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Abstract

The dynamics of the transition processes in plasma turbulence described by the nonlinear Langevin equation (1) is studied. We show that intermittent or global transitions between metastable states can appear. The conditions for the generation of these transitions and their statistical characteristics are determined.

Keywords: plasma turbulence, stochastic transitions, statistical approaches, Langevin equation, subcritical excitation, intermittence.

1 Introduction

The statistical approach of plasma turbulence was much developed in the last years [1], [2], [3]. An important application of these methods is the study of the transition and bifurcation phenomena which have been observed in magnetized plasmas [1], [4]. In this theoretical framework a statistical theory of subcritically excited strong turbulence in inhomogeneous plasma was recently developed [5], [6], based on the renormalization and random coupling model. It was shown, that self-sustained strong turbulence can be produced by subcritically excited modes [7], [8], [9]. The current-diffusive interchange mode, for example, generates turbulence in inhomogeneous plasma through this non-linear statistical mechanism. In an unified treatment of thermal and turbulent fluctuations it was shown that the coherent part determines a nonlinear drag while the incoherent part has the effect of a noise in the evolution of a test mode. A Langevin equation was deduced for describing the evolution of the turbulence amplitude, which includes both the thermal and the turbulent noise in the presence of turbulent and collisional drag [7]. Several studies of other modes [10], of the L-H transition in tokamak plasma and of transport barriers [11] lead to the conclusion that this statistical approach has a wide applicability in plasma and fluid turbulence. The behavior described by the non-linear Langevin equation appears to be generic for a wide class of processes in plasma turbulence. Moreover, this method of analyzing strong turbulence shows a direction to extend principles of statistical physics as Kubo formula and Prigogine's principle of minimum entropy production rate to non-equilibrium and non-linear systems.

In these models the quantity of interest x (a mode amplitude, or the turbulence energy, or the electric field) is a stochastic function of time. Its evolution is shown to be governed by a Langevin equation containing a nonlinear damping term and a noise [7]. The latter can have several sources and is usually modelled by a white Gaussian noise with amplitude that can depend on x. The main interest is to determine the probability density function for this quantity. Due to the nonlinearity of the damping or of the noise amplitude, the probability density is non-Gaussian. It was determined by deriving a Fokker-Planck equation from the Langevin equation. The solution of this equation was obtained in the asymptotic stationary regime and the transition probabilities were determined. The transition between metastable states were studied with this type of stationary solution for several systems and the dependences on the physical parameters were determined. Non-Gaussian probability density functions having two maxima or power law tails were obtained. However, these were obtained only in the limit of stationary state, and the study of dynamical evolution has not yet been fully addressed.

In this paper we study other characteristics of the transitions in strongly turbulent plasmas, which are essentially related to the dynamics of this process. We determine the time evolution of the system, the possible stationary states and the conditions for reaching these states. The two moment Lagrangian correlation and the frequency spectrum that characterizes the transition process are determined. To this aim we analyze the Langevin equation rather than the Fokker-Planck equation and we determine moments of the stochastic function $x(t)$.

The paper is organized as follows. The physical model, the basic equation and the method of study are presented in Sec. 2. The results are analyzed in Sec. 3 where intermittent and global transitions are evidenced. Statistical properties related to Lagrangian correlations and frequency spectra are determined in Sec. 4. The conclusions are summarized in Sec. 5.

2 Basic equation and statistical approach

The following nonlinear Langevin equation was deduced [7] in the studies of strong turbulence:

$$\frac{dx}{dt} + \Lambda(x)x = R\omega(t) \tag{1}$$

where $\Lambda(x)$ is a deterministic amplification and $\omega(t)$ is a Gaussian white noise with amplitude R. The diffusion coefficient determined by the noise in the absence of the deterministic amplification is $D_0 = R^2$. We consider $R = const.$ but the study can be easily extended to x dependent amplitude. The essential feature of this equation is contained in the deterministic term $\Lambda(x)x$ which has three zeros such that in the absence of noise Eq. (1) has two stable fixed points and one unstable fixed point. The function $\Lambda(x)$ is modelled by

$$\Lambda(x) = (a - x)(1 - x) \tag{2}$$

where $0 < a < 1$. Thus multiple stationary states appear : all trajectory determined from Eq. (1) with initial condition $x_0 < a$ evolve to the stable point $x = 0$, those with $x_0 > a$ go asymptotically to $x = 1$ and there is an unstable equilibrium at $x = a$. The noise makes the fixed points metastable and transitions can appear between the two points $x = 0$ and $x = 1$. As shown in [8], [11] such transitions from a fundamental state with $x \simeq 0$ to an excited state with $x \simeq 1$ explain important plasma processes as subcritical excitation of turbulence or generation of a radial electric field and of H mode in tokamak.

We are interested here in the dynamics of the process of transition described by Eq. (1). This evolution is well represented by the time dependence of the average $X(t) = \langle x(t) \rangle$, which is actually an approximation of the probability of finding the system in the excited state at time t. $\langle ... \rangle$ represents statistical average over the realizations of the noise. We study directly the Langevin equation (1) by developing the hierachy of equations for the moments and using the cumulant expansion procedure. We show that the truncation of the moment system by neglecting the cumulants of order higher than n converges rapidly if the process is not very slow. The main features of the evolution already appear in the lowest approximation ($n = 2$) and starting with $n = 4$ the corrections becomes of the order of 1% for the main range of the parameters of interest. The results presented in the next section are in the 6th cumulant approximation.

