§29. Analytically Conserved Quantity of Lines of Force in Helical Systems

Watanabe, T., Yoshida, M., Masuzaki, S., Emoto, M., Nagayama, Y.

Equation of lines of force,

$$\frac{\mathrm{d}\boldsymbol{x}(\ell)}{\mathrm{d}\ell} = \frac{\boldsymbol{B}(\boldsymbol{x}(\ell))}{|\boldsymbol{B}(\boldsymbol{x}(\ell))|} \tag{1}$$

is a canonical dynamical system. In helical system, there are no trivial conserved quantity for eq.(1). Then, numerical computaons are requested for the analysis of the magnetic surface. The most difficult and most necessary work is to specify the shape and position of the last closed magnetic surface (LCFS). Lines of force near the LCFS weave countless numbers of small island with chaotic field lines of long connection length. Therefore, an accurate position of LCFS cannot be specified by a simple numerical computation.

Magnetic flux, $\iint_{\Omega} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$ is conseved in a flux tube Ω due to divergent free nature of $\boldsymbol{B}$. We have been able to prove the existence of the conserved quantity to not only the magnetic flux tube but also a line of force. We can distinguish closed magnetic surface ($\iota/2\pi$ = irational number), island ($\iota/2\pi$ = rational number) and chaotic field lines of the lines of force given by eq.(1), with the help of the conserved quantity. The rotational transform on the magnetic axis, $\iota_{\mathrm{ax}}/2\pi$, can also be determined.

The conserved quantity exists in the variational equation for the lines of force eq.(1).

$$\frac{\mathrm{d}\delta\boldsymbol{x}}{\mathrm{d}\ell} = (\delta\boldsymbol{x}\cdot\boldsymbol{\nabla})\left\{\frac{\boldsymbol{B}(\boldsymbol{x})}{|\boldsymbol{B}(\boldsymbol{x})|}\right\}, \ \delta\boldsymbol{x}^i = \begin{pmatrix} \delta x^i \\ \delta y^i \\ \delta z^i \end{pmatrix}. \tag{2}$$

Direct manipulation leads the following relation due to $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$.

$$|\boldsymbol{B}(\boldsymbol{x})| \times \det \begin{pmatrix} \delta x^1 & \delta x^2 & \delta x^3 \\ \delta y^1 & \delta y^2 & \delta y^3 \\ \delta z^1 & \delta z^2 & \delta z^3 \end{pmatrix} = \text{const.} \tag{3}$$

When we use the rotating helical coordinate system (X, Y, ϕ) and toroidal angle ϕ is treated as an independent variable, the conservative quantity (3) reduce to

$$\boldsymbol{B_\phi}(X, Y, \phi) \times \det \begin{pmatrix} \delta X^1 & \delta X^2 \\ \delta Y^1 & \delta Y^2 \end{pmatrix} = \text{const.} \tag{4}$$

We solve the variational equation eq.(2) with two independent initial condition

$$\begin{pmatrix} \delta X^1(0) = 1 \\ \delta Y^1(0) = 0 \end{pmatrix}, \quad \begin{pmatrix} \delta X^2(0) = 0 \\ \delta Y^2(0) = 1 \end{pmatrix}.$$

After one and N toroidal turn, we get,

$$\begin{pmatrix} \delta X^1(2\pi) & \delta X^2(2\pi) \\ \delta Y^1(2\pi) & \delta Y^2(2\pi) \end{pmatrix} \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} (= W),$$

$$\begin{pmatrix} \delta X^1(2N\pi) & \delta X^2(2N\pi) \\ \delta Y^1(2N\pi) & \delta Y^2(2N\pi) \end{pmatrix} = W^N \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We get the relation $\det(W) = 1$ on the magnetic axis due to eq.(4) and $B_\phi(X, Y, 2\pi) = B_\phi(X, Y, 0)$. Then, eigen value of the matrix W become $\exp(\pm i\mu)$,

$$\mu = \tan^{-1}\left\{\frac{\sqrt{4 - (a_{11} + a_{22})^2}}{a_{11} + a_{22}}\right\}, \tag{5}$$

and we get the following relations,

$$U\, W^N\, U^{-1} = \begin{pmatrix} \exp\{i\, N\mu\} & 0 \\ 0 & \exp\{-i\, N\mu\} \end{pmatrix}.$$

Then, we can conclude that, the rotational transform at magnetic axis is given by

$$\frac{\iota_{\mathrm{ax}}}{2\pi} = \pm\frac{\mu}{2\pi} + N_\omega + p = , \ \text{for} \left\{ \begin{array}{c} a_{21} > 0 \\ \text{otherwise} \end{array} \right. \tag{6}$$

where N_ω is poloidal rotation number of $\delta\boldsymbol{x}$ and $p\,(= 5)$ is the helical pitch number due to the rotating helical coordinate system.

Length of the eigen vector of the variational trace, eq.(2), $U \begin{pmatrix} \delta X^1(2\,N\pi) & \delta X^2(2\,N\pi) \\ \delta Y^1(2\,N\pi) & \delta Y^2(2\,N\pi) \end{pmatrix}$, remain constant on the magnetic axis.

Recursive nature of $|\delta\boldsymbol{x}(\phi)|$ on any rational surfaces is approved by similar argument for the case of magnetic axis. A numerical example is shown in Fig.1

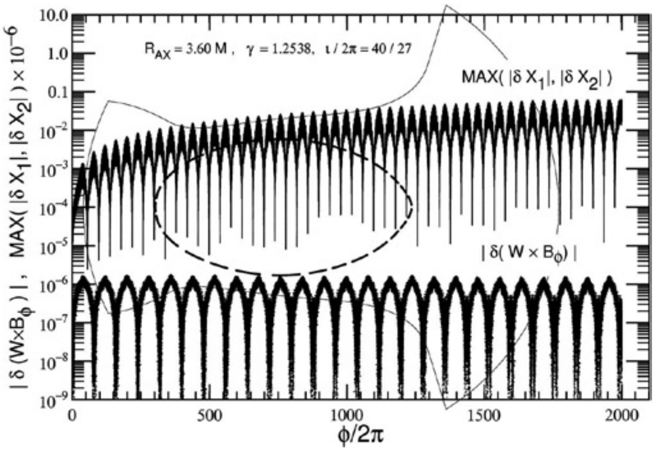

Fig. 1: Recursive nature of variational trace on an island ($\iota/2\pi = 40/27$). Poincar'e plots of eq.(1) and cross section of vacuum vessel are slso shown.

A variational trace on a magnetic surface grows linearly with ϕ, because,

$$\delta\boldsymbol{x}(\phi) \simeq \delta\left[\rho\begin{pmatrix} \cos\left(\frac{\iota(\rho)}{2\pi}\phi\right) \\ \sin\left(\frac{\iota(\rho)}{2\pi}\phi\right) \end{pmatrix}\right] \Longrightarrow$$

$$\frac{|\boldsymbol{\delta x(\phi)}|}{\delta\rho} \simeq \sqrt{1 + \rho^2\left(\frac{\iota'(\rho)}{2\pi}\phi\right)^2}.$$

Variational trace of lines of force on chaotic field line region show exponential growth, because of unstable orbit of chaotic lines of force.