§3. MHD Equilibrium with Chaotic Magnetic Field

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A numerical calculation of the equilibrium magnetic field is usually the first step in analyzing plasma behavior. This is a comparatively simple task for a perfectly axisymmetric tokamak or any system with a continuous symmetry, as the symmetry guarantees that a nested, continuous family of flux surfaces exists, i.e., the magnetic field is integrable. This is a result of the fact that a toroidal magnetic field is analogous to a time-dependent, one degree of freedom Hamiltonian system and by Noether's theorem, which states that a Hamiltonian with an ignorable coordinate possess an invariant of the motion. By exploiting axisymmetry, the ideal equilibrium equation can be reduced to the Grad–Shafranov equation, and equilibrium solutions can generally be found numerically.

Perturbations to an axisymmetric system, either from internal plasma motions or coil alignment errors, lead to the formation of magnetic islands, chaotic field-lines, and the destruction of flux surfaces. However, from the Kolmogorov–Arnold–Moser (KAM) theorem, we know that under certain conditions for a Hamiltonian system slightly perturbed from an integrable case, the strongly irrational flux surfaces are likely to survive. We can expect that a realistic tokamak, therefore, will possess a finite-measure of KAM surfaces in addition to the islands and chaotic volumes. This is fortunate, as it is primarily the existence of flux surfaces that results in plasma confinement.

On the other hand, under some conditions, applied resonant magnetic perturbations (RMPs) can *advantageously* be used to suppress edge-localized-modes (ELMs). It is plausible to expect that such perturbations will result in the formation of magnetic islands at the rational surfaces, and the overlap of these islands will cause chaotic fields, particularly near the plasma edge. Some understanding of the impact of applied magnetic perturbations may be gleaned, at least in the low pressure case, by superimposing the equilibrium and error fields. The degree of magnetic chaos can then be determined by field-line tracing. Such an approach, however, cannot account for the self-consistent plasma response. To what extent the field becomes chaotic or whether ideal plasma flows will respond by shielding out the error fields remains unclear. The importance of computing non-axisymmetric equilibria with chaotic fields is emphasized by noting that it is likely that ITER will employ RMP methods to suppress ELMs.

Stellarators are intrinsically nonaxisymmetric and thus generally possess nonintegrable fields. Stellarators are designed to have "good-flux-surfaces" as much as possible, but despite one's best efforts, without a continuous symmetry, perfectly integrable fields cannot be achieved. Also, computational evidence suggests that as the plasma pressure increases, stellarator fields become increasingly chaotic. To understand the impact magnetic islands and chaotic fields have on plasma confinement, for both realistic tokamaks and stellarators, a computational algorithm that solves for the plasma equilibrium in the presence of islands and chaotic fields, and a significant volume of robust KAM surfaces, is required.

A given magnetic field may be a continuous, smooth function of space, but it also may be "chaotic." The term chaotic is really a description of the magnetic field-lines, i.e., the phase space of the magnetic field. The behavior of the field-lines of a chaotic field depends sensitively on position, not only in the sense that nearby trajectories may separate exponentially at a rate given by the Lyapunov exponent (butterfly-effect), but also in the sense that irregular, chaotic trajectories lie arbitrarily close to regular trajectories and invariant flux surfaces.

A chaotic magnetic field has a *fractal* phase space structure. The fractal structure arises when an integrable field is generally perturbed, as the rational flux surfaces and irrational flux surfaces break apart quite differently. Quoting Grad, "What is pathological is the question that is asked, viz., what is the position of a magnetic field-line after *infinitely* many circuits?" Some field-lines trace out structures that are infinitely complex, such as the unstable manifold and the irregular trajectories, which seem to come arbitrarily close to every point in a fractal volume. Interspersed between these irregular field-lines are periodic orbits; arbitrarily small, high-order island chains; and irrational field-lines, which may or may not trace out smooth flux surfaces.

Ideal force balance has the consequence that the pressure is constant along the infinite length of every field-line. The structure of the pressure is exactly tied to the structure of the magnetic field.

In his work it is argued that for a chaotic magnetic field, a continuous, nontrivial pressure that satisfies $\mathbf{B}\cdot\nabla p = 0$ must also be fractal. Various objections to computational algorithms were discussed that seek solutions to ideal force balance, with continuous pressure and chaotic fields. The derivation of ideal force balance from a minimization principle was reviewed, but discard this as a practical numerical approach for treating chaotic fields, as ideal variations do not allow the topology of the field to change. The solubility conditions on magnetic differential equations are reviewed and applied to chaotic fields. Moreover, the fractal structure of the phase-space of chaotic fields is reviewed. Since the structure of the pressure is tied to the structure of the field, we conclude that a nontrivial, continuous pressure has an uncountable infinity of discontinuities in the pressure gradient and so therefore must the current. Thus, ideal force balance cannot serve as a coherent mathematical foundation for a computational algorithm. The problems caused by the pathological structure of the solution are not easy to remedy by *ad hoc* adjustments to an iterative algorithm and lead to convergence problems. Finally, we suggest that it is preferable instead to seek solutions to a well-posed *nonideal* equilibrium model, and we discuss various algorithmic approaches aimed at solving for such an equilibrium. Practical algorithmic approach will shown near future by using HINT code.