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Unified Linear Response Function for the Stationary Zonal Flow and the Geodesic Acoustic Mode

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Abstract

This paper presents a formulation of linear response function of electrostatic potential to the nonlinear drives, which is important in considering the self-regulation of the turbulence. There are two well-known branches, stationary zonal flow and Geodesic Acoustic mode, which are known to have weak damping rate and therefore have been believed to regulate the turbulences. Since the frequencies of these modes are in so different regimes, these branches have been analyzed separately with proper approximations. This paper gives unified expressions of these response functions. Drift kinetic equation is integrated along particle orbits by expressing particle orbits in Fourier series expansion. The obtained formula disclosed some interesting aspects of the response function, for example, the existence of a low frequency Geodesic Acoustic Mode and the harmonic resonance effects of individual particles.

I. Introduction

The stationary zonal flow and the Geodesic Acoustic Mode (GAM) are gathering attentions, for they are supposed to have weak damping and therefore potentially regulate the turbulence and reduce the associated anomalous transports. Winsor et al. [1] pointed out possible existence of GAM and many theoretical works followed it. However, the frequency range of validity is limited for they are mostly based on the MHD equations, and applied primarily to tokamaks. In our previous paper [2-3], the theory of GAM oscillation was extended to helical systems. Drift kinetic equation was used there in order to incorporate the kinetic effects. However, since parallel velocity was assumed constant in these works, some kinetic effects were not properly incorporated. Recently, the stationary zonal flow response function was extended to helical systems by Sugama et al.[4] by use of the gyro-kinetic equation. The response of the plasma flow to the nonlinear drive is in itself interesting subject of research and therefore, some works are dedicated to clarification of the transient responses by use of drift kinetic equation [5] and gyro-kinetic equation [6-7]. Particularly, the latter two papers suggested that damping is small for the stationary zonal flow and the residual flow surviving through the damping is interpreted in terms of neo-classical polarization screening.

This paper presents a formulation that unifies these previous works. Since extension of the GAM theories to helical system has been dealt with in our previous work [2], we limit the scope of this paper to axisymmetric device. For reference, Rosenbluth [8] and Mikhailovskii [9] employed the drift-kinetic equation incorporating the toroidal effect in the analyses of waves in the plasmas. Though these papers successfully predicted the combined effects of toroidal effects and collisions, they are not formulated in a convenient way to describe the zonal flows and GAM oscillations. Recent work by Sugama et al. [10] developed a new formulation of the response function for GAM oscillation including finite orbit effects. However, since $\upsilon_{\parallel}$ is assumed constant in these theories, some parallel finite Larmor radius effect and mode coupling is missing. Present paper intends to improve the previous our work [2-3] by including all these effects. In order to extend the theory to tokamaks of arbitrary cross-section, the formulation is made in the coordinate-independent form. The general formulation is presented in section 2. Some of the properties of the response function are shown in section 3: In section 3A, the formula is reduced in the low frequency range is and exact agreement with Rosenbluth Hinton neoclassical shielding effect is confirmed. In section 3B, the formula is applied to GAM frequency range and comparison is made to the previous work. In section 4, the obtained formula is reduced with constant velocity approximation for a simple tokamak to confirm that it is sound extension of the latter. The formulation in section 2 requires subsidiary formulas related to particle orbits expressed in the Fourier components. They are presented in Appendix I~V.

II. Formulation

As in the previous paper [2], we start with the drift kinetic equation with independent variables (w,μ), kinetic energy and magnetic moment, respectively.

$$\frac{\partial f}{\partial t} + (\vec{\upsilon}_{\parallel} + \vec{\upsilon}_D)\cdot\nabla f = e_i (ik_\psi \phi_0)\left[(\vec{\upsilon}_{\parallel} + \vec{\upsilon}_D)\cdot\vec{\nabla}\psi\right]\frac{\partial f}{\partial w}\exp(ik_\psi \psi - i\omega t) \\ + S_{m_0}(w,\mu,m_0,k_{\psi,0},\omega_0)\exp(ik_{\psi,0}\psi - i\omega_0 t) \qquad (1)$$

Here, f, $\vec{\upsilon}_D$, and $\vec{\upsilon}_{\parallel}$ are the distribution function, drift velocity, and parallel velocity of the particles. The electric field has been assumed to be electrostatic, characterized by the frequency ω, the radial wave number k_ψ, and the amplitude of the perturbation ϕ_0, respectively. The S_{m_0} on the right hand side is the non-linear drive due to the turbulence. Here, ω_0, $k_{\psi,0}$, and m_0 are the frequency, radial wave number, and poloidal mode number of non-linear drive. The two terms on the right hand side is regarded as the external force and the two terms on the left hand side compose a propagator.

A problem existing in the previous works [2,11] is in the approximation applied to the propagator, $\frac{\partial f}{\partial t}+(\vec{\upsilon}_{\parallel}+\vec{\upsilon}_D)\cdot\nabla f$. In these works, some of the finite Larmor radius effects were ignored assuming moderate radial wave number for zonal flows. Recent experiments, however, shows that the wave number can be of the order of inverse poloidal Larmor radius [12,13,14]. Therefore, the finite orbit effect is also taken into consideration in this paper. In the previous papers an approximation $(\vec{\upsilon}_{\parallel}+\vec{\upsilon}_D)\cdot\nabla f \approx (\vec{\upsilon}_{\parallel})\cdot\nabla f$ was made. The $(\vec{\upsilon}_D)\cdot\nabla f$ term is retained in the present analysis and the radial wave number is explicitly included in the assumed form of the potential $\phi \propto \exp(ik_{\psi,0}\psi - i\omega_0 t)$. In the previous works $\vec{\upsilon}_{\parallel} = \vec{b}\sigma\sqrt{2(E-\mu B(\vec{r})/m}$ was treated as independent of time in order to simplify the calculation; Therefore, previous works miss some important mode coupling and finite orbit effects. Thus this paper deals with $\vec{\upsilon}_{\parallel} = \vec{b}\sigma\sqrt{2(E-\mu B(\vec{r})/m}$ as dependent on particle position and therefore time dependent.

Here, the solution to Eq.(1) is obtained by an integration along the trajectories of particles time dependently:

$$\begin{aligned} f &= -\int_{-\infty}^{t} dt'\left[e_i(\vec{\upsilon}_D(t'))\cdot\vec{E}(t')\right]\frac{\partial f_0}{\partial w} = ie_i k_{\psi}\phi_0\int_{-\infty}^{t} dt'\left[(\vec{\upsilon}_D(t'))\cdot\nabla\psi\right]\frac{\partial f_0}{\partial w}\exp(-i\omega t'+ik_{\psi}\psi(t')) \\ &= e_i\phi_0\int_{-\infty}^{t} dt'\left[\frac{d\psi}{dt'}+i\omega\right]\exp(-i\omega t'+ik_{\psi}\psi(t'))\frac{\partial f_0}{\partial w} \\ &= \left[e_i\phi_0\right]\frac{\partial f}{\partial w}\exp(-i\omega t+ik_{\psi}\psi)[1+i\omega\int_0^{\infty}\exp(i\omega(t'')-ik_{\psi}(\psi(t)-\psi(t-t''))dt''] \end{aligned} \tag{2}$$

In order to perform the integration, the integrand is series expanded in the power of $(\psi(t)-\psi(t''))$:

$$\begin{aligned} I &= \sum_n \frac{1}{n!}(-ik_{\psi})^n I_n \\ I_n &= \int_0^{\infty}\exp(i\omega(t''))(\psi(t)-\psi(t-t''))^n dt'' = \int_0^{\infty}(-1)^n\left(\frac{I\upsilon_{\parallel}}{\omega_c}-\frac{I\upsilon_{\parallel}(t-t'')}{\omega_c(t-t'')}\right)^n\exp(i\omega(t''))dt'' \end{aligned} \tag{3}$$

The second equality is due to toroidal momentum conservation $\psi + I\upsilon_{\parallel}/\omega_c = const.$