The moments of the stochastic function $x(t)$, the solution of the Langevin equation (1), can be systematically obtained from the explicit form of (1)

$$\frac{dx}{dt} = -x^3 + (a+1)x^2 - ax + R\omega(t). \tag{3}$$

where the right hand term is $v(x) \equiv -\Lambda(x)x$, the rate of variation of $x(t)$. The solution of Eq. (3) in a realization of the noise will be called trajectory. The initial condition in all realizations is $x(0) = 0$. Multiplying this equation with x^{n-1} and performing the average one obtains

$$\frac{1}{n}\frac{dM_n}{dt} = -M_{n+2} + (a+1)M_{n+1} - aM_n + R\left\langle x^{n-1}\omega\right\rangle \tag{4}$$

where $M_n \equiv \langle x^n(t)\rangle$ is the moment of order n of the trajectory. The average in the last term of Eq.(4), $C_n(t) \equiv \langle x^n(t)\omega(t)\rangle$, is determined using the formal solution of Eq.(1)

$$x(t) \cong x(t-\delta t) + v\left(x(t-\delta t)\right)\delta t + \int_{t-\delta t}^{t} R\omega(\tau)d\tau. \tag{5}$$

for small δt. In order to obtain $C_1(t)$, this equation is multiplied with $\omega(t)$, is averaged over the noise and the limit $\delta t \to 0$ is performed. Since the average of the product of any function of $x(\tau)$ with a white noise $\omega(t)$ is zero at $\tau < t$, the averages obtained from the first two terms in the r.h.s. of Eq. (5) are zero. The last term leads to $R\int_{t-\delta t}^{t}\langle\omega(t)\omega(\tau)\rangle d\tau = R\int_{t-\delta t}^{t}\delta(\tau-t)d\tau = R/2$. Thus

$$C_1(t) \equiv \langle x(t)\omega(t)\rangle = \frac{R}{2}. \tag{6}$$

The higher order correlation are obtained using the identity

$$\lim_{\delta t\to 0}\left\langle\left(\int_{t-\delta t}^{t}\omega(\tau)d\tau\right)^{k}\omega(t)\right\rangle = \frac{1}{2}\delta_{k,1} \tag{7}$$

which holds for Gaussian white noise. The correlation C_n is calculated as

$$C_n(t) = \lim_{\delta t\to 0}\left\langle\left[x(t-\delta t) + v\left(x(t-\delta t)\right)\delta t + R\int_{t-\delta t}^{t}\omega(\tau)d\tau\right]^{n}\omega(t)\right\rangle \tag{8}$$

and since the averages of powers of the first two terms in Eq. (5) multiplied with the third one are zero (because $\tau > t-\delta t$), this expression reduces, after straightforward calculations and using Eq. (7), to

$$C_n(t) = \frac{R}{2}\lim_{\delta t\to 0} n\left\langle\left[x(t-\delta t) + v\left(x(t-\delta t)\right)\delta t\right]^{n-1}\right\rangle = n\frac{R}{2}\left\langle x^{n-1}(t)\right\rangle. \tag{9}$$

The moment equation (4) can be written as:

$$\frac{1}{n}\frac{dM_n}{dt} = -M_{n+2} + (a+1)M_{n+1} - aM_n + (n-1)\frac{R^2}{2}M_{n-2}. \tag{10}$$

and thus a system of exact equations is obtained for the moments. The source in this system is determined by the noise and appears only in the equation forM_2. Since $M_2 = M_1^2 + \langle\delta x^2(t)\rangle$, where $\delta x(t) \equiv x(t) - \langle x(t)\rangle$ is the fluctuation of the stochastic solution, the noise drives the dispersion $\langle\delta x^2(t)\rangle$ of the trajectories and this dispersion generates the whole chain of moments through the coupling of the moments. The moment coupling is determined by the nonlinear deterministic term in the r.h.s. of Eq. (3) which thus represents the mechanism for transforming the Gaussian white noise into a non-Gaussian probability distribution of the solutions of the Langevin equation.

The system (10) has to be truncated and closed at an order n by providing approximations for the two supplementary moments M_{n+1} and M_{n+2} appearing in the last two equations. This is done by neglecting the cumulants of order higher than n and by approximating the two moments M_{n+1} and M_{n+2} using the first n cumulants [12]. Due to the closure the system of moments (10) is actually nonlinear. The moments M_n can be expressed in terms of the moments of the fluctuation $\delta x(t)$ (or central moments) as

$$M_n = \sum_{k=1}^{n} \binom{n}{k} M_1^{n-k} \left\langle \delta x^k \right\rangle \tag{11}$$

and $\left\langle \delta x^k \right\rangle$ are determined by the cumulants κ_i with $i \leq k$. The following expressions can be obtained for the first eight central moments in terms of cumulants:

$$\langle \delta x \rangle = 0, \quad \left\langle \delta x^2 \right\rangle = \kappa_2, \quad \left\langle \delta x^3 \right\rangle = \kappa_3, \tag{12}$$

$$\left\langle \delta x^4 \right\rangle = 3\kappa_2^2 + \kappa_4, \quad \left\langle \delta x^5 \right\rangle = 10\kappa_2\kappa_3 + \kappa_5, \tag{13}$$

$$\left\langle \delta x^6 \right\rangle = 15\kappa_2\kappa_4 + 15\kappa_2^3 + 10\kappa_3^2 + \kappa_6, \tag{14}$$

$$\left\langle \delta x^7 \right\rangle = 21\kappa_2\kappa_5 + 105\kappa_2^2\kappa_3 + 35\kappa_3\kappa_4 + \kappa_7, \tag{15}$$

$$\left\langle \delta x^8 \right\rangle = 28\kappa_2\kappa_6 + 210\kappa_2^2\kappa_4 + 105\kappa_2^4 + 35\kappa_4^2 + 56\kappa_3\kappa_5 + 280\kappa_2\kappa_3^2 + \kappa_8. \tag{16}$$

The successive approximations were considered, for $n = 2$, 3, 4, 5, and 6. For instance, in the case $n = 5$ the system contains the Eqs. (10) for $n = 1, ..., 5$ and M_6, M_7 are determined in the fifth cumulant approximation using Eq. (11) with $\langle \delta x^6 \rangle$ and $\langle \delta x^7 \rangle$ determined from Eqs. (14), (15) where $\kappa_6, \kappa_7 = 0$.

