§12. Cyclic Reduction Method for Block Pentagonal Matrix

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Cyclic reduction method for trigonal matrix can be found in the standard text book. Here we consider a problem with block pentagonal matrix.

First a problem with pentagonal matrix of dimension $n=2^k-1$, $\mathbf{M}\mathbf{x}=\mathbf{f}$, is considered.

c_1	d_1	e_1						x_1		f_1
b_2	c_2	d_2	e_2					x_2		f_2
a_3	b_3	c_3	d_3	e_3				x_3		f_3
	a_4	b_4	c_4	d_4	e_4			x_4	=	f_4
		a_5	b_5	c_5	d_5	e_5		x_5		f_5
			…	…	…	…		…		…
				a_n	b_n	c_n		x_n		f_n

For a fixed even i, we write

$a_{i-2}x_{i-4}+b_{i-2}x_{i-3}+c_{i-2}x_{i-2}+d_{i-2}x_{i-1}+e_{i-2}x_i=f_{i-2}$ ----(1)

$a_{i-1}x_{i-3}+b_{i-1}x_{i-2}+c_{i-1}x_{i-1}+d_{i-1}x_i+e_{i-1}x_{i+1}=f_{i-1}$ ----(2)

$a_ix_{i-2}+b_ix_{i-1}+c_ix_i+d_ix_{i+1}+e_ix_{i+2}=f_i$ ----(3)

$a_{i+1}x_{i-1}+b_{i+1}x_i+c_{i+1}x_{i+1}+d_{i+1}x_{i+2}+e_{i+1}x_{i+3}=f_{i+1}$ ----(4)

$a_{i+2}x_i+b_{i+2}x_{i+1}+c_{i+2}x_{i+2}+d_{i+2}x_{i+3}+e_{i+2}x_{i+4}=f_{i+2}$ ----(5)

By multiplying k_1,k_2,k_3,k_4,k_5 to Eqs.(1)-(5), the condition to annihilate coefficients of x_j with odd j are,

$k_1b_{i-2}+k_2a_{i-1}=0$ (for x_{i-3}) ---------(1*)

$k_1d_{i-2}+k_2c_{i-1}+k_3b_i+k_4a_{i+1}=0$ (for x_{i-1}) ---------(2*)

$k_2e_{i-1}+k_3d_i+k_4c_{i+1}+k_5b_{i+2}=0$ (for x_{i+1}) --------(3*)

$k_4e_{i+1}+k_5d_{i+2}=0$ (for x_{i+3}) --------(4*)

From (1*) and (4*), we have

$k_1= -k_2(a_{i-1}/b_{i-2})$, -------(1**)

$k_5= -k_4(e_{i+1}/d_{i+2})$, ------(2**)

and substituting these to (2*) and (3*) gives

$k_3= -k_4[e_{i-1}d_{i+2}a_{i+1}b_{i-2}+(c_{i+1}d_{i+2}-e_{i+1}b_{i+2})(a_{i-1}d_{i-2}-c_{i-1}b_{i-2})]/[e_{i-1}d_{i+2}b_{i-2}b_i+d_id_{i+2}(a_{i-1}d_{i-2}-c_{i-1}b_{i-2})]$ --------(3**)

Choosing $k_2=b_{i-2}$ and $k_4=d_{i+2}$ gives $k_1= -a_{i-1}$, $k_5= -e_{i+1}$, and $k_3= - [e_{i-1}d_{i+2}a_{i+1}b_{i-2} + (c_{i+1}d_{i+2}-e_{i+1}b_{i+2}) (a_{i-1}d_{i-2}-c_{i-1}b_{i-2})] / [e_{i-1}b_{i-2}b_i+d_i(a_{i-1}d_{i-2}-c_{i-1}b_{i-2})]$. By using these k_1-k_5, sum of eqs.(1)-(5) becomes

$[k_1a_{i-2}]x_{i-4} + [k_1c_{i-2}+ k_2b_{i-1}+ k_3a_i]x_{i-2} +[k_1e_{i-2}+ k_2d_{i-1}+ k_3c_i +k_4b_{i+1}+ k_5a_{i+2}]x_i+ [k_3e_i+ k_4d_{i+1}+ k_5c_{i+2}]x_{i+2}+[k_5e_{i+2}]x_{i+4} =k_1f_{i-2}+k_2f_{i-1}+k_3f_i+k_4f_{i+1}+k_5f_{i+2}.$ ---------(4**)