For $n=0$, we obtain

$$I_0 = \int_0^{\infty} \exp(i\omega(t''))dt'' = \frac{1}{-i\omega} \quad (4)$$

, which exactly cancels the first term in the bracket in Eq.(2). Therefore, we need to calculate I_n only for $n \neq 0$. In order to proceed further, the integrand in Eq.(3) is Fourier decomposed. Since $I\upsilon_{\parallel} / \omega_c$ is a periodic function of time with period $T(w,\mu;\psi) = 2\pi / \Omega(w,\mu;\psi)$ whether it is a passing particle or a trapped particle, the adoption of Fourier expansion is adequate:

$$(\frac{I\upsilon_{\parallel}}{\omega_c} - \frac{I\upsilon_{\parallel}(t-t'')}{\omega_c(t-t'')})^n = \sum_m H_{n,m} \exp(im\Omega t'') \quad (5)$$

where

$$H_{n,m} = \frac{1}{T}\int_0^T (\frac{I\upsilon_{\parallel}}{\omega_c} - \frac{I\upsilon_{\parallel}(t-t'')}{\omega_c(t-t'')})^n \exp(-im\Omega t'')dt'' . \quad (6)$$

Since $(\frac{I\upsilon_{\parallel}}{\omega_c} - \frac{I\upsilon_{\parallel}(t-t'')}{\omega_c(t-t'')})^n = 0$ at $t'' = t$, the following sum rules are obtained from the inspection of Eqs. (5) and (6).

$$\sum_m H_{n,m} = 0 . \quad (7)$$

In order to study the properties of $H_{n,m}$, We introduce in Appendix-I, $K(\theta) = \int_0^{\theta} (dl/d\theta)/\upsilon_{\parallel} d\theta$ specifying the particle orbits with their initial conditions $t' = K(\theta') + (t - K(\theta))$. Here, $K(\theta)$ physically means the time for a particle to travel from equatorial plane $\theta = 0$ to $\theta = \theta$, which depends on (μ, w). In association, we define

$$b_{n,m}(w,\mu) = (1/T)\int_0^T (\frac{I\upsilon_{\parallel}(K^{-1}(t))}{\omega_c(K^{-1}(t))})^n \exp(-im\Omega t)dt , \quad (8)$$

for passing particles with respect to those moving in positive direction. The same definition is used for trapped particles which however is defined with respect to those moving in positive direction at t=0. Their detailed properties are shown in Appendix-I.

In Appendix-II, $H_{n,m}$ are expressed in terms of $b_{n,m}$ with details of their analytic properties. One of the useful properties obtained working with $b_{n,m}$ are:

$$H_{n,m} = -2ih_{n,m}(\theta)[\sin m\Omega K_+(\theta)] \quad \textit{for odd } n \quad (9)$$

and

$$H_{n,m} = 2h_{n,m}(\theta)[\cos m\Omega K_{+}(\theta)] \quad \textit{for even } n \quad . \quad (10)$$

Here,

$$h_{n,m}(\theta, w, \mu) \equiv \sum_{l=0}^{n} C_{n,l}(-1)^{l}\left(\frac{I\upsilon_{\parallel}(\theta(t))}{\omega_{c}(\theta(t))}\right)^{n-l} b_{l,-m} \, . \quad (11)$$

and it has even dependence on θ (see Appendix-II).

Once, the Fourier components $H_{n,m}$ are calculated, it is easy to integrate Eq.(2) to obtain the distribution function of each order of n .

$$f_{n} = +\left[e_{i}\phi_{0}\right]\frac{\partial f}{\partial w} i\omega \frac{1}{n!}(ik_{\psi})^{n} \exp(-i\omega t + ik_{\psi}\psi)\left[\frac{H_{n,0}}{-i\omega} + \sum_{m\neq 0} \frac{H_{n,m}}{-i(\omega + m\Omega)}\right]. \quad (12)$$

We introduce the following notations for brevity:

$\hat{L}_{V} \cdot \equiv \int \sqrt{g}\, d\theta d\zeta \, \cdot$, $\hat{L}_{i,w} \cdot \equiv \sum_{\sigma} (2\pi / m_{i}^{2}) B dw d\mu / \left|\upsilon_{\parallel}\right| \cdot$, $\hat{L}_{i} \cdot \equiv \hat{L}_{V} \cdot \hat{L}_{i,w} \cdot$,

and $\hat{\hat{L}}_{i,w} \cdot \equiv \int (2\pi / m_{i}^{2}) B dw d\mu / \left|\upsilon_{\parallel,i}\right|$.

The flux surface averaged (=integrated) charge q_{ind} is obtained by operating $e_{i}\hat{L}_{i} \cdot \equiv e_{i}\hat{L}_{V} \cdot \hat{L}_{i,w}$ $\equiv e_{i}\int \sqrt{g}\, d\theta d\zeta (2\pi / m_{i}^{2}) B dw d\mu / \left|\upsilon_{\parallel}\right|$ to the distribution function Eq.(12); $\hat{\hat{L}}_{i} = \hat{L}_{V} \cdot \hat{\hat{L}}_{i,w}$ is used instead of $\hat{L}_{i}$ where the summation over σ is included in $H_{n,m}$ (see Appendix II). Similarly we define corresponding notations for electrons changing the subscripts from i to e .

We express the induced charge as

$$q_{i,induced} = \sum_{n} \chi_{i,n}\phi_{0} \quad (13)$$

using the susceptibility $\chi_{i,n}$ defined by

$$\chi_{i,n=2n'} = 4\pi e_{i}^{2}(ik_{\psi})^{n}\frac{1}{n!}\hat{\hat{L}}_{i} \cdot \frac{\partial f}{\partial w}\sum_{m=-\infty}^{m=\infty} H_{n,m}(\theta, w, \mu)\frac{i\omega}{-i(\omega + m\Omega)} \, . \quad (14)$$

The essential difference of this formulation from other works is that $k_{\parallel}\upsilon_{\parallel} \sim \upsilon_{\parallel} / qR$ included in previous works are replaced by $\Omega(\mu, w)$, the period of the particle, without assuming constancy of

$k_{\parallel}\upsilon_{\parallel} \sim \upsilon_{\parallel} / qR$; $\Omega(\mu, w)$ is independent of θ. Therefore, mode coupling and higher harmonic effects caused by the variation of the velocity change are correctly incorporated.

It is found that f_n s with odd integer n have odd dependences on θ and make no contribute to the ion response; since $\sqrt{g}$ and B have even dependence on θ, they vanish after integration over θ included in $\hat{\hat{L}}_i \cdot$.

The induced charge due to the classical polarization $q_{classical}$ is employed from the well-known formula in wave physics:

$$4\pi q^{pol}_{classical} = \chi^{pol}_{classical}\phi_0 = 4\pi e_i^2 (ik_\psi)^2 \hat{L}_i \cdot (|\nabla\psi|)^2 \frac{1}{2}\frac{\upsilon_\perp^2}{\omega^2{}_{c,i}}\frac{1}{T} f_M \phi_0 = -k^2_{D,i} V' \frac{\overline{(|\nabla\psi|)^2}}{\omega^2{}_{c,i}} \frac{1}{2}(\upsilon_T)^2 (k_\psi)^2 \phi_0$$

(15)

So far, calculus was shown for ions. For electrons, we assume that the response is adiabatic and adopt the following form (see appendix-III):

$$f_e = -(e_i / e_e)(n_{i,n}(\theta) - < n_{i,n}(\theta) >)(f_{e,0} / n_{e,0}) \quad (16)$$

where $n_{i,n}(\theta, t) = \hat{L}_{i,w} \cdot f_{i,n}$.

The electron response is then written in the form

$$4\pi q_{e,induced} = \chi_e \phi_0 = \sum_n \chi_{e,n} \phi_0 \quad (17)$$

with the electron response function $\chi_{e,n}$ defined by

$$\chi_{e,n} = 4\pi \frac{1}{i\omega} e_e \hat{L}_e \cdot (ik_\psi)(\vec{\upsilon}_{d,e} \cdot \vec{\nabla}\psi)[(-\frac{e_i}{e_e})(f_{e,0} / n_{e,0})](n_{i,n}(\theta) - < n_{i,n}(\theta) >) \quad (18)$$

Using Fourier components $H_{n,m}$, the $\chi_{e,n}$ is expressed as follows:

$$\chi_{e,n=2n'} = -4\pi e_i^2 (ik_\psi)^{2n'} \hat{L}_e \cdot (\vec{\upsilon}_{d,e} \cdot \vec{\nabla}\psi)[(f_{e,0} / n_{e,0})]\hat{\hat{L}}_{i,w} \cdot \frac{\partial f_{i,0}}{\partial w} \frac{1}{(2n'-1)!} \sum_{m=-\infty}^{m=\infty} H_{2n'-1,m}(\theta; w, \mu) \frac{1}{-i(\omega + m\Omega)}$$

(19)

Thus, the electron response is expressed in the power of $(k_\psi)^2$ as ion response is. However, since $(\vec{\upsilon}_{d,e}\cdot\vec{\nabla}\psi)$ in this expression has odd dependence on θ, electron response Eq. (19) contributes through $H_{n=(2n'-1),m}$ with odd n, which has odd dependence on θ as shown in Eq.(9). It is noted that the potential form of Eq.(16), has θ dependence which causes $\vec{E}_\theta\times\vec{B}$ drift. We dropped these terms in the formulation, assuming that electron and ion contributions cancel each other.