The results obtained for the average $X(t) \equiv M_1(t)$ and the dispersion $g(t) \equiv \langle \delta x^2(t) \rangle = M_2 - M_1^2$ for two typical sets of parameters are presented in Fig. 1. They show that the truncation of the system by neglecting the cumulants of the order higher than n converges rapidly. The differences between the approximations are larger for larger values of the amplitude of the noise. One can see that even the lowest 2-cumulant approximation already shows qualitatively the characteristics of the solution. We note however that the accuracy of the cumulant method depends on the characteristic time for the evolution of the system. Higher order approximations are necessary when the evolution is slow. Such a very small evolution appear for $a < 0.5$ in the limit $R \to 0$. The sixth cumulant approximation is not sufficient in this limit.

This fast convergence of the cumulant approximation is due to the following two aspects. First is the important property of the cumulants, namely that adding

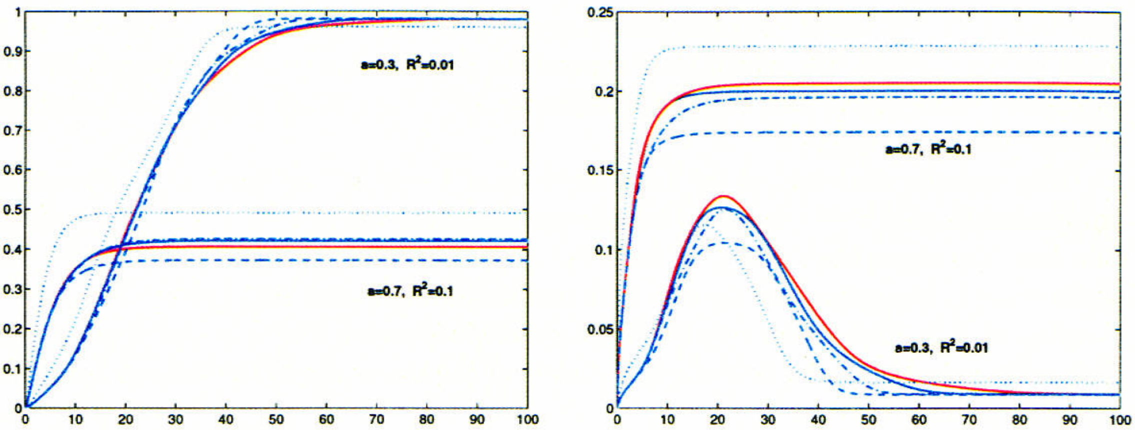


Figure 1: Comparative results of the successive approximations for the average solution $X(t)$ (a) and for the dispersion $g(t)$ (b): 2-cumulants (dotted), 3-cumulants (dashed), 4-cumulants (dot-dashed), 5-cumulants (solid) and 6-cumulants (red). Two typical set of parameters are considered $a = 0.3$, $R^2 = 0.01$ and $a = 0.7$, $R^2 = 0.1$.

a cumulant in the characteristic function does not change the values of the lower order cumulants and moments, although it can strongly change the shape of the distribution function. Secondly, the dynamics determined by Eq. (10) couples only a few moments and the approximation involved in the closure of the system affects directly only the last two equation for the highest moments.

Remark 1 *The moment system (10) is very suitable for a study of closure methods. It is rather simple and a large number of equations can be solved. This permits to push the errors involved in the truncation at high order moments and to have a rather correct evolution of the first moments at small time. One can easily try various closure procedures for systems with 20 equations or more. Most of them lead to correct evaluations at small time but to a fast, explosive behavior at later times. The error propagation and amplification is so strong that even some stationary state cannot be obtained. The fast convergence and the stability of the cumulant truncation appear quite impressive in this context.*

3 Intermittent and global transitions

Three types of evolutions were obtained for the system described by the Langevin equation (1), (2). We present here results obtained numerically in the sixth cumulant approximation.

The first type of evolution is shown in Fig.2, for $a = 0.7$ and $R^2 = 0.01$. The moments (Fig. 2a) and the cumulants (Fig. 2b) grow monotonically and saturate. The average $X(t)$ is small and the high order cumulants are negligible. Consequently $x(t)$ remains around the stable point $x = 0$ which means that a very small number of trajectories perform the transition to the excited state. The distribution of $x(t)$ is close to a Gaussian with small dispersion around the initial position. The fluctuations $\delta x(t)$ have an amplitude comparable with that of the noise: $\langle \delta x^2 \rangle \cong R^2/2\Lambda(0)$.

Actually, the solution obtained in this case is close to the analytical solution corresponding to constant Λ in Eq. (1). Physically, this is the case of a very low amplitude noise which cannot induce the transition to the excited state.

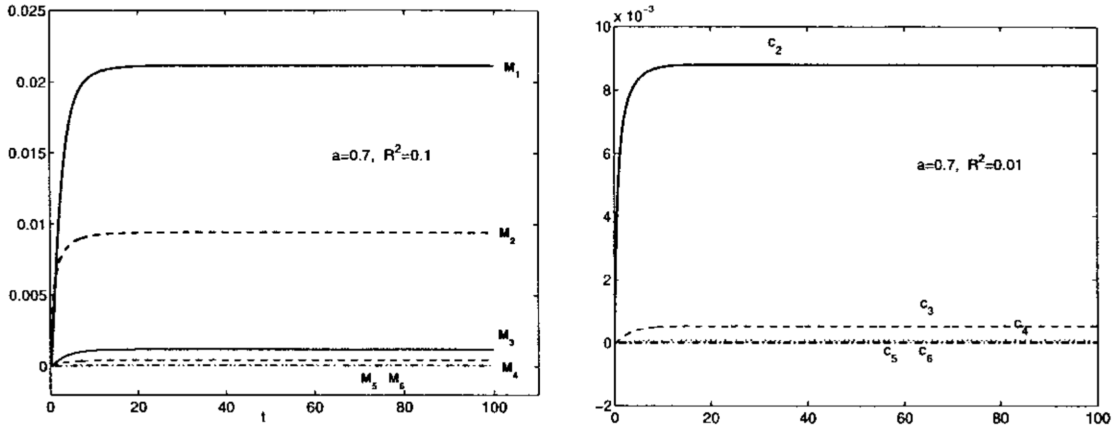


Figure 2: The evolution of the moments (a) and of the cumulants (b) for a = 0.7 and R^2 = 0.01.

