{\partial Y^2}\langle \tilde{y}^2 \rangle \right\}^2 + \left(\frac{\partial \Lambda_X}{\partial Y}\right)^2 \left(\frac{\partial (Y\Lambda_Y)}{\partial Y}\right)^{-2} X^2 g_{Y*}^2 . \tag{19}$$

This expression is derived in the appendix.

Substituting Eq.(13) into Eq.(18), one has the renormalized driving/damping rate as

$$\hat{\Lambda}_X = \Lambda_X(X, Y_*) + \frac{1}{4}\left(\frac{\partial^2 \Lambda_X}{\partial Y^2}\right)\left(\frac{\partial (Y\Lambda_Y)}{\partial Y}\right)^{-1} g_{Y*}^2 . \tag{20}$$

Up to the first order correction of $g_Y^2{}_*$, the renormalized noise $\hat{g}_X^2$ of Eq.(19) is rewritten as

$$\hat{g}_X^2 = g_X(X, Y_*)^2 + (A_1 + A_2)\, g_{Y*}^2 , \tag{21a}$$

where

$$A_1 = \frac{1}{2} g_X(X, Y_*)\frac{\partial^2 g_X}{\partial Y^2}\left(\frac{\partial (Y\Lambda_Y)}{\partial Y}\right)^{-1} \tag{21b}$$

$$A_2 = \left(\frac{\partial \Lambda_X}{\partial Y}\right)^2 \left(\frac{\partial (Y\Lambda_Y)}{\partial Y}\right)^{-2} X^2 . \tag{21c}$$

Equations (17), (20) and (21) form the evolution equation of X with the correction of the statistical variance $\tilde{y}$. Here the hierarchy between X and Y is employed according to the time scales of Λ_X and Λ_Y.

Equation (17) is solved, and the PDF of the variable X is derived as

$$P_{st}(X) \propto \hat{g}_X^{-1} \exp\left(- \int_0^X 2\, \hat{g}_X^{-2}\, \hat{\Lambda}_X X \, dX \right) . \tag{22}$$

(Procedure to obtain the stationary solution of PDF from Langevin equation is explained in [9].) This integral is taken along the path which satisfies Eq. (8). If one writes the correction by $\tilde{y}$ explicitly, one obtains

$$\hat{P}_{\rm st}(X) \propto \hat{g}_X^{-1}\exp\left(-\int_0^X \frac{2\Lambda_X(X,Y_*) + \frac{1}{2}\left(\frac{\partial^2 \Lambda_X}{\partial Y^2}\right)\left(\frac{\partial(Y\Lambda_Y)}{\partial Y}\right)^{-1} g_{Y*}^2}{g_X(X,Y_*)^2 + (A_1 + A_2)\, g_{Y*}^2} X\, dX \right) \tag{23}$$

The adiabatic approximation which neglects $\tilde{y}$, Eq.(9), gives

$$P_{\rm st}(X) \propto g_X^{-1}\exp\left(-\int_0^X \frac{2\Lambda_X(X,Y_*)}{g_X(X,Y_*)^2} X\, dX \right). \tag{24}$$

The terms in Eq.(23) which include g_{Y*}^2 are found as a new correction in this article. Comparing Eqs.(23) and (24), one finds that the statistical variance $\tilde{y}$ provides a higher order correction with respect to the expansion parameter ε. In the numerator, the first order correction $O(\varepsilon)$ appears. In the denominator of Eq.(23), the second order corrections with respect to ε are included. The A_1 term in the denominator of Eq.(23) is $O(\varepsilon^2)$, or higher, because $\partial^2 g_X/\partial Y^2$ has a coefficient of the order of ε^2. (In the case of Eq.(5a), it vanishes.) The A_2 term is $O(\varepsilon^2)$. The simplest result Eq.(24) is valid as the lowest order estimate. We note that the new correction terms in the denominator can have an influence on the tail of PDF. The term $(\partial\Lambda_X/\partial Y)^2 X^2$ can have a higher order dependence on X in comparison with g_X^2. If it is so, the tail could be chopped-off owing to this correction term.

The most probable state for X is given by the peak of the PDF, $\hat{P}_{\rm st}(X)$. It is given by the equation

$$\hat{\Lambda}_X X + \hat{g}_X \partial \hat{g}_X/\partial X = 0 \,. \tag{25}$$

As is the case of Eq.(8), Eq.(25) includes the correction by the noise term in comparison with Eq.(6b).