The source term on the right hand side of Equation (1) causes the following external charges.

$$q_{i,e,,turbulence}=(e_{i,e})\hat{L}\cdot\int_{-\infty}^{t}dt'S_{i,e}(w,\mu,m_0,k_{\psi,0},\omega_0)\exp(im_0\theta(t')+ik_{\psi,0}\psi(t')-i\omega_0 t') \quad (20)$$

The sum of the contributions from ions and electrons, $q_{turebulence}=q_{i,turebulence}+q_{e,turebulence}$, gives the total external charge. Eq.(20) may be calculated by use of similar algorism that is applied to the calculation of induced charge. Exact calculation of the drive term is not however discussed in this paper, for $S_{i,e}(w,\mu,m_0,k_{\psi,0},\omega_0)$ is not always specified.

Gathering all the terms and assuming quasi-neutrality,

$$q=q_{classical}+(q_{i,induced}+q_{e,induced})+q_{turebulence}=0 \quad (21)$$

We obtain the following response function:

$$\phi_0=\frac{-4\pi q_{turbulence}}{\chi^0_{classical}+\sum_{n=2n'}\chi_{i,n}+\sum_{n=2n'}\chi_{e,n}} \qquad (22)$$

The denominator of this equation, D, consists of the following terms:

$$\chi^0_{classical}=4\pi e_i^2\hat{\hat{L}}_i\cdot[\ +\frac{\partial f_{i,0}}{\partial w}\sum_\sigma\ (|\nabla\psi|)^2\frac{1}{2}\frac{\upsilon_\perp^2}{\omega^2_{c,i}}(k_\psi)^2\ ] \quad (23)$$

and

$$\begin{aligned}\sum_{n=2n'}\chi_{i,n}&=4\pi e_i^2\hat{\hat{L}}_i\cdot[\ \frac{\partial f_{i,0}}{\partial w}\sum_{n=2n'}\frac{1}{(2n')!}(-1)^{n'}(k_\psi)^{2n'}\sum_{m=-\infty}^{m=\infty}H_{2n',m}(\theta,w,\mu)\frac{i\omega}{-i(\omega+m\Omega)}\ ]\\&=4\pi e_i^2\hat{\hat{L}}_i\cdot[\ \frac{\partial f_{i,0}}{\partial w}\sum_{n=2n'}\frac{1}{(2n')!}(-1)^{n'}(k_\psi)^{2n'}\sum_{m\neq0}H_{2n',m}(\theta,w,\mu)\frac{-im\Omega}{-i(\omega+m\Omega)}\ ]\end{aligned} \quad (24)$$

$$\sum_{n=2n'} \chi_{e,n} = -4\pi e_i^2 \hat{L}_e \cdot (\vec{\upsilon}_{d,e} \cdot \vec{\nabla}\psi)[(f_{e,0}/n_{e,0})]$$
$$\times\hat{\hat{L}}_{i,w} \cdot \frac{\partial f_{i,0}}{\partial w} \sum_{n=2n'\geq 0} (-1)^{n'} (k_\psi)^{2n'} \frac{1}{(2n'-1)!} \sum_{m=-\infty}^{m=\infty} H_{2n'-1,m}(\theta;w,\mu) \frac{1}{-i(\omega+m\Omega)} \quad (25)$$

The second transformation in Eq.(24) is due to the sum rule, $\sum_{m=-\infty}^{m=\infty} H_{2n',m}(\theta,w,\mu)=0$.

Eq. (23-25) is presented as a sum of the terms of the form,

$$\frac{H_{n,m}(\theta,w,\mu)}{i(\omega+m\Omega(w,\mu))}. \quad (26)$$

The index of the summation n physically designate the order of Finite Orbit Effects associated with the finite radial wave number as demonstrated in Ref [10]. Equations (23-25) contain also summation over m for each order of n, which is caused by the inclusion of the variation of $\upsilon_{\parallel}$. Thus it is found in this paper that variation of $\upsilon_{\parallel}$ causes many harmonics, which may affect damping of GAM oscillation as well as finite orbit effects.

Though Eqs.(23-25) contain somewhat complicated integrations, some reduction can be made in the numerical calculations: By introducing variable $k=\mu/w$ and using $\upsilon=\sqrt{2w/m_i}$ in place of w we can separate the two variables in $\Omega(w,\mu)$: $\Omega(w,\mu)\equiv\frac{\upsilon}{\upsilon_T}\Omega(k)$. Similarly, we find that $\tilde{K}(\theta,k)\equiv\Omega K(\theta,w,\mu)$ is independent of $\upsilon=\sqrt{2w/m_i}$.

Further, we define

$$\tilde{b}_{l,m}(k)\equiv b_{l,m}(I\upsilon_T/\omega_{c,0})^{-l}(\upsilon/\upsilon_T)^{-l} \quad (27)$$

and

$$\tilde{h}_{n,m}(\theta,k)\equiv\sum_l C_{n,l}(-1)^l\left(\frac{\sqrt{1-kB(\theta)}}{\omega_c(\theta)/\omega_{c,0}}\right)^{n-l}\tilde{b}_{l,-m}(k) \quad (28)$$

in order to separate variables:

$$H_{n,m}(\theta,w,\mu)=(I\upsilon_T/\omega_{c,0})^n(\upsilon/\upsilon_T)^n\tilde{h}_{n,m}(\theta,k)\cos(m\tilde{K}(\theta,k)) \quad \textit{for even } n$$
$$H_{n,m}(\theta,w,\mu)=(I\upsilon_T/\omega_{c,0})^n(\upsilon/\upsilon_T)^n\tilde{h}_{n,m}(\theta,k)\sin(m\tilde{K}(\theta,k)) \quad \textit{for odd } n \quad (29)$$

The integration remains still complex but stays in a tractable level. In order to integrate Eqs.(23-25) over υ, we introduce the dispersion function Z_n;

$$Z_n(\zeta) \equiv \frac{1}{\sqrt{\pi}} \int_0^{\infty} dx \exp(-x^2) \frac{(x)^n}{(x-\zeta)} \quad . \ (30)$$

For instance, the neoclassical ion response is written in the simplified form after the integration over w_i .

$$\chi_{ion} = -k_{D,i}^2 \sum_{n=2n'} \frac{1}{(2n')!} (-1)^{n'} (\frac{I\upsilon_T}{\omega_{c,0}} k_\psi)^{2n'} \hat{L}_V \cdot \int \ Bdk \cdot (\frac{1}{\sqrt{1-kB}}) \\ \times [\sum_{m\neq 0} \ 2\tilde{h}_{2n',m}(\theta,k) \cos(m\tilde{K}(\theta,k))] \times Z_{2n'+3}(-\zeta_{m,k}) \qquad (31)$$

Similar calculations lead to a simplification of electron contributions Eq.(25) : after conductiong integrations over w_e and k_e , and w_i we obtain.

$$\chi_{electron} = -(\frac{e_i}{e_e}) \frac{T_e}{T_i} (k_{D,i}^2) \sum_{n=2n'\geq 0} \ (-1)^{n'} (k_\psi)^{2n'} \frac{1}{(2n'-1)!} (I\upsilon_T / \omega_{c,0})^{2n'} \\ \times [L_V \cdot [\frac{\omega_{c,0}}{\omega_c} \frac{1}{B^3} (\vec{B}_p) \cdot \vec{\nabla} B^2] \cdot \sum_{m=-\infty}^{m=\infty} \ \int dk \cdot \frac{B}{\sqrt{1-kB}} \tilde{h}_{2n'-1,m}(\theta,k) \sin(m\tilde{K}(\theta,k)) \times (\frac{\upsilon}{m\Omega}) Z_{2n'+1}(-\zeta_{m,k})]$$

(32)

The notable differences of the electron term from that of ions are: 1) The former has the multiplication factor T_e / T_i , which takes important roles depending on experimental conditions. 2) $Z_{2n'+1}$ appears in the former, whereas $Z_{2n'+3}$ appears in the latter due to the different kinetics between them. Since $[\frac{\omega_{c,0}}{\omega_c} \frac{1}{B^3} (\vec{B}_p) \cdot \vec{\nabla} B^2]$ and $\tilde{h}_{2n'-1,m}(\theta,k)$ are both odd functions of θ, we find electrons contributes through even dependent parts of $Z_{2n'+1}(-\zeta_{m,k})$ on $\zeta_{m,k}$. The Eqs. (31) and (32) still have numerical integrations over k and θ, which however are inevitable containing important physics related to mode coupling.