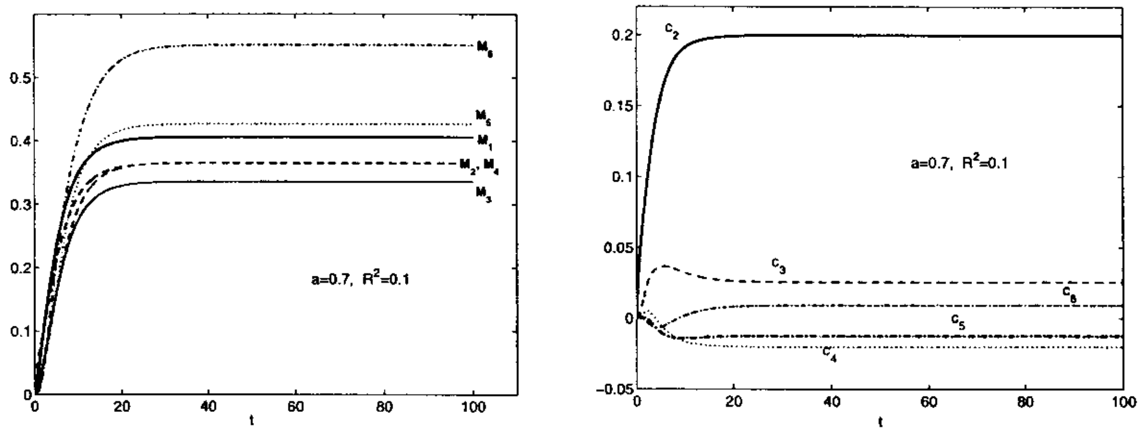
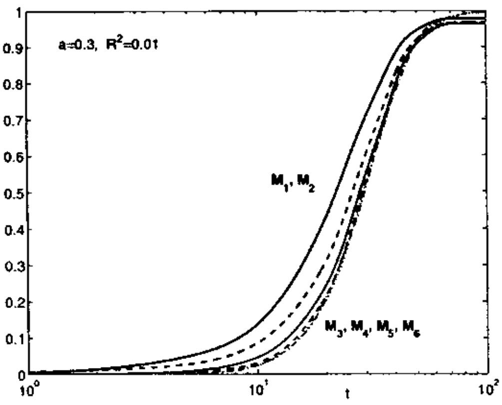


Figure 3: The evolution of the moments (a) and of the cumulants (b) for a = 0.7 and $R^2 = 0.1$.

The second type of evolution is presented in Fig. 3 for $a = 0.7$ and $R^2 = 0.1$. The moments (Fig. 3a) grow monotonically and saturate but at much larger values. At fixed shape of the nonlinear term (fixed a), these asymptotic values depend on the amplitude of the noise: they increase when the amplitude R increases. The evolution to the stationary state is faster when the amplitude of the noise is higher. The cumulants (Fig.3b) have more complicated evolution before reaching a stationary state at values which are sensibly different from zero for all of them and depend on the amplitude of the noise. This means that the probability distribution is not Gaussian. The asymptotic value of $X(t)$ is much larger than in the previous case but remains smaller than a, even at high amplitude of the noise showing that the trajectories are more attracted by the stable point $x = 0$ and a smaller part of the trajectories perform transitions to the stable point $x = 1$. A statistical mixture of states with $x(t) \sim 0$ and $x(t) \sim 1$ exists and transitions between one state and



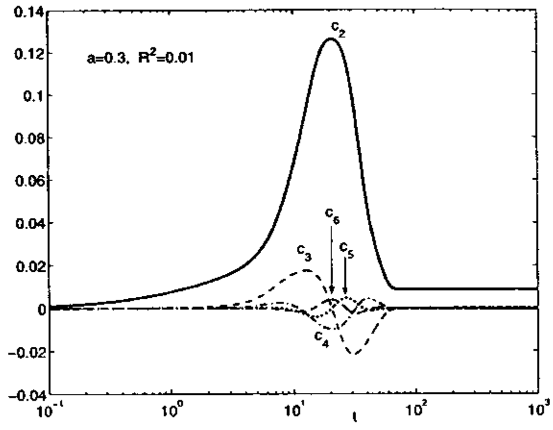


Figure 4: The evolution of the moments (a) and of the cumulants (b) for $a = 0.3$ and $R^2 = 0.01$.

the other appear. The numbers of trajectories in the two metastable states are of the same order, but with the fundamental state more present. The physical interpretation of this type of states, which appear at higher amplitude of the noise, consists in the intermittent behavior of the turbulent system. The noise determines transitions to the excited state but also backward transition to the fundamental state. Consequently, the evolution of the system consists of a random sequence of time intervals in the fundamental and excited states, with fast transition between these two metastable states.