Taking the right hand side to be new f_i, and defining new coefficients a_{i-4}, b_{i-2}, c_i, d_{i+2}, and e_{i+4} for x_{i-4}, x_{i-2}, x_i, x_{i+2}, and x_{i+4}, we have a half size matrix problem,

c_2	d_2	e_2						x_2
b_4	c_4	d_4	e_4					x_4
a_6	d_6	d_6	d_6	e_6				x_6
	a_8	b_8	c_8	d_8	e_8			x_8
		…	…	…	…	…		…
				$a_{n'-2}$	$b_{n'-2}$	$c_{n'-2}$	$d_{n'-2}$	$x_{n'-3}$
					$a_{n'}$	$b_{n'}$	$c_{n'}$	$x_{n'}$

$= (f_2,f_4,\dots,f_{n'-1})^T$,

with matrix dimension $n'=2^{k-1}-1$. This process can be repeated k times to obtain $c_jx_j=f_j$ with $j=2^{k-1}$. Then all the x_j with even j is determined by (1*) and (4*), and then they are used again to obtain x_j with odd j.

Next we consider a problem with block pentagonal matrix,

$\mathbf{c}_1$	$\mathbf{d}_1$	$\mathbf{e}_1$						$\mathbf{x}_1$		$\mathbf{f}_1$
$\mathbf{b}_2$	$\mathbf{c}_2$	$\mathbf{d}_2$	$\mathbf{e}_2$					$\mathbf{x}_2$		$\mathbf{f}_2$
$\mathbf{a}_3$	$\mathbf{b}_3$	$\mathbf{c}_3$	$\mathbf{d}_3$	$\mathbf{e}_3$				$\mathbf{x}_3$		$\mathbf{f}_3$
	$\mathbf{a}_4$	$\mathbf{b}_4$	$\mathbf{c}_4$	$\mathbf{d}_4$	$\mathbf{e}_4$			$\mathbf{x}_4$	=	$\mathbf{f}_4$
		$\mathbf{a}_5$	$\mathbf{b}_5$	$\mathbf{c}_5$	$\mathbf{d}_5$	$\mathbf{e}_5$		$\mathbf{x}_5$		$\mathbf{f}_5$
			…	…	…	…		…		…
				$\mathbf{a}_n$	$\mathbf{b}_n$	$\mathbf{c}_n$		$\mathbf{x}_n$		$\mathbf{f}_n$

where again n is an odd number, $n=2^k-1$, and each **a**,**b**,**c**,**d**,**e** is a dense, m x m matrix, and **x** and **f** are vector with m components. Then m x m matrix $\mathbf{k}_1$ to $\mathbf{k}_5$ are multiplied to eqs.(1)-(5), to see that eqs.(1)-(5) are only replaced by a matrix-vector product, for example, $a_{i-2}x_{i-4}$ becomes $\mathbf{a}_{i-2}\,\mathbf{x}_{i-4}$. Then $\mathbf{k}_1$ to $\mathbf{k}_5$ to annihilate the odd $\mathbf{x}_j$ are

$\mathbf{k}_1= -\mathbf{k}_2\,\mathbf{a}_{i-1}\,\mathbf{b}_{i-2}^{-1}$, ------(1***)

$\mathbf{k}_5= -\mathbf{k}_4\,\mathbf{e}_{i+1}\,\mathbf{d}_{i+2}^{-1}$, ------(2***)

to obtain $\mathbf{k}_3= -\mathbf{k}_4\,\mathbf{p}\,\mathbf{q}^{-1}$ ------(3***)

with $\mathbf{p}=\mathbf{a}_{i+1}\,\mathbf{r}^{-1} - \mathbf{s}\,\mathbf{e}_{i-1}^{-1}$ and $\mathbf{q}=\mathbf{b}_i\,\mathbf{r}^{-1} - \mathbf{d}_i\,\mathbf{e}_{i-1}^{-1}$ and $\mathbf{r}=\mathbf{c}_{i-1} - \mathbf{a}_{i-1}\,\mathbf{b}_{i-2}^{-1}\,\mathbf{d}_{i-2}$ and $\mathbf{s}=\mathbf{c}_{i+1} - \mathbf{e}_{i+1}\,\mathbf{d}_{i+2}^{-1}\,\mathbf{b}_{i+2}$.

Choosing $\mathbf{k}_2$ and $\mathbf{k}_4$ be unit matrix, $\mathbf{k}_1$,..,$\mathbf{k}_5$ are obtained by inversing matrixes in eqs.(1***)-(3***) five times. Then we have a equation corresponding to eq.(4**) in matrix-vector version, and the problem is reduced to be half size. The same procedure will be repeated k times to obtain a m x m matrix problem, $\mathbf{c}_j\mathbf{x}_j=\mathbf{f}_j$ with $j=2^{k-1}$. Although number of matrix inversion $\sim\Sigma_{j=1}^{k}[5(2^{k-j-1})]$ needed is not so different from the total block number $\sim[5(2^k-1)]$, the reduction method has independent loops to be suitable for parallelization.