Section 3. Relationship to preceding works:

A. Neoclassical Screening Effect

In the very low frequency range, as considered by Rosenbluth [5-6], the $\chi_2^{m=0}$ gives larger contribution than other terms by factors $m\Omega / \omega$. Electron response can be ignored in the low frequency range for its small mass ratio. Therefore, Eq. (22) reduces to

$$\phi_0 = \frac{-4\pi q_{turbulence}}{\chi^0_{classical} + \chi^{m=0}_{i,2}} . \ (33)$$

Using the identities

$$\hat{L}\cdot \equiv \int \sqrt{g} d\theta d\zeta \sum_\sigma \frac{2\pi}{m^2} B dw d\mu / |\upsilon_{\parallel}| \cdot A = \int \frac{dl_p}{|\upsilon_p|} \sum_\sigma \int \frac{2\pi}{m^2} dw d\mu \bar{A}, \quad (34)$$

$\chi_2^{m=0}$ is transformed to the following form (see Appendix-IV):

$$\chi_2^{m=0} = 4\pi(+ie_i^2\omega)(ik_\psi)^2 \frac{1}{-i\omega} \int \frac{dl_p}{|\upsilon_p|} \int \sum_\sigma \frac{2\pi}{m^2} dw d\mu \frac{\partial f}{\partial w} [\overline{\left(\frac{I\upsilon_{\parallel}}{\omega_c}\right)^2} - \left(\overline{\frac{I\upsilon_{\parallel}}{\omega_c}}\right)^2] \ . \quad (35)$$

Thus it has been shown that general expression Eq. (33) is reduced to Eq.(14) in ref.[5] and thus $H_2^{m=0}$ is identified as the neoclassical polarization effect itself [4]. It is noted that the two terms in Eq.(35) in the bracket effectively cancels each other and leaves small residue for passing particles. For trapped particles however $\overline{I\upsilon_{\parallel} / \omega_c} = 0$ and such cancellation does not occur. Therefore, trapped particles are considered to take more important roles though they are smaller in population. Using the numerical results in ref. (5), one can reach the following familiar form for a simple model tokamak.

$$\phi_0 = -4\pi q_{turbulence} / (\chi^0_{classical}(1 + 1.6q^2 / \sqrt{\varepsilon})) \qquad (36)$$

Equation (35) and (36) are important results that new formulations have to be reduced to in the low frequency range in order for them to be correct.

B) Application to Geodesic Acoustic Mode.

The geodesic acoustic mode response function is obtained in the lowest order from the term proportional to $(\frac{I\upsilon_T}{\omega_{c,0}} k_\psi)^2$. The reduction is obtained from Eq.(22), which gives unified expression of the response function valid through zero to GAM frequency range. In the GAM frequency range, $\chi_2^{m\neq 0}$ are retained as well as $\chi_2^{m=0}$.

$$\phi_0 = \frac{-4\pi q_{turbulence}}{\chi^0_{classical} + (\chi^{m=0}_{i,2} + \chi^{m\neq 0}_{i,2}) + \chi^{M\neq 0}_{e,2}} \quad (37)$$

$$\chi_{i,2}^{m=0} = -4\pi(e_i^{\ 2})(k_\psi)^2 \frac{1}{2}\hat{\hat{L}}_i \cdot \frac{\partial f}{\partial w} H_{2,0}(\theta, w, \mu)\frac{i\omega}{-i\omega}$$
$$\chi_{i,2}^{m\neq 0} = -4\pi(e_i^{\ 2})(k_\psi)^2 \frac{1}{2}\hat{\hat{L}}_i \cdot \frac{\partial f}{\partial w}\sum_{m\neq 0} H_{2,m}(\theta, w, \mu)\frac{i\omega}{-i(\omega+m\Omega)} \quad (38)$$

$$\chi_{e,n=2}^{m=0} = 4\pi e_i^2(k_\psi)^2 \hat{L}_e \cdot (\vec{\upsilon}_{d,e}\cdot\vec{\nabla}\psi)[(f_{e,0}/n_{e,0})]\hat{\hat{L}}_{i,w}\cdot\frac{\partial f}{\partial w}H_{n=1,m=0}(\theta, w, \mu)\frac{1}{-i\omega} = 0$$
$$\chi_{e,n=2}^{m\neq 0} = 4\pi e_i^2(k_\psi)^2 \hat{L}_e \cdot (\vec{\upsilon}_{d,e}\cdot\vec{\nabla}\psi)[(f_{e,0}/n_{e,0})]\hat{\hat{L}}_{i,w}\cdot\frac{\partial f}{\partial w}\sum_{m\neq 0} H_{n=1,m\neq 0}(\theta, w, \mu)\frac{1}{-i(\omega+m\Omega)}$$
(39)

The dispersion relation is obtained by the putting the denominator to be zero to determine the oscillation frequency and the damping rate of GAM, i.e., non-trivial solutions are sought.

$$D \equiv \chi_{classical}^0 + (\chi_2^{m=0} + \chi_2^{m\neq 0}) + \chi_e = 0\,. \qquad (40)$$

Using Eqs.(15), (38), and(39), the dispersion relation is obtained in the following form:

$$D = 4\pi(e_i^{\ 2})(k_\psi)^2 \hat{\hat{L}}_i \cdot \frac{\partial f}{\partial w}$$
$$[\sum_\sigma (|\nabla\psi|)^2 \frac{1}{2}\frac{\upsilon_\perp^2}{\omega^2{}_{c,i}} + \frac{1}{2}H_{2,0}(\theta, w, \mu) + \sum_{m\neq 0}\frac{1}{2}H_{2,m}(\theta, w, \mu)\frac{-i\omega}{-i(\omega+m\Omega)}]$$
$$+[4\pi(e_i^{\ 2})(k_\psi)^2 \hat{L}_e \cdot (\vec{\upsilon}_{d,e}\cdot\vec{\nabla}\psi)[(f_{e,0}/n_{e,0})]\hat{\hat{L}}_{i,w}\cdot\frac{\partial f_{i,0}}{\partial w}\sum_{m\neq 0} H_{n=1,m\neq 0}(\theta, w, \mu)\frac{1}{-i(\omega+m\Omega)}]$$
(41)

Eq.(41) suggests that the oscillation frequency is determined by equating the sum of classical- and neoclassical-) polarization currents(the first-and the second terms in the bracket) with other geodesic current (the third term in the bracket). An alternative form is, however, obtained by using the sum rule, $\sum_m H_{2,m} = 0$, and eliminating terms which are odd function in θ.

$$D = 4\pi(e_i^{\ 2})(k_\psi)^2 \hat{\hat{L}}_i \frac{\partial f_{i,0}}{\partial w}$$
$$[\sum_\sigma (|\nabla\psi|)^2 \frac{1}{2}\frac{\upsilon_\perp^2}{\omega^2{}_{c,i}} + \sum_{m\neq 0}\frac{1}{2}H_{2,m}(\theta, w, \mu)\frac{+im\Omega}{-i(\omega+m\Omega)}] + \chi_e = 0 \qquad (42)$$

Or, equivalently, using the properties that $H_{2,m}(\theta,w,\mu)=H_{2,-m}(\theta,w,\mu)$ and they are real, ion terms are further transformed as follows:

$$D = 4\pi e_i^{\,2}(k_\psi)^2\hat{\hat{L}}_i\cdot\frac{\partial f}{\partial w}[\ \ +\sum_\sigma\ (|\nabla\psi|)^2\frac{1}{2}\frac{\upsilon_\perp^2}{\omega^2{}_{c,i}}+\frac{1}{2}\sum_{m\neq 0}\ H_{2,m}(\theta,w,\mu)\frac{(m\Omega)^2}{(\omega^2-m^2\Omega^2)}]+\chi_e=0$$

(43)

In the formula given in Eq.(42) and (43), the frequency of the oscillation is determined by equating the classical polarization current (the first term in the bracket) with the geodesic current (the second term in the bracket). This formula has essentially the same structure that we obtained in the previous work [2]. The present paper thus provides the unified expression of response functions in the wide frequency range from stationary zonal flow to Geodesic Acoustic Modes.