Based on these results one can calculate the transition probability in the turbulence with multiple scale lengths. Let us consider the case that there are at least three solutions that satisfy Eqs.(25) and (8). Figure 2 illustrates the case that Eqs.(25) and (8) have three solutions. Two of them (being labeled "A" and "B") are stable, and the intermediate one is unstable. PDF has two peaks: one is around "A" and the other at "B".

The dominant one, "A" or "B", and the transition rate between these two are calculated by use of the PDF. A nonlinear potential is introduced as

$$S(X) = \int_{X_A}^{X} 2\, \hat{g}_X^{-2}\, \hat{\Lambda}_X X \, \mathrm{d}X \tag{26}$$

where the integral is taken along the path Eq.(8) (i. e., the solid line in Fig.2). Using this potential, the PDF is rewritten as $P_{\mathrm{st}}(X) \propto \hat{g}_X^{-1} \exp\left(-S(X)\right)$, and the transition probability is deduced as in [9]. The transition rate from the state "A" to "B" is given by

$$r_{\mathrm{A} \to \mathrm{B}} = \frac{\sqrt{\Lambda_{\mathrm{A}} \Lambda_{\mathrm{C}}}}{2\pi} \exp\left(-S(X_{\mathrm{C}})\right), \tag{27}$$

and that from "B" to "A" as

$$r_{\mathrm{B} \to \mathrm{A}} = \frac{\sqrt{\Lambda_{\mathrm{B}} \Lambda_{\mathrm{C}}}}{2\pi} \exp\left(S(X_{\mathrm{B}}) - S(X_{\mathrm{C}})\right), \tag{28}$$

respectively. (Note that $S(X_{\mathrm{A}}) = 0$ holds by definition.) In Eqs.(27) and (28), the time rates $\Lambda_{\mathrm{A,B,C}}$ are given as $\Lambda_{\mathrm{A,B,C}} = 2X \left| \partial \hat{\Lambda}_X / \partial X \right|$ at $X = X_{\mathrm{A}}$, $X = X_{\mathrm{B}}$ and $X = X_{\mathrm{C}}$. (This partial derivative is taken along the path Eq.(8).) The dominant dependence of the transition rates comes from the exponential parts in Eq.(28). Equations (27) and (28) are the extension of the result in [9] to the cases for the turbulence with multiple scale lengths.

The probability that the state is found in the "A"-state is given as

$$P_{\mathrm{A}} = r_{\mathrm{B} \to \mathrm{A}} / \left(r_{\mathrm{A} \to \mathrm{B}} + r_{\mathrm{B} \to \mathrm{A}}\right). \tag{29a}$$

That for the "B"-state is

$$P_{\mathrm{B}} = r_{\mathrm{A} \to \mathrm{B}} / \left(r_{\mathrm{A} \to \mathrm{B}} + r_{\mathrm{B} \to \mathrm{A}}\right). \tag{29b}$$

The state "A" is dominant if $P_A > 1/2$, and "B" is dominant if $P_B > 1/2$, respectively. The condition $P_A = P_B$ is rewritten as $r_{\mathrm{L} \to \mathrm{H}} = r_{\mathrm{H} \to \mathrm{L}}$. From Eqs.(27) and (28), this condition is given as

$$S(X_B) = \frac{1}{2} \ln \left(\Lambda_A / \Lambda_B\right) . \tag{30}$$

Apart from a weak logarithmic term, it is approximated as $S(X_B) = 0$. When this condition is satisfied, the two states appear with the same probability. This condition dictates the phase limit in the parameter space.

VI. Summary

In this article, the stochastic equations were analyzed for the turbulence which is composed of two kinds of modes with different scale lengths. The time scale hierarchy was employed, and the statistical properties were examined. The influence of the stochastic variance of rapidly-changing variable was investigated in using the adiabatic approximation. It is confirmed that this contribution remains to be a higher order correction with respect to the ordering parameter which is a basis for the time-scale hierarchy. These results show that the adiabatic approximation is valid for the study of multiple-scale turbulence in plasmas. The statistical results, such as the probability density function of a stationary state, the transition probability between different states and the selection rule of a branch, etc., are obtained for turbulent fluctuations which have two components with different scales. They are the generalization of the cases in previous analysis [9] where the turbulence is characterized by one scale length.