The third type of evolution is represented in Fig. 4, for $a = 0.3$ and $R^2 = 0.01$. All the moments are comparable and have similar evolution. They remain small for some time, then they grow and saturate at a value close to one (Fig. 4a). The second cumulant (Fig. 4b) grows slowly, then very fast during the time interval when the moments grow and after that it decays rapidly and saturates at a small value. The other cumulants remain zero at small time, have strong variations during the increase of the moments and eventually they decay to zero. This clearly shows that the probability distribution evolves from the initial $\delta-$ function to an expanding Gaussian which moves toward $x = 1$. When it reaches the unstable point $x = a$, the distribution is strongly distorted and expands ($g(t)$ is more than 10 times larger than R^2 for the case presented in Fig. 4b). The distribution continues its displacement toward $x = 1$ and, when the unstable point is bypassed, it recovers the Gaussian shape and the dispersion decays to a small value of the order R^2 (corresponding to constant Λ). Thus, in this case an almost global transition to the excited state $x = 1$ was performed due to the noise. A stationary and stable excited state is generated by the noise in these conditions, with backward transitions having very small probability.

The dependence of the average at stationarity, X_{st}, on the noise amplitude in represented in Fig. 5a. The almost global transition can be observed in the upper curve with $a = 0.3$. It has an upper threshold in the noise amplitude R. For noise amplitude smaller this value, the global transition to the excited state always appears as seen by the value of X_{st} which is very close to 1. At higher amplitude of the noise, the average stationary value decreases showing that the trajectories are around both stable points. An intermittent behavior of the turbulence similar to that presented in Fig. 3 appears. The stationary value of the average corresponding to the previous case (Fig. 3) is also represented in Fig. 5a (lower curve for $a = 0.7$). One can see that the global transition (with $X_{st} \cong 1$) does not appear in this case: at low amplitude of the noise the upper state is almost empty and at higher amplitude a statistical mixture of states appears with intermittent behavior.

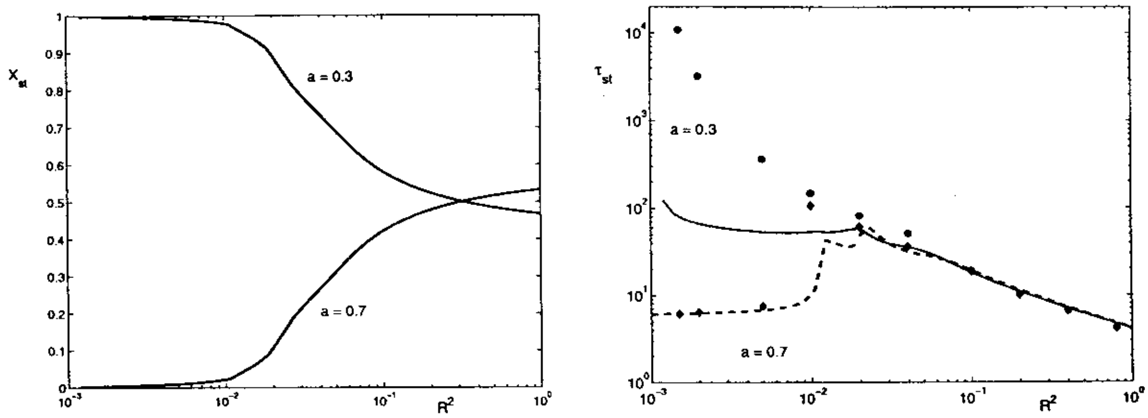


Figure 5: Effects of the noise amplitude: (a) the stationary (asymptotic) value of the average as a function of noise amplitude R^2 for two values of a; (b) the time for reaching the stationary state as a function of R^2 for $a = 0.3$ (circles) and $a = 0.7$ (diamonds). Circles and diamonds are determined by numerical simulation of the stochastic trajectories. The results obtained with the 6th cumulant approximation (dashed and solid lines) are correct when $\tau_{st} \lesssim 30$.

The noise amplitude R strongly influences the characteristic time of the evolution of the system. This influence is represented in Fig. 5b there the time of reaching the stationary state, τ_{st}, is plotted as function of R^2 for $a = 0.3$ and 0.7. We have chosen a physical definition for τ_{st} as the time during which the average $X(t)$ reaches 95% of its stationary value. One can see that in the case of global transition ($a = 0.3$, $R^2 \lesssim 0.01$) this time is very long and it rapidly increases when R decreases. In the intermittent case ($R^2 \gtrsim 0.01$), τ_{st} is practically independent on the shape of the amplification rate (parameter a) and decreases approximately as $\tau_{st} \sim R^{-1.5}$. We note that the results in Fig. 5b represented by circles and stars are obtained from numerical simulation of the stochastic trajectories. The dashed and the solid lines are the results of the 6th cumulant approximation. One can see that at small noise amplitudes this approximation is not sufficient. Actually, the validity of the cumulant expansion depends on the characteristic evolution time of the system. Figure 5b shows that this method used in the 6th order gives good results for $\tau_{st} \lesssim 30$. At slower evolution ($\tau_{st} > 30$) higher order cumulants have enough time to develop through the coupling determined by the nonlinear term in Eq. (1) and thus they cannot be neglected.

In conclusion, intermittent or global transitions to the excited metastable state can appear depending on the parameters a and R. At the same values of the noise amplitude, completely different evolution of the system was obtained (note that the amplitude of the noise in Figs. 3 and 4 is the same). The strong difference between the two cases is determined by the shape of the nonlinear growth term $x\Lambda(x)$ and more exactly by the relative values of the derivative of this growth term,

$v_1 \equiv -d(x\Lambda(x))/dx$, in the two stable points. This parameter characterizes the stability of a fixed point, which is stronger at large negative values of v_1. For the model considered here, Eq. (2), $v_1(0) = -a$ and $v_1(1) = -(1-a)$ and thus for $a < 0.5$ the point $x = 1$ is more stable than $x = 0$ while for $a > 0.5$ the initial point $x = 0$ is more stable.