Section 4. Reduction to the constant velocity approximation:

Under the assumption that particle velocity does not change, the formula given above is analytically reduced to familiar forms. As shown in the previous section, we analyze the second order term in $(\frac{I\upsilon_T}{\omega_{c,0}}k_\psi)^2$ by putting $n=2$ in Eqs.(24).

$$\chi_i = k_{D,i}^2(\frac{I\upsilon_T}{\omega_{c,0}}k_\psi)^2\hat{L}_V\cdot(\int\ Bdk\cdot\frac{1}{\sqrt{1-kB}})\times[\sum_{m\neq 0}\ \tilde{h}_{2,m}(\theta,k)\cos(m\tilde{K}(\theta,k))]\times Z_{2n'+3}(-\zeta_{m,k})$$

(44)

For the simple tokamak case with circular cross section, the Jacobean is expressed as $\sqrt{g}=\frac{dr}{d\psi}rR_0(1+\varepsilon\cos\theta)$. Under the approximation that velocity does not change while along the trajectory, we may put $\cos(m\tilde{K}(\theta,k))\rightarrow\cos(m\theta)$. In this simplest case, it is shown in Appendix-V that $h_{2,-1}=-(I\upsilon_T/\omega_{c,0})^2(\upsilon/\upsilon_T)^2\varepsilon^2\cos\theta$ and $\tilde{h}_{2,2}(\theta,k)=+(I\upsilon_T/\omega_{c,0})^2(\upsilon/\upsilon_T)^2(-1)^2\frac{1}{4}\varepsilon^2$.

Gathering these analytic expressions we may write

$$[\sum_{m\geq 1} \tilde{h}_{2n',m}(\theta,k)\cos(m\tilde{K}(\theta,k))]\times[Z_{2n'+3}(-\zeta_{m,k})+Z_{2n'+3}(+\zeta_{m,k})]$$
$$=-[\varepsilon^2\cos\theta\cos(\theta)]\times[Z_{2n'+3}(-\zeta_{1,k})+Z_{2n'+3}(+\zeta_{1,k})]+[\frac{1}{4}\varepsilon^2\cos(2\theta)]\times[Z_{2n'+3}(-\zeta_{2,k})+Z_{2n'+3}(+\zeta_{2,k})]$$

(45)

On integration over k, we replace $\zeta_{m,k}$ with $\zeta_{m,k=0}$ in order to keep consistency with the assumption that velocity dose not change. Therefore we obtain $(\int Bdk\cdot\frac{1}{\sqrt{1-kB}})=2$.

It is shown that the second term makes no contribution as an average is taken by applying $\hat{L}_V\ \cdot$

$$\chi_{ion}=-k_{D,i}^2(\frac{I\upsilon_T}{\omega_{c,0}}k_\psi)^2\hat{L}_V\cdot[\varepsilon^2\cos\theta\cos(\theta)]\times[Z_{2n'+3}(-\zeta_{1,k})+Z_{2n'+3}(+\zeta_{1,k})]$$
$$=-\frac{1}{2}k_{D,i}^2(\frac{I\upsilon_T}{\omega_{c,0}}k_\psi)^2V_0'\varepsilon^2[Z_{2n'+3}(-\zeta_{1,k})+Z_{2n'+3}(+\zeta_{1,k})]$$

(46)

where $V_0'\approx(2\pi)^2 rR_0\frac{dr}{d\psi}$.

For electrons we start with Eq.(32):

$$\chi_{electron}=(\frac{e_i}{e_e})\frac{T_e}{T_i}(k_{D,i}^2)\sum_{n=2n'\geq 0}(k_\psi)^2(I\upsilon_T/\omega_{c,0})^2$$
$$\times[L_V\cdot[\frac{\omega_{c,0}}{\omega_c}\frac{1}{B^3}(\vec{B}_p)\cdot\vec{\nabla}B^2]\cdot\sum_{m=\pm1}\int dk\cdot\frac{B}{\sqrt{1-kB}}\tilde{h}_{1,m}(\theta,k)\sin(m\tilde{K}(\theta,k))\times(\frac{\upsilon}{m\Omega})Z_{2n'+1}(-\zeta_{m,k})]$$

(47)

and use $\sin(m\tilde{K}(\theta,k))=\sin(m\theta)$, $[\frac{\omega_{c,0}}{\omega_c}\frac{1}{B^3}(\vec{B}_p)\cdot\vec{\nabla}B^2]\approx+\frac{\overline{B}_p}{B_{t,0}}2\frac{1}{R}\sin\theta$, and $\tilde{h}_{1,m}(\theta,k)=\frac{1}{2}\varepsilon$.

Writing that

$$\sum_{m=\pm1}\tilde{h}_{1,m}(\theta,k)\sin(m\tilde{K}(\theta,k))\times(\frac{\upsilon}{m\Omega})Z_{2n'+1}(-\zeta_{m,k})]$$
$$=\frac{1}{2}\varepsilon\sin(\theta)\times(\frac{\upsilon}{\Omega})[Z_{2n'+1}(-\zeta_{m=1,k})+Z_{2n'+1}(-\zeta_{m=-1,k})] \quad (48)$$

, we obtain

$$\chi_{electron} = (\frac{e_i}{e_e})\frac{T_e}{T_i}(k_{D,i}^2)(k_\psi)^2(I\upsilon_T/\omega_{c,0})^2$$
$$\times[\hat{L}_V \cdot \frac{\overline{B}_p}{B_{t,0}} 2\frac{1}{R_0}\sin\theta \cdot \frac{1}{2}\varepsilon\sin(\theta)\times(\frac{\upsilon}{\Omega})[Z_{2n'+1}(-\zeta_{m=1,k}) + Z_{2n'+1}(-\zeta_{m=-1,k})] \qquad . (49)$$
$$= \frac{1}{2}(\frac{e_i}{e_e})\frac{T_e}{T_i}(k_{D,i}^2)(k_\psi)^2(I\upsilon_T/\omega_{c,0})^2 \times \varepsilon^2 V'[Z_{2n'+1}(-\zeta_{m=1,k}) + Z_{2n'+1}(-\zeta_{m=-1,k})]$$

where an interpretation was made that $\Omega = \frac{\upsilon}{qR_0}$. Gathering Eqs.(46) and (49), Eq.(40) is expressed in terms of the dispersion functions:

$$D = \chi_{classical} + \chi_{ion} + \chi_{electron} = \chi_{classical} + (k_{D,i}^2)(k_\psi)^2(I\upsilon_T/\omega_{c,0})^2\varepsilon^2 V'$$
$$\times\frac{1}{2}\left[-[Z_{2n'+3}(-\zeta_{m=1,k}) + Z_{2n'+3}(-\zeta_{m=-1,k})] + (\frac{e_i}{e_e})\frac{T_e}{T_i}[Z_{2n'+1}(-\zeta_{m=1,k}) + Z_{2n'+1}(-\zeta_{m=-1,k})]\right].$$

(50)

The dispersion relation is therefore:

$$D = (k_{D,i}^2)(k_\psi)^2 \times V'$$
$$\times\frac{1}{2}\left[-\frac{\overline{(|\nabla\psi|)^2}}{\omega^2{}_{c,i}}(\upsilon_T)^2 + \varepsilon^2(I\upsilon_T/\omega_{c,0})^2[(\frac{1}{\zeta})^2\frac{15}{8} + (\frac{1}{\zeta})^4\frac{15\times 7}{16} + \cdots] + (-\frac{e_i}{e_e})\frac{T_e}{T_i}[\frac{3}{4}(\frac{1}{\zeta})^2 + \frac{15}{8}(\frac{1}{\zeta})^4]\right] = 0$$

(51)

The lowest order solution is

$$\omega^2 = \zeta^2(\frac{\upsilon_T}{qR})^2 = \frac{\varepsilon^2(I\upsilon_T/\omega_{c,0})^2}{\frac{\overline{(|\nabla\psi|)^2}}{\omega^2{}_{c,i}}(\upsilon_T)^2}(\frac{\upsilon_T}{qR})^2[(\frac{15}{8} + \frac{3}{4}(-\frac{e_i}{e_e})\frac{T_e}{T_i}] \approx (\frac{\upsilon_T}{R})^2[(\frac{15}{8} + \frac{3}{4}(-\frac{e_i}{e_e})\frac{T_e}{T_i}]$$

(52)

, which reproduces the formula obtained under the assumption that velocity is constant along the trajectory.