Appendix: Response to rapidly-changing stochastic variable

The rapidly varying element is expressed as $Y = Y_* + \tilde{y}$, where Y_* is determined by an average of adiabatic approximation. The stochastic part $\tilde{y}$ may obey the stochastic equation

$$\frac{\partial}{\partial t} \tilde{y} + \nu \tilde{y} = g_{Y*} w(t) . \tag{A1}$$

Here ν and g_{Y*} are determined by use of Y_*, e.g., $\nu = \partial(Y \Lambda_Y)/\partial Y$ at $Y = Y_*$. The fluctuating part $\tilde{y}$ is given from Eq.(A1) as

$$\tilde{y} = g_{Y*} e^{-\nu t} \int_0^t \mathrm{d}s \, e^{\nu s} w(s) \tag{A2}$$

yielding $\langle \tilde{y}^2 \rangle = g_{Y*}^2 / 2\nu$.

We study the response of slowly-changing variable against the perturbation with $\tilde{y}$. The equation

$$\frac{\partial}{\partial t} X + \Lambda X = \alpha \tilde{y} \tag{A3}$$

is studied. The condition $\Lambda << \nu$ holds, representing the time scale hierarchy. This stochastic equation is solved as

$$X(t) = e^{-\Lambda t} \int_0^t \mathrm{d}s \frac{\alpha g_{Y*}}{\nu - \Lambda} e^{\Lambda s} w(s) - e^{-\nu t} \int_0^t \mathrm{d}s \frac{\alpha g_{Y*}}{\nu - \Lambda} e^{\nu s} w(s) \ . \tag{A4}$$

The contribution to the stochastic deviation of X is calculated from Eq.(A4). In the lowest order with respect to the smallness parameter $\Lambda / \nu \sim O(\varepsilon)$, one has $\langle X^2 \rangle = \alpha^2 g_{Y*}^2 / 2\Lambda\nu^2$. That is, in the range of $\Lambda t \sim O(1)$, X is approximately given as

$$X(t) \simeq e^{-\Lambda t} \int_0^t \mathrm{d}s \, \nu^{-1} \alpha \, g_{Y*} e^{\Lambda s} w(s) \ . \tag{A5}$$

This means that $\tilde{y}$ in the stochastic equation (A3) is modelled by the term $\alpha \nu^{-1} g_{Y*} w(t)$. If one studies the equation

$$\frac{\partial}{\partial t} X + \Lambda_X X = g_X w(t) + \alpha \tilde{y} \tag{A6}$$

with $\Lambda_X << \nu$, it is modelled by a stochastic equation which has two independent white noise terms in the RHS. That is, one has

$$\frac{\partial}{\partial t} X + \Lambda_X X = \hat{g}_X w(t) \tag{A7}$$

with

$$\hat{g}_X^2 = g_X^2 + \alpha^2 g_{Y*}^2 \nu^{-2} \ . \tag{A8}$$

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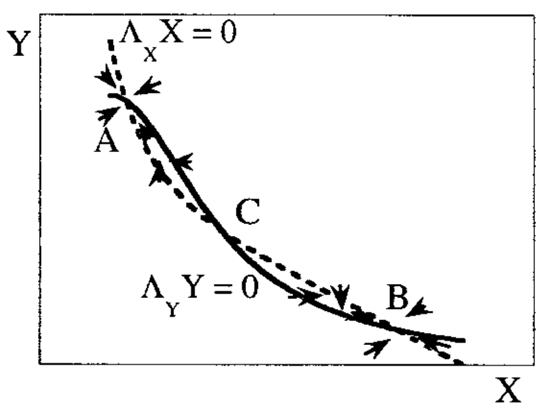

Fig.1 Schematic drawing for a deterministic view when multiple stationary solutions are allowed. Conditions (6a) and (6b) are shown by the dashed line and solid line, respectively. Flow vector $(\partial X/\partial t, \partial Y/\partial t)$ is shown by arrows. "A" and "B" indicate stable fixed point.

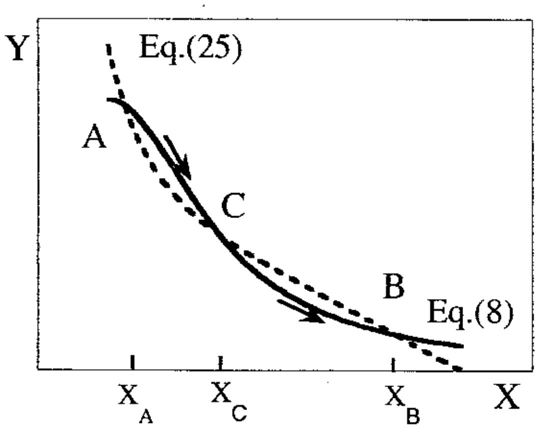

Fig.2 Schematic drawing for a stochastic model when multiple stationary solutions are allowed. Conditions (25) and (8) are shown by the dashed line and solid line, respectively. "A" and "B" indicate peaks of the PDF.