§19. Conservation Laws for the Boltzmann-Poisson-Ampère System of Equations

Sugama, H., Watanabe, T.-H., Nunami, M.

In conventional works based on the Lagrangian/Hamiltonian gyrokinetic formulations [1-2], turbulent transport and conservation laws are investigated mainly for collisionless plasmas because the formulation and Noether's theorem are originally applicable to conservative systems without collisions. In order to investigate collisional (classical and neoclassical) and turbulent transport processes simultaneously based on the gyrokinetic model, it is important to clarify how collisions modify conservation laws for particles, momentum, and energy. In this work, we consider the Boltzmann-Poisson-Ampère system of equations, which provide the basis of approximate description by the collisional electromagnetic gyrokinetic system of equations for strongly magnetized plasmas, in order to show how conservation laws are modified by collisions and, if any, external sources.

Time evolution of the distribution function $f_a(\mathbf{x}, \mathbf{v}, t)$ for particle species a is described by the Boltzmann kinetic equation,

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \frac{e_a}{m_a}\left\{\mathbf{E}(\mathbf{x},t) + \frac{1}{c}\mathbf{v}\times\mathbf{B}(\mathbf{x},t)\right\}\cdot\frac{\partial}{\partial \mathbf{v}}\right] f_a(\mathbf{x},\mathbf{v},t) = \mathcal{K}_a(\mathbf{x},\mathbf{v},t), \tag{1}$$

where $\mathcal{K}_a(\mathbf{x}, \mathbf{v}, t)$ denotes the rate of change in the distribution function f_a due to Coulomb collisions and it may also include other parts representing external particle, momentum, and/or energy sources. The electromagnetic fields $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ are written as $\mathbf{E} = -\nabla\phi - c^{-1}\partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla \times \mathbf{A}$, where the electrostatic potential ϕ and the vector potential $\mathbf{A}$ are determined by Poisson's equation,

$$\nabla^2\phi(\mathbf{x},t) = -4\pi\sum_a e_a \int f_a(\mathbf{x},\mathbf{v},t)d^3\mathbf{v}, \tag{2}$$

and Ampère's law,

$$\nabla^2\mathbf{A}(\mathbf{x},t) = -\frac{4\pi}{c}\mathbf{j}_T, \tag{3}$$

respectively. Here, the Coulomb (or transverse) gauge condition $\nabla\cdot\mathbf{A} = 0$ is used and the current density $\mathbf{j} \equiv \sum_a e_a n_a \mathbf{u}_a \equiv \sum_a e_a \int f_a(\mathbf{x},\mathbf{v},t)\mathbf{v}d^3\mathbf{v}$ is written as $\mathbf{j} = \mathbf{j}_L + \mathbf{j}_T$, where the subscripts L and T represent the longitudinal and transverse parts, respectively. Equations (1)–(3) are the governing equations for the Boltzmann-Poisson-Ampère system.

Suppose that f_a, ϕ, and $\mathbf{A}$ which satisfy Eqs. (1)–(3) are given. Then, for the electromagnetic fields $\mathbf{E} = -\nabla\phi - c^{-1}\partial\mathbf{A}/\partial t = \mathbf{E}_L + \mathbf{E}_T$ and $\mathbf{B} = \nabla\times\mathbf{A}$ given from ϕ and $\mathbf{A}$, we consider a distribution function f_a^V to satisfy the Vlasov equation that is given from Eq. (1) with the right-hand side replaced by 0. We also assume f_a^V to coincide instantaneously with f_a at a given time t_0 so that $f_a^V(\mathbf{x},\mathbf{v},t_0) = f_a(\mathbf{x},\mathbf{v},t_0)$. Therefore, equations obtained from Eqs. (2) and (3) with f_a replaced by f_a^V also hold at t_0. In other words, f_a^V, ϕ, and $\mathbf{A}$ satisfy the Vlasov-Poisson-Ampère system of equations at t_0. Note that the Vlasov-Poisson-Ampère system of equations can be derived from the variational principle using the action integral defined by Eq. (1) in Ref.[3] where Noether's theorem is used to obtain conservation laws of energy and momentum. Then, expressing the action integral a infinitesimal time interval in terms of f_a instead of f_a^V, we can show effects of collisions and external sources on conservation laws for the Boltzmann-Poisson-Ampère system.

Defining the kinetic energy density and flux by $\mathcal{E}_p = \sum_a \int d^3\mathbf{v} f_a(\mathbf{x},\mathbf{v},t)\frac{1}{2}m_a|\mathbf{v}|^2$ and $\mathbf{Q}_p = \sum_a \int d^3\mathbf{v} f_a(\mathbf{x},\mathbf{v},t)\frac{1}{2}m_a|\mathbf{v}|^2\mathbf{v}$, respectively, the energy balance equation is derived as

$$\frac{\partial}{\partial t}\left(\mathcal{E}_p + \frac{|\mathbf{E}_L|^2 + |\mathbf{B}|^2}{8\pi}\right) + \nabla\cdot\left(\mathbf{Q}_p + \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} - \frac{1}{4\pi}\frac{\partial\phi}{\partial t}\mathbf{E}_T\right) = \sum_a \int d^3\mathbf{v}\,\mathcal{K}_a\frac{1}{2}m_a|\mathbf{v}|^2. \tag{4}$$

Next, the momentum balance equation is obtained as

$$\frac{\partial}{\partial t}(\mathbf{P}_p + \mathbf{P}_f) + \nabla\cdot(\mathbf{\Pi}_p + \mathbf{\Pi}_f) = \sum_a \int d^3\mathbf{v}\,\mathcal{K}_a m_a\mathbf{v}, \tag{5}$$

where the particle parts $\mathbf{P}_p$ and $\mathbf{\Pi}_p$ of the momentum density and the pressure tensor are defined by $\mathbf{P}_p = \sum_a \int d^3\mathbf{v} f_a(\mathbf{x},\mathbf{v},t)m_a\mathbf{v}$ and $\mathbf{\Pi}_p = \sum_a \int d^3\mathbf{v} f_a(\mathbf{x},\mathbf{v},t)m_a\mathbf{v}\mathbf{v}$, respectively, and the field parts $\mathbf{P}_f$ and $\mathbf{\Pi}_f$ are given by $\mathbf{P}_f = (\mathbf{E}_L\times\mathbf{B})/(4\pi c)$, and $\mathbf{\Pi}_f = (1/8\pi)(|\mathbf{E}_L|^2 + 2\mathbf{E}_L\cdot\mathbf{E}_T + |\mathbf{B}|^2)\mathbf{I} - (1/4\pi)(\mathbf{E}_L\mathbf{E}_L + \mathbf{E}_L\mathbf{E}_T + \mathbf{E}_T\mathbf{E}_L + \mathbf{B}\mathbf{B})$, respectively. If $\mathcal{K}_a$ is given by the Coulomb collision term only, the right-hand sides of Eqs. (4) and (5) vanish so that the energy and momentum balance equations for the Boltzmann-Poisson-Ampère system take the same forms as those for Vlasov-Poisson-Ampère system [3].

1) H. Sugama, T.-H. Watanabe, M. Nunami, Phys. Plasmas **21**, 012515 (2014).
2) H. Sugama, Phys. Plasmas **7**, 466 (2000).
3) H. Sugama, T.-H. Watanabe, M. Nunami, Phys. Plasmas **20**, 024503 (2013).