Conclusion.

The response function of radial electric field to external drive was investigated in this paper. This paper presents an analytic formulation to numerically calculate the response without setting the assumption that particle does not change speed due to magnetic in-homogeneity. Also, Finite Larmor radius effects have been taken into consideration by assuming radial wave number. The formula therefore contains the Larmor radius effects of full orders. The velocity change of particles (particularly barely trapped particles) gives additional linear mode coupling and consequently causes higher harmonic resonances.

This formulation satisfies exactly the result obtained by Rosenbluth and Hinton in the low frequency range as well as it gives correct kinetic responses for geodesic acoustic mode.

Therefore, the formula obtained in this paper is a proper unification of those derived for stationary zonal flow and for Geodesic Acoustic Mode.

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Appendix－I : Modeling Particle Orbits

The following form of the particle orbit is assumed relating the position of the particle θ' with time t' :

$$t' = \int_0^{\theta'} \frac{dl/d\theta}{\upsilon_{\parallel}} d\theta + \alpha \quad \text{(A-1-1)}$$

In order to determine α , the initial condition of the particles are used; particles exist at poloidal angle θ at time t yielding

$$\alpha = t - \int_0^{\theta} \frac{dl/d\theta}{\upsilon_{\parallel}} d\theta = t - K(\theta) \text{, (A-1-2)}$$

where

$$K(\theta) \equiv \int_0^{\theta} \frac{dl/d\theta}{\upsilon_{\parallel}} d\theta \quad . \quad \text{(A-1-3)}$$

Substituting (A-1-2) into (A-1-1), we obtain

$$t' = \int_0^{\theta'} \frac{dl/d\theta}{\upsilon_{\parallel}} d\theta + \alpha = \int_0^{\theta'} \frac{dl/d\theta}{\upsilon_{\parallel}} d\theta + (t - K(\theta)) \quad \text{(A-1-4)}$$

, which is solved for θ' yielding

$$\theta' = K^{-1}(\int_0^{\theta} \frac{dl/d\theta}{\upsilon_{\parallel}} d\theta - (t - t')) \text{ . (A-1-5)}$$

For the uses in Appendix-II, we define $b_{l,m}$ by

$$(\frac{I\upsilon_{\parallel}(K^{-1}(t))}{\omega_c(K^{-1}(t))})^l \equiv \sum b_{l,m} \exp(im\Omega t) \quad \text{. (A-1-8)}$$

The $b_{l,m}$ s are calculated with respect to particles moving in positive direction while for trapped particles they are calculated for those moving in positive direction at time t=0.

Appendix II. Properties of $H_{n,m}$:

Now we define H_n by

$$H_n = (\frac{I\upsilon_{\parallel}}{\omega_c} - \frac{I\upsilon_{\parallel}(t-t')}{\omega_c(t-t')})^n \quad \text{(A-2-1)}$$

and decompose it in Fourier series:

$$H_n = \sum_m H_{n,m}(\theta)\exp(im\Omega t') \quad \text{(A-2-2)}$$

$$H_{n,m} \equiv \frac{1}{T}\int_0^T H_n \exp(-im\Omega t)dt' = \frac{1}{T}\int_0^T (\frac{I\upsilon_{\parallel}}{\omega_c} - \frac{I\upsilon_{\parallel}(t-t')}{\omega_c(t-t')})^n \exp(-im\Omega t)dt' \,. \quad \text{(A-2-3)}$$

The integrand in Eq. (A-2-1), H_n is expanded using two term expansion coefficients $C_{l,n}$:

$$H_n = (\frac{I\upsilon_{\parallel}}{\omega_c} - \frac{I\upsilon_{\parallel}(t-t')}{\omega_c(t-t')})^n = \sum C_{l,n}(-1)^l (\frac{I\upsilon_{\parallel}(\theta(t))}{\omega_c((\theta(t))})^{n-l} (\frac{I\upsilon_{\parallel}(\theta(t-t'))}{\omega_c((\theta(t-t'))})^l \quad \text{(A-2-4)}$$

Substituting Eq. (A-2-4) into Eq. (A-2-2), we obtain

$$H_{n,m} \equiv \frac{1}{T}\sum_l C_{l,n}(-1)^l (\frac{I\upsilon_{\parallel}(\theta(t))}{\omega_c((\theta(t))})^{n-l} \int_0^T (\frac{I\upsilon_{\parallel}(\theta(t-t'))}{\omega_c((\theta(t-t'))})^l \exp(-im\Omega t')dt' \,. \quad \text{(A-2-5)}$$

A. Passing particles:

For passing particles, an index $\sigma = \pm 1$ is introduced to designate the direction of particles. By writing

$$(\frac{I\upsilon_{\parallel}(\theta(t-t'))}{\omega_c((\theta(t-t'))})^l = \sum_m \sigma^l b_{l,m} \exp(im(K_\sigma(\theta) - t') \,. \quad \text{(A-2-6)}$$

in terms of $b_{l,m}$ defined in Appendix-I and from the definition Eq.(A-2-3), the following equation is obtained :

$$H_{n,m}(\sigma) = \sum_l C_{n,l}\sigma^l (-1)^l (\sigma \frac{I\upsilon_{\parallel}(\theta_+(t))}{\omega_c((\theta_+(t))})^{n-l} b_{n,-m} \exp(-im\Omega(\sigma K_+(\theta)) \quad \text{(A-2-7)}$$

with

$$h_{n,m} \equiv \sum_l C_{n,l}(-1)^l (\frac{I\upsilon^{+}_{\parallel}(\theta(t))}{\omega_c((\theta(t))})^{n-l} b_{l,-m}(w,\mu) \,. \quad \text{(A-2-8)}$$

Due to the properties given in Appendix-I, $b_{l,m} = b_{l,-m}$, $b_{l,m} = b^*{}_{l,-m}$ and $b_{l,m} = real$, the summation over $\sigma = \pm 1$ gives the following convenient form:

$$\sum_\sigma H_{n,m}(\sigma) = h_{n,m} \sum_\sigma \sigma^n \exp(-im\sigma K_+(\theta)) = h_{n,m} \sum_\sigma [\exp(-imK_+(\theta)) + (-1)^n \exp(imK_+(\theta))]$$

(A-2-9)

,i.e.,

$$\begin{aligned} H_{n,m}(\sigma) &= -2ih_{n,m}[\sin m\Omega K_+(\theta)] \quad \textit{for odd } n \\ H_{n,m}(\sigma) &= 2h_{n,m}[\cos m\Omega K_+(\theta)] \quad \textit{for even } n \end{aligned} \qquad \text{(A-2-10)}$$

Thus, it has been shown for odd n that $\sum_\sigma H_{n,m}(\sigma)$ is an odd function of θ. On the contrary, for even number of n, $\sum_\sigma H_{n,m}(\sigma)$ is an even function of θ.

Trapped particles:

For trapped particles also, $H_{n,m}$ is expressed as follows with the definition of $b_{l,-m}$ given by Eq.(A-1-8):

$$H_{n,m} = \sum_\sigma \sum_{l=0}^{n} C_{n,l}(-1)^l \left(\frac{I\upsilon_\parallel(\theta(t))}{\omega_c((\theta(t))}\right)^{n-l} b_{l,-m} \exp(-im\Omega K_\sigma(\theta)) \quad \text{(A-2-11)}$$

There exist two $K(\theta)$s for given values of θ, designated by the subscript $\sigma(=\pm)$ which are associated with the direction of the parallel velocity $\upsilon_\parallel = \pm|\upsilon_\parallel|$ at initial condition, $(\theta = \theta, t = t)$. They have an interrelationship:

$$K_{\sigma_-}(\theta) = \frac{T}{2} - K_{\sigma_+}(\theta) \quad \text{(A-2-13)}$$

In calculating $H_{n,m} = \sum_\sigma H_{n,m}(\sigma)$ as the sum of these two states, the following properties are used: 1) For even number of l, $b_{l,m}$ takes non-zero values only for even number of m and 2) For odd number of l, $b_{l,m}$ takes non-zero values only for odd number of m.