The conditions for the global transition to the exited state can be observed in Fig. 6a where the stationary value of the average amplitude X_{st} is plotted as a function of a for several values of the noise amplitude. One can see that the global transition is not possible at $a > 0.5$. It appears at small noise amplitude for $a < 0.5$. When the threshold of the noise is attained, the global transition is not more possible, first at $a \lesssim 0.5$ and then neither at small a. At higher noise amplitude or for $a > 0.5$, the system has an intermittent behavior with trapping intervals around each of the metastable points and fast transitions.

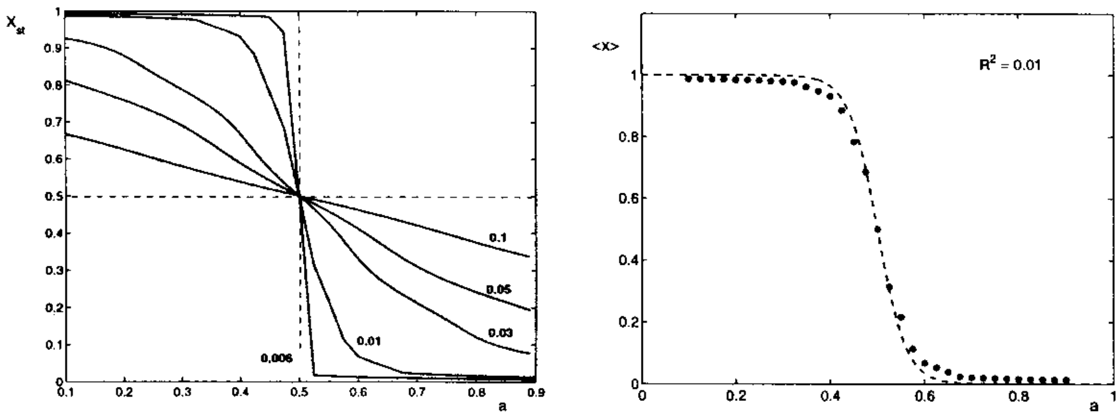


Figure 6: (a) The stationary value of the average as a function of a for several noise amplitudes: the values of R^2 label the curves. (b) Comparison of the present method (closed dots) with Eq. (19) (dashed line).

It is worthwhile to compare the present results with those obtained by an alternative method. In the steady state, a statistical estimate for $\langle x \rangle$ has been obtained by employing a Fokker-Planck equation, which is derived from Eq. (1). By introducing a potential function

$$S(X) = \int_0^x 2R^{-2}\Lambda x' dx' \tag{17}$$

the probability that the state is found near $x = 1$ is given as

$$P_{x=1} = \frac{1}{1 + \exp\left[S(1)\right]} \tag{18}$$

keeping the dominant exponential dependence (see [7], [11]). In this case we have $\langle x \rangle = x P_{x=1}$ and

$$\langle x \rangle = \frac{1}{1 + \exp\left[S(1)\right]} \tag{19}$$

This alternative evaluation is compared with the results in this article. Figure 6b illustrates the results of this cumulant method (closed dots) and formula (19) (dashed line) for fixed value or R^2. It shows that the agreement between the two methods is good. The method presented in [7], [11] provides a simple formula for evaluating the average in the steady state, while the present method allows the study of the dynamical behavior associated with the transition.

4 Lagrangian correlation and transition spectra

More details about the dynamics of the transition process can be obtained from Lagrangian correlations and frequency spectra. They are determined from the correlations of the trajectory fluctuations at two time moments, $G(t,t') \equiv \langle \delta x(t) \delta x(t') \rangle$. This is a symmetrical function of the two time arguments, with a maximum at $t = t'$, where $G(t,t) = g(t)$. The Lagrangian correlation of the rate of variation of $x(t)$, $v(x,t) = -x\Lambda(x) + R\omega(t)$ is defined by

$$L(t,t') \equiv \langle (v(x(t),t) - \langle v(x(t),t) \rangle) \; (v(x(t'),t') - \langle v(x(t'),t') \rangle) \rangle \tag{20}$$

and its Fourier transform represents the frequency spectrum of the process. For a stationary process this function depends on $\tau = |t - t'|$. The Lagrangian correlation describes the coherence of the stochastic evolution: it is delta-function in the case of a non-correlated noise and a function extended to $\tau \to \infty$ for a deterministic evolution.

The equation for the fluctuations $\delta x(t) = x(t) - X(t)$ obtained from Eq.(1) in each realization of the noise is

$$\frac{d\delta x}{dt} = v\left(X\right) + \delta x \; v_1\left(X\right) + \frac{1}{2}\delta x^2 \; v_2\left(X\right) + \frac{1}{6}\delta x^3 \; v_3\left(X\right) + R\omega - \frac{dX}{dt} \tag{21}$$

where $v_1 = dv/dx$, $v_2 = d^2v/dx^2$ and $v_3 = d^3v/dx^3$. This equation is exact for the velocity represented by a polynomial of order 3. Multiplying this equation with $\delta x(t')$ and averaging, one obtains

$$\frac{\partial G(t,t')}{\partial t} = G(t,t') \left[v_1\left(X(t)\right) + \frac{1}{2} g(t) v_3\left(X(t)\right) \right] + \mathcal{C}(t;t') \tag{22}$$

where the cumulants of third and forth order, were neglected for simplicity. This equation is valid for $t' > t$ and the last term is the fluctuation-noise correlation $\mathcal{C}(t;t') \equiv \langle \omega(t) \delta x(t') \rangle$. An equation for this function can be obtained using Eq. (21) and the same approximation:

$$\frac{\partial \mathcal{C}(t;t')}{\partial t'} = \mathcal{C}(t;t') \left[v_1\left(X(t')\right) + \frac{1}{2} g(t') v_3\left(X(t')\right) \right] + R^2 \delta(t-t'). \tag{23}$$

