§31. Differential-geometirical Structure of Hall Magnetohydrodynamic Systems

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The dynamics of a dissipationless incomplessible Hall magnetohydrodynamic (HMHD) medium is formulated as geodesics on a semidirect product group of two volume preserving diffeomorphism groups, say G and H, equipped with the right invariant Riemannian metric given by total plasma energy and the associated Levi-Civita connection.

The group G is the function space of the Lagrangian history of fluid particles. Each element of G, say $\vec{X}(\vec{a};t)$, is such a triplet of functions that gives the position of a fluid particle at the time t initially ($t = 0$) located at $\vec{a}$ and advected by an assigned flow history. The group operation of G is given by their composite: $\vec{X}_1 \circ \vec{X}_2 = \vec{X}_1(\vec{X}_2(\vec{a};t_2);t_1)$.

On the other hand, the element of group H is the Lagrangian history of integral lines of current field, say $\vec{Y}(\vec{a},t;s)$, which is also given by a triplet of functions, where t and s respectively denote time and line parameter, and implies the position of current line element at (t,s) which is initially ($(t,s) = (0,0)$) located at $\vec{a}$.

The whole group operation of $G \ltimes H$ is given by

$$\left(\vec{X}_1,\vec{Y}_1\right) \circ \left(\vec{X}_2,\vec{Y}_2\right) = \Big(\vec{X}_1(\vec{X}_2(\vec{q};t_2);t_1), \vec{Y}_1(\vec{X}_1(\vec{Y}_2(\vec{A}_1(\vec{q};t_1),t_2;s_2);t_1),t_1;s_1)\Big), \tag{1}$$

where $\vec{A}_1$ is the inverse of $\vec{X}_1$.

Following the lines given in Vizman[1)], we obtain the Lie bracket on $\mathfrak{g} \ltimes \mathfrak{h}$ as follows:

$$[(\boldsymbol{u}_1,-\alpha\boldsymbol{j}_1),(\boldsymbol{u}_2,-\alpha\boldsymbol{j}_2)] = \bigg(\nabla\times(\boldsymbol{u}_1\times\boldsymbol{u}_2), -\alpha\Big(-\alpha\nabla\times(\boldsymbol{j}_1\times\boldsymbol{j}_2) + \nabla\times(\boldsymbol{u}_1\times\boldsymbol{j}_2) - \nabla\times(\boldsymbol{u}_2\times\boldsymbol{j}_1)\Big)\bigg). \tag{2}$$

To derive the HMHD systems, the Riemannian metrics on $\mathfrak{g}$ and $\mathfrak{h}$ must be defined as follows:

$$\langle\boldsymbol{u}_1|\boldsymbol{u}_2\rangle_{\mathfrak{g}} = \int \boldsymbol{u}_1(\vec{x})\cdot\boldsymbol{u}_2(\vec{x})\,\mathrm{d}^3\vec{x}, \tag{3}$$

$$\langle-\alpha\boldsymbol{j}_1|-\alpha\boldsymbol{j}_2\rangle_{\mathfrak{h}} = \frac{1}{\alpha^2}\int \left((\nabla\times)^{-1}(-\alpha\boldsymbol{j}_1(\vec{x}))\right)\cdot\left((\nabla\times)^{-1}(-\alpha\boldsymbol{j}_2(\vec{x}))\right)\mathrm{d}^3\vec{x}, \tag{4}$$

where $(\nabla\times)^{-1}$ is the inverse of curl operator.

The first variation of action given by

$$S := \frac{1}{2}\int_0^1 dt\Big(\langle\boldsymbol{u}|\boldsymbol{u}\rangle_{\mathfrak{g}} + \langle-\alpha\boldsymbol{j}|-\alpha\boldsymbol{j}\rangle_{\mathfrak{h}}\Big) \tag{5}$$

yields the following equations:

$$\dot{\boldsymbol{u}}(t) = P_\sigma\left(\boldsymbol{u}\times(\nabla\times\boldsymbol{u}) + \boldsymbol{j}\times\boldsymbol{b}\right), \tag{6}$$

$$-\alpha\dot{\boldsymbol{j}}(t) = \nabla\times\nabla\times((-\alpha\boldsymbol{j})\times(-\alpha\boldsymbol{b})) + \nabla\times\nabla\times(\boldsymbol{u}\times(-\alpha\boldsymbol{b})). \tag{7}$$

In case of $\mathbb{T}^3$ or E^3, the generalized Elsasser variables which was introduced by Galtier[2)] yield remarkably simple expressions of basic formulas and equations such as structure constants of Lie algebra, equation of motion, Riemannian curvature tensor, etc. The detailed mathematical investigation is now underway.

1) Vizman, C. "Geodesics and curvature of semidirect product groups", *Rendiconti del Circolo Matematico di Palermo*, Serie II, Supplemento, (2001) vol.66 pp.199-206.

2) Galtier, S. "Wave turbulence in incompressible Hall magnetohydrodynamics" *J. Plasma Phys.*, (2006) vol.72, pp.721-769