Dividing the sum over l into even- and odd- l.

$$H_{n,m} = \sum_{\sigma} \sum_{l=2l'+1}^{n} C_{n,l} (-1)^{l} \sigma^{n-(2l'+1)} (\frac{I\left|\upsilon_{\|}(\theta(t))\right|}{\omega_{c}((\theta(t))})^{n-(2l'+1)} b_{2l'+1,-(2m'+1)} \exp(-im\Omega K_{\sigma}(\theta))$$
$$+\sum_{\sigma} \sum_{l=2l'}^{n} C_{n,l} (-1)^{l} \sigma^{n-(2l')} (\frac{I\left|\upsilon_{\|}(\theta(t))\right|}{\omega_{c}((\theta(t))})^{n-(2l')} b_{2l',-(2m')} \exp(-im\Omega K_{\sigma}(\theta))$$

(A-2-14)

$H_{n,m}$ is subject to the following transformation:

$$H_{n,m} = \sum_{l=2l'+1}^{n} C_{n,l} (-1)^{l} (\frac{I\left|\upsilon_{\|}(\theta(t))\right|}{\omega_{c}((\theta(t))})^{n-(2l'+1)} b_{2l'+1,-(2m'+1)}$$
$$\times[\sigma_{+}{}^{n-(2l'+1)} \exp(-im\Omega K_{+}(\theta)) - \sigma_{-}{}^{n-(2l'+1)} \exp(+im\Omega K_{+}(\theta))]$$
$$+\sum_{l=2l'}^{n} C_{n,l} (-1)^{l} (\frac{I\left|\upsilon_{\|}(\theta(t))\right|}{\omega_{c}((\theta(t))})^{n-(2l')} b_{2l',-(2m'+1)}$$
$$\times[\sigma_{+}{}^{n-(2l')} \exp(-im\Omega K_{+}(\theta)) + \sigma_{-}{}^{n-(2l')} \exp(+im\Omega K_{+}(\theta))$$

Gathering the summation over $l = 2l'$ and $l = 2l'+1$, we finally obtain the formula:

$$H_{n,m} = -2ih_{n,m}[\sin(m\Omega(K_{+}(\theta))] \quad \textit{for odd n}$$
and (A-2-15)
$$H_{n,m} = 2h_{n,m}[\cos(m\Omega(K_{+}(\theta))] \quad \textit{for even n}$$

where

$$h_{n,m} = \sum_{l=0}^{n} C_{n,l} (-1)^{l} (\frac{I\left|\upsilon_{\|}(\theta(t)\right|)}{\omega_{c}((\theta(t))})^{n-l} b_{l,-m} \quad \text{(A-2-16)}$$

Thus it has been shown that $H_{n,m}$s due to passing particles and trapped particles are cast into the same form and have the following properties: For odd number of n, $H_{n,m} = \sum_{\sigma} H_{n,m}(\sigma)$ has only odd dependence on θ. On the contrary, for even number of n, $H_{n,m} = \sum_{\sigma} H_{n,m}(\sigma)$ has only even dependence on θ.

Appendix-III, Derivation of the electron term, Eq. (16):

We start with the drift kinetic equation

$$\frac{\partial f_e}{\partial t}+(\vec{\upsilon}_{\parallel}+\upsilon_D)\cdot\vec{\nabla}f_e=-e_e(\vec{\upsilon}_{\parallel}+\upsilon_D)\cdot\nabla(\phi-<\phi>)\frac{1}{T}f_{e,0}-e_e(\upsilon_D)\cdot\nabla<\phi>\frac{1}{T}f_{e,0}\,. \quad \text{(A-3-1)}$$

It is assumed that electrons move along the magnetic surface in the lowest order and drifts across the magnetic surface in the next order. Therefore, for first order, we take the second term on the LHS and first term on the RHS as the dominant terms. Assuming that $(\vec{\upsilon}_{\parallel})\cdot\vec{\nabla} \gg (\upsilon_D)\cdot\vec{\nabla}$, Eq. (A-3-1) is reduced to the following simpler form:

$$(\vec{\upsilon}_{\parallel})\cdot\vec{\nabla}(f_e+e_e(\phi-<\phi>)\frac{1}{T_e}f_{0,e})=0\,. \quad \text{(A-3-2)}$$

The following solution is obtained from Eq. (A-3-2), known as adiabatic response.

$$f_e=-\frac{e_e}{T_e}\left[(\phi-<\phi>)\right]f_{e,0} \quad \text{(A-3-3)}$$

Equation (A-3-3) limits the functional form of the electron distribution, i.e., it is expressed as a product of the Maxwell distribution function and the factor that gives θ dependence. Imposing neutrality condition along the magnetic lines of forces, the electron distribution function assumes the following forms:

$$f_e=-\frac{e_i}{e_e}(f_{e,0}/n_{e,0})\sum_n \tilde{n}_{i,n}(\theta,t) \quad \text{(A-3-4)}$$

Here, $n_{i,1}(\theta,t)$ is obtained by integrating $f_{i,1}$ revealing up-down asymmetry:

$$n_{i,n}(\theta,t)=\hat{\hat{L}}_{i,w}\cdot f_{i,n} \quad \text{(A-3-5)}$$

where

$$f_{i,n}=+\left[e_i\phi_0\right]\frac{\partial f}{\partial w}i\omega\exp(-i\omega t+ik_\psi\psi)(+ik_\psi)\left[\frac{H_{n,0}}{-i\omega}+\sum_{m\neq 0}\frac{H_{n,m}}{-i(\omega-m\Omega)}\right] \quad \text{(A-3-6)}$$

Equation (A-3-4) can be integrated in the velocity space and can be shown to satisfy the quasi-neutrality condition:

$$n_{e,1}(\theta,t)=\int\sum_\sigma\frac{2\pi}{m_e^{\,2}}\frac{Bdwd\mu}{\left|\upsilon_{\parallel}\right|}[-\frac{e_i}{e_e}(f_{e,0}/n_{e,0})]n_{i,1}(\theta,t)=-\frac{e_i}{e_e}n_{i,1}(\theta,t) \quad \text{(A-3-7)}$$

By using Eq. (A-3-4), the solution to Eq. (A-3-1) is obtained for each order of ε.

$$f_{e,n}=\frac{1}{i\omega}\int(ik_\psi)(\vec{\upsilon}_{d,e}\cdot\vec{\nabla}\psi)[-\frac{e_i}{e_e}(f_{e,0}/n_{e,0})]n_{i,n}(\theta,t) \quad \text{(A-3-8)}$$

The electron charge integrated in the flux surface, Eq.(17) in the text, is thus obtained.

Appendix IV. Proof of Eq. (26) in the text:

The $H_{2,0}$ included in the expression of $\chi_{i,2}^{m=0}$ is expressed in terms of time averages as follows:

$$H_{2,0} = \frac{1}{T}\int_0^T \left(\frac{I\upsilon_{\|}}{\omega_c} - \frac{I\upsilon_{\|}(t-t'')}{\omega_c(t-t'')}\right)^2 dt'' = \left(\frac{I\upsilon_{\|}(\theta)}{\omega_c(\theta)}\right)^2 + \overline{\left(\frac{I\upsilon_{\|}(t-t'')}{\omega_c(t-t'')}\right)^2} - 2\frac{I\upsilon_{\|}(\theta)}{\omega_c(\theta)}\overline{\frac{I\upsilon_{\|}(t-t'')}{\omega_c(t-t'')}}$$

(A-4-1)

Taking account of that $\overline{\left(\frac{I\upsilon_{\|}(t-t'')}{\omega_c(t-t'')}\right)^2}$ and $\overline{\frac{I\upsilon_{\|}(t-t'')}{\omega_c(t-t'')}}$ are the functions of (w,μ) only, the order of the integrations is changed allowing the following transformations for arbitral function of θ, $G(\theta)$:

$$\int \sqrt{g}\, d\theta d\zeta \sum_{\sigma} \int \frac{2\pi}{m^2} dw d\mu \frac{B}{|\upsilon_{\|}|} \cdot G(\theta; w,u) = \sum_{\sigma} \int \frac{2\pi}{m^2} dw d\mu \oint \frac{dl_p}{|\upsilon_p|} \cdot G(\theta; w,u)$$
$$= \sum_{\sigma} \int \frac{2\pi}{m^2} dw d\mu T(w,\mu) \overline{G(\theta; w,u)}$$