The solution of this equation is

$$\mathcal{C}(t;t') = \Theta(t'-t) R^2 \exp\left[\int_{t'}^{t} \lambda(\theta) d\theta \right] \tag{24}$$

where $\Theta(t'-t)$ is the step function, which is equal to 1 for $t' > t$ and is zero otherwise, and

$$\lambda(\theta) \equiv v_1\left(X(\theta)\right) + \frac{1}{2} g(\theta) v_3\left(X(\theta)\right). \tag{25}$$

Using Eq. (24), the solution of Eq. (22) in terms of the average $X(t)$ is

$$G(t,t') = R^2 \int_0^t d\xi \Theta(t'-\xi) \exp\left[\int_\xi^t \lambda(\theta)d\theta + \int_\xi^{t'} \lambda(\theta)d\theta\right] d\xi. \quad (26)$$

The Lagrangian correlation of the rate of variation of the fluctuations (20) is obtained from Eq. (26) by time derivatives

$$L(t,t') = \lambda(t)g(t,t')\lambda(t') + \lambda(t) \exp\left[\int_{t'}^t \lambda(\theta)d\theta\right] + R^2\delta(t-t') \quad (27)$$

for $t' \geq t$. Equations (26) and (27) show that the process is not stationary at small time but it reaches a stationary state because $\lambda(t)$ saturates at a negative value as seen from Eq. (23) with the time derivative equal to zero. The first two terms in the Lagrangian correlation (27) (denoted by $L_n(t,t')$) are determined by the interaction of the noise with the nonlinear amplification rate while the last term is the direct contribution of the noise. L_n is negative, showing that the deterministic part of the process contributes to compensate the effect of the noise. The nonlinear term hiders the diffusive evolution corresponding to the white noise producing the trapping of the stochastic trajectory $x(t)$.

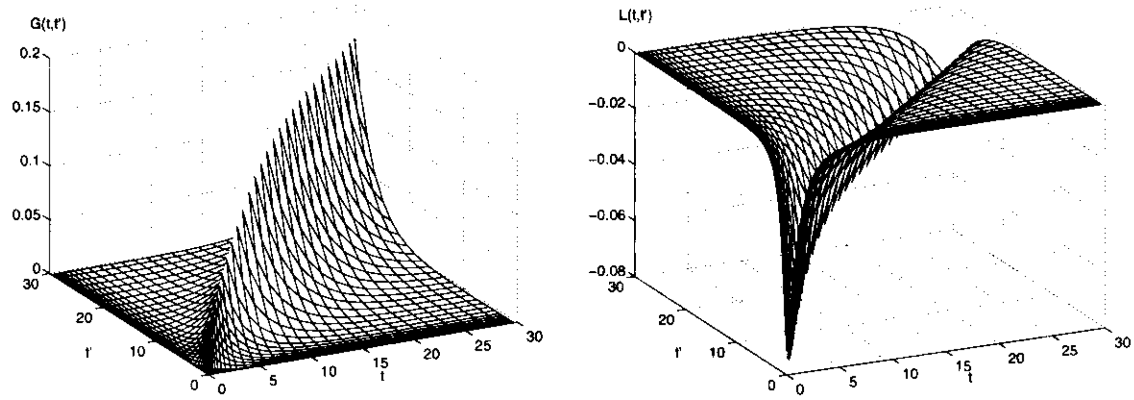


Figure 7: The correlation of the trajectory fluctuations $G(t,t')$ (a) and (b) the Lagrangian correlation (27) for the intermittent transition ($a = 0.7$).

Typical results obtained for the correlations $G(t,t')$ and $L(t,t')$ are presented in Figs. 7 and 8. Significant differences can be observed between the intermittent and global transitions. In the case of intermittent transitions between the two metastable states, $g(t)$, the maximum of $G(t,t')$, and its width are continuously growing up to stationary values (Fig. 7a). The Lagrangian correlation L_n (Fig. 7b) has a negative minimum at $t = t'$ which is very deep at small time and it continuously rises and becomes larger up to the saturation. This shows that the trapping effect is stronger at small time and at later time it is reduced by the action of the noise. The stationary correlations has an exponential shape

$$L_n(t,t') \to L_n(t-t') \approx \exp\left(-\frac{|t-t'|}{\tau_c}\right). \quad (28)$$

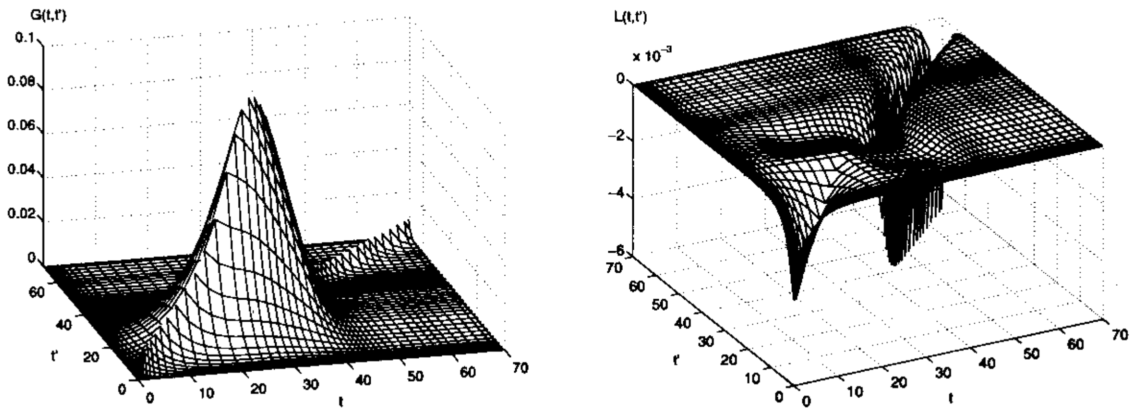


Figure 8: The correlation of the trajectory fluctuations $G(t,t')$ (a) and (b) the Lagrangian correlation (27) for the global transition ($a = 0.3$).

