§33. Taylor's Relation of Turbulent Energy Dissipation

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In one of the most outstanding papers on turbulence, G. I. Taylor [1)] suggested that the energy dissipation rate ϵ (per unit time and mass) of a turbulent flow is determined by its root-mean-square velocity u' and the characteristic length scale ℓ as

$$\epsilon \sim \frac{u'^3}{\ell} \,. \tag{1}$$

Here, we investigate whether the non-dimensional dissipation coefficient

$$C_\epsilon = \frac{\epsilon \ell}{u'^3} \tag{2}$$

is universal or not, when we adopt the integral length of the longitudinal velocity correlation function as ℓ. In the previous experimental and numerical studies, both of the universality (in the high Reynolds number limit) and non-universality of C_ϵ have been claimed.

In the followings, we examine the Taylor relation (1) from a new perspective, i.e. in terms of the statistics of the velocity stagnation points. We consider the turbulence whose energy spectrum $E(k)$ is proportional to k^{-p} in the wavenumber region $\ell^{-1} \lesssim k \le \eta^{-1}$. (Here, η is the Kolmogorov length.) Then, it can be shown [2)] that the number density of the stagnation points of the velocity field coarse-grained at ℓ_c is

$$n_s = C_s \, \frac{1}{\ell^3} \left(\frac{\ell}{\ell_c}\right)^{D_s} , \quad D_s = \frac{3(3-p)}{2} \,. \tag{3}$$

Here, C_s is a non-dimensional constant. On the other hand, according to the theorem by Rice [3)], if the velocity and its spatial derivative are normally distributed, the Taylor length λ of the velocity field is expressed as

$$\lambda = C_\lambda \left[n_s(\ell_c = \eta) \right]^{-1/3} . \tag{4}$$

Above relation implies that the Taylor length λ is proportional to the mean distance between the stagnation points. This is important in the current context because the energy dissipation rate ϵ in statistically isotropic turbulence is expressed in terms of λ as

$$\epsilon = 15 \nu \, u'^2 / \lambda^2 \tag{5}$$

where ν is the kinematic viscosity of the fluid. Then, recalling $\eta = \nu^{3/4} \, \epsilon^{-1/4}$, we obtain, from (3)–(5),

$$\epsilon = \left[15 u'^2 {C_\lambda}^{-2} {C_s}^{2/3} \ell^{-2+2D_s/3} \nu^{1-D_s/2} \right]^{1/(1-D_s/6)} . \tag{6}$$

It is interesting to observe that when $E(k)$ is the Kolmogorov spectrum (i.e. $p = 5/3$ and $D_s = 2$), (6) reduces to the Taylor relation (1) with the relationship between the coefficients

$$C_\epsilon = (15)^{3/2} \, {C_\lambda}^{-3} \, C_s \,. \tag{7}$$

Our main claim is that C_ϵ is *not* universal because it explicitly depends on C_s as seen in (7). Here, we note, from (3), that the coefficient C_s is related to the number of stagnation points at the largest scale ℓ; $C_s = n_s(\ell_c = \ell)\ell^3$. Therefore, C_s must depend on the turbulent structure at the largest scale (i.e. boundary condition, external forcing, and so on). This means that C_ϵ as well as C_s are non-universal.

In order to verify the non-universality of C_ϵ, we have conducted a series of direct numerical simulations of isotropic turbulence of an incompressible fluid by changing the large-scale structures systematically. More precisely, the behaviour of the energy spectrum $E(k) \sim k^q$ in the low wavenumber range ($k \ll \ell^{-1}$) is controlled, since it can be shown analytically [4)] that C_s (and therefore C_ϵ) is a function of q as

$$C_\epsilon \sim C_s \sim \left[\frac{3(q+1)}{6q+10} \right]^{\frac{3}{2}} \frac{\frac{1}{q} + \frac{3}{5}}{\frac{1}{q+1} + \frac{3}{2}} \tag{8}$$

in the high Reynolds number limit. In Fig. 1, the dissipation coefficient C_ϵ in the statistically stationary regime is plotted as the function of the Reynolds number R_λ based on the Taylor length. It is clearly observed that the coefficient depends on the large-scale structure, i.e. the shape of the energy spectrum in the low wavenumber region. The non-universality is likely to survive even for larger R_λ, since the dependence is consistent with our prediction (8).

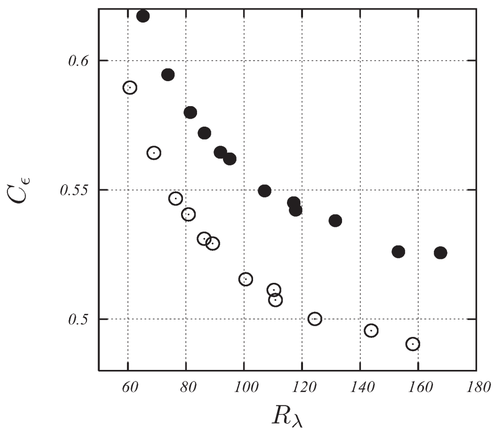

Fig. 1 Energy dissipation coefficient C_ϵ of isotropic turbulence as the function of the Reynolds number. Results of direct numerical simulations for different shapes of the energy spectrum $E(k) \sim k^q$ in the low wavenumber range. Solid circles, $q = 2$; open circles $q = 4$.

1) Taylor, G. I.: Proc. Roy. Soc. London A **151** (1935) 421.
2) Rice, S. O.: Bell Syst. Tech. J. **23** (1944) 282.
3) Dávila, J. and Vassilicos, J. C.: Phys. Rev. Lett. **91** (2003) 144501.
4) Goto, S. and Vassilicos, J.C.: **6** (2008) in preparation.