(A-4-2)

Using Eqs. (A-4-1) and (A-4-2), the following expression is obtained:

$$\chi_2^{m=0} = (+ie_i^2\omega)(ik_\psi)^2 \frac{1}{-i\omega} \sum_{\sigma} \int \frac{2\pi}{m^2} dw d\mu \frac{\partial f}{\partial w} T(w,\mu) \left[\overline{\left(\frac{I\upsilon_{\|}}{\omega_c}\right)^2} - \left(\overline{\frac{I\upsilon_{\|}}{\omega_c}}\right)^2\right] \qquad \text{(A-4-3)}$$

This is equivalent to the more familiar form,

$$\chi_2^{m=0} = (+ie_i^2\omega)(ik_\psi)^2 \frac{1}{-i\omega} \oint \frac{dl_p}{B_p} \sum_{\sigma} \int \frac{2\pi}{m^2} dw d\mu \frac{\partial f}{\partial w} \left[\overline{\left(\frac{I\upsilon_{\|}}{\omega_c}\right)^2} - \left(\overline{\frac{I\upsilon_{\|}}{\omega_c}}\right)^2\right]. \qquad \text{(A-4-4)}$$

Thus, it has been shown that $\chi_2^{m=0}$ represents the same physics that was referred to as the neoclassical shielding effect in reference [5].

Appendix V Analytic expressions of $\tilde{h}_{n,m}(\theta,k)$ for a simple case.

Here, we tabulate a few terms of $\tilde{h}_{n,m}(\theta,k)$ in the analytic forms under the assumption that the particles do not change velocity along the trajectories. Under this assumption all the particles are regarded as passing particles and therefore we may put $\sqrt{1-kB(\theta(t-t''))} = \sqrt{1-kB(\theta)}$. Then the definition Eq.(28) is interpreted as:

$$\tilde{h}_{n,m}(\theta,k) \approx (\sqrt{1-kB(\theta)})^n \sum_l C_{n,l}(-1)^l (\frac{B_0}{B})^{n-l} \tilde{\tilde{b}}_{l,-m}(k)(I\upsilon_T / \omega_{c,0})^{n-l}(\upsilon/\upsilon_T)^{n-l} \quad \text{(A-5-1)}$$

where

$$\tilde{\tilde{b}}_{l,-m}(k) \equiv \int (1+\varepsilon\cos(\Omega(t-t''))^l \exp(im\Omega t'') \quad \text{(A-5-2)}$$

In Eq. (A-5-1), $(\sqrt{1-kB(\theta)})^n$ may be put equal unity under the assumption that the particles do not change velocity along the trajectories.

I. $\tilde{h}_{2,\pm1}(\theta,k)$

$$\begin{aligned}
&\tilde{h}_{2,\pm1}(\theta,k)(\sqrt{1-kB(\theta)})^{-2} \approx \sum_{l=0,1,2} C_{2,l}(-1)^l (\frac{B_0}{B})^{2-l} \tilde{\tilde{b}}_{l,-1}(k) \\
&= C_{2,1}(-1)^1(1+\varepsilon\cos\theta)\tilde{\tilde{b}}_{1,\mp1}(k) + C_{2,2}(-1)^2 \tilde{\tilde{b}}_{2,\mp1}(k) + C_{2,0}(-1)^0 \tilde{\tilde{b}}_{0,\mp1}(k) \leftrightarrow 0 \quad \text{(A-5-3)} \\
&\approx -\varepsilon^2 \cos\theta
\end{aligned}$$

where, we have used $\tilde{\tilde{b}}_{1,\mp1}(k) \sim \frac{1}{2}\varepsilon$ and $\tilde{\tilde{b}}_{2,\mp1}(k) \sim \varepsilon$.

II. $\tilde{h}_{2,\pm2}(\theta,k)$

$$\begin{aligned}
&\tilde{h}_{2,\pm2}(\theta,k)(\sqrt{1-kB(\theta)})^{-2} \approx \sum_{l=0,1,2} C_{2,l}(-1)^l (\frac{B_0}{B})^{2-l} \tilde{\tilde{b}}_{l,\mp2}(k) \\
&\equiv C_{2,1}(-1)^1(1+\varepsilon\cos\theta)\tilde{\tilde{b}}_{1,\mp2}(k) \leftrightarrow 0 \\
&+C_{2,2}(-1)^2 \tilde{\tilde{b}}_{2,\mp2}(k) + C_{2,0}(-1)^0 \tilde{\tilde{b}}_{0,\mp2}(k) \leftrightarrow 0 \quad \text{(A-5-4)} \\
&= C_{2,2}(-1)^2 \frac{1}{4}\varepsilon^2
\end{aligned}$$

, where we have used $\tilde{\tilde{b}}_{2,\mp2}(k) = \frac{1}{4}\varepsilon^2$.

III. $\tilde{h}_{2,\pm 0}(\theta,k)$

$$\tilde{h}_{2,0}(\theta,k)(\sqrt{1-kB(\theta)})^{-2} \approx \sum_{l=0,1,2} C_{2,l}(-1)^l(\frac{B_0}{B})^{2-l}\tilde{\tilde{b}}_{l,0}(k)$$
$$= C_{2,0}(-1)^0(1+\varepsilon\cos\theta)^2\tilde{\tilde{b}}_{0,0}(k) + C_{2,1}(-1)^1(1+\varepsilon\cos\theta)\tilde{\tilde{b}}_{1,0}(k) + C_{2,2}(-1)^2\tilde{\tilde{b}}_{2,0}(k)$$
$$\approx C_{2,0}(-1)^0(1+2\varepsilon\cos\theta+\varepsilon^2\cos^2\theta) + C_{2,1}(-1)^1(1+\varepsilon\cos\theta) + C_{2,2}(-1)^2(1+\frac{1}{2}\varepsilon^2)$$
$$= \varepsilon^2(\frac{1}{2}+\cos^2\theta)$$
(A-5-5)

where we have used $\tilde{\tilde{b}}_{0,0}(k)=1$, $\tilde{\tilde{b}}_{1,0}(k)=1$, and $\tilde{\tilde{b}}_{2,0}(k)=1+\frac{1}{2}\varepsilon^2$.

The Fourier components $\tilde{\tilde{b}}_{n,m}$ are given as follows:

$$\tilde{\tilde{b}}_{0,0}(k) = \frac{1}{T}\int dt'' = 1$$
$$\tilde{\tilde{b}}_{0,\pm 1}(k) = \frac{1}{T}\int dt'' \exp(\mp i\Omega t'') = 0$$
$$\tilde{\tilde{b}}_{0,\pm 2}(k) = \frac{1}{T}\int dt'' \exp(\mp i2\Omega t'') = 0$$
$$\tilde{\tilde{b}}_{1,0}(k) = \frac{1}{T}\int dt''(1+\varepsilon\cos(\Omega(t-t'')) = 1$$
$$\tilde{\tilde{b}}_{1,\pm 1}(k) = \frac{1}{T}\int dt''(1+\varepsilon\cos(\Omega(t-t''))\exp(\mp i\Omega t'') = \frac{1}{2}\varepsilon$$
$$\tilde{\tilde{b}}_{1,\pm 2}(k) = \frac{1}{T}\int dt''(1+\varepsilon\cos(\Omega(t-t''))\exp(\mp i2\Omega t'') = 0$$
$$\tilde{\tilde{b}}_{2,0}(k) = \frac{1}{T}\int dt''(1+\varepsilon\cos(t-t''))^2 = 1+\frac{1}{2}\varepsilon^2$$
$$\tilde{\tilde{b}}_{2,\pm 1}(k) = \frac{1}{T}\int dt''(1+\varepsilon\cos(t-t''))^2\exp(\mp i\Omega t) = \varepsilon$$
$$\tilde{\tilde{b}}_{2,\pm 2}(k) = \frac{1}{T}\int dt''(1+\varepsilon\cos(t-t''))^2\exp(\mp i2\Omega t) = \frac{1}{2}\varepsilon^2 \qquad \text{(A-5-6)}$$