In the case of a global transition (Fig. 8a), the correlation $G(t,t')$ shows a strong increase of its maximum and of its width when the average is around the unstable point, $X(t) \cong a$. Later in the evolution the correlation strongly decreases and becomes very narrow. The Lagrangian correlation $L(t,t')$ represented in Fig. 8b has a deep negative minimum at small time (as in the previous case) but as $X(t)$ approaches the unstable point it becomes practically zero showing that at this stage the deterministic process cannot determine an opposition to the noise and, due to this, the correlation of the fluctuations $G(t,t')$ is strongly growing and expanding. Later a small negative correlation L_n appears on large time intervals which develops a minimum at $t = t'$. This minimum becomes at stationarity very narrow and deep showing a strong trapping around the $x = 1$. The stationary Lagrangian correlation is exponential as in the previous case, Eq. (28), but the correlation time is much smaller.

The correlation time τ_c was determined for the global and the intermittent transition. In the first case it is practically independent of the noise amplitude and can be approximated by $\tau_c \cong |v_1(1)|^{-1} = (1-a)^{-1}$. In the intermittent state the correlation time depends on the noise amplitude and is only weakly influenced by the parameter a of the deterministic term. The correlation time as function of the noise amplitude is represented in Fig. 9 for the intermittent transition. At small amplitude the correlation time is constant and is given by $\tau_c \cong |v_1(0)|^{-1} = a^{-1}$. At such a weak noise the system remains in the fundamental state (only a negligible number of trajectories perform transitions). The increase of the noise amplitude determines a strong increase of the correlation time τ_c. This shows that large coherent displacements can appear and that the system has an intermittent evolution with trapping in the two metastable states. At larger amplitudes of the noise the correlation time decays due to the decay of the trapping time. The intermittent behavior of the system is progressively lost and the evolution becomes random. Thus, the intermittent transitions appear for a limited interval of the noise amplitude, as the global transitions.

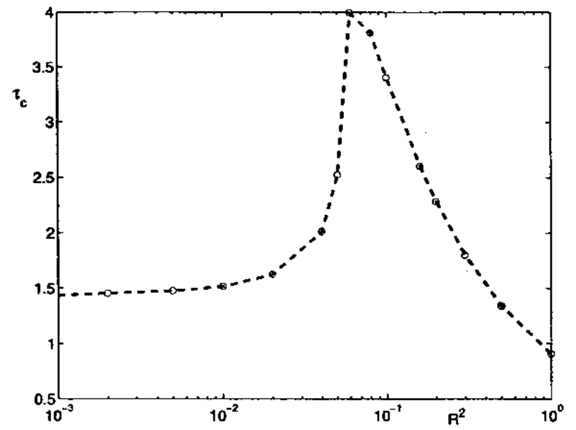


Figure 9: The correlation time τ_c for the intermittent transition as a function of the noise amplitude.

The spectrum of the stationary state can be approximated by

$$S(\omega) \sim 1 - \frac{1}{(\tau_c \omega)^2 + 1} \tag{29}$$

where the first term is the contribution of the white noise and the second term is the Fourier transform of the nonlinear contribution L_n given in Eq. (28). One can see that the nonlinearity eliminates the small frequencies ($S(\omega) \cong 0$ for $|\omega| \ll \tau_c^{-1}$). The stationary spectrum has the same expression for intermittent and global transitions. There is however a significant difference due to the correlation time τ_c which is much smaller for the global transition than for the intermittent one. Consequently, a large range of frequencies around $\omega = 0$ is eliminated in the spectrum of the global transition. This means that only the components of the Lagrangian velocity with large frequencies remain and thus the displacements in the stationary state are all small. In the intermittent state, much smaller frequency components of the Lagrangian velocity are present determining large displacements (between the two metastable points).

5 Conclusions

The dynamics of the transitions in plasma turbulence based on the nonlinear Langevin equation (1) was studied. It was shown that two different types of transitions can be generated by the noise from the fundamental metastable state to the upper state: intermittent and global transitions.

In the intermittent transition the evolution of the system in each realization consists of a random sequence of trapping intervals in the fundamental or in the excited state separated by fast transitions. The probability distribution function of the stochastic function of time $x(t)$ is non-Gaussian during the evolution and in the stationary state. The dispersion of the trajectories increases with the noise amplitude and has weak dependence on the details in the shape of the nonlinear term (on the parameter a). The correlation time of the Lagrangian rate of variation of $x(t)$ shows that the intermittent transitions appear for a limited interval of the values of the noise amplitude. At weak noise the system remains in the fundamental state and at too large noise amplitude the evolution is random and metastable states do not exist.

The global transitions lead to stable excited states and the evolution in most of the realizations consists of a single transition and permanent trapping of the system around the upper state. Such transitions strongly depend on the nonlinear deterministic term (on a). They appear when the upper point is more stable than the lower one and for the noise amplitude smaller than a threshold value (which depend on a). The noise amplitude (below this value) does not influence the stationary state but it determines the characteristic transition time. The latter is large compared to the time of reaching the stationary state in intermittent transitions and increases rapidly to infinity when $R \to 0$. The statistical properties of the global transition are different of those of the intermittent transition. The probability density shows a strong non-Gaussian transitory state with large fluctuations (when the average is around the unstable point, $X(t) = a$) but at latter times the fluctuation amplitude strongly decays and the distribution becomes Gaussian. The correlation time of the Lagrangian rate of variation of $x(t)$ increases very strongly around the unstable point and then it decays at a much smaller value determined by the stability parameter of the upper state. In the stationary frequency spectrum a large range of frequencies around $\omega = 0$ is eliminated by the action of the nonlinear term showing that large displacements (backward transitions) have very small probabilities and the system is stable in the excited state.

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