§18. On the Numerical Integral for the Trapped Particles in (ε, μ) Coordinates

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When (ε, μ) coordinates are used as the independent velocity variables of the gyrokinetic equation, the differential volume of the velocity space is,

$$d^3v = \pi \left(\frac{2}{m}\right)^{3/2} d\varepsilon B d\mu \frac{1}{\sqrt{\varepsilon - \mu B}} \frac{1}{2} \sum_{\sigma=\pm}, \qquad (1)$$

which is singular in the turning points of the trapped particles where $|v_{\|}| = \sqrt{(\varepsilon - \mu B)2/m} = 0$ holds. Here $\varepsilon = (1/2)mv^2$ and $\mu = mv_\perp^2/(2B)$. Since B includes the spatial dependence, the Jacobian does as well. In the ballooning representation, it is the θ.

As an example of the direct application of the velocity integral with Eq.(1), the normalized Maxwellian $F_M = (\sqrt{\pi} v_T)^{-3} e^{-\varepsilon/T}$ is integrated, and is shown as a function of θ in Fig.1. Given a constant ε and μ, the Jacobian is divergent near the points θ that satisfy $\varepsilon - \mu B(\theta) = 0$, and the integral curve along a field line shows the jagged spikes. The resulting total integral deviates from 1 everywhere. It can be seen that the number of spikes is depending on the number of discrete μ taken into account, and the sharpness of the spikes is depending on how the θ happens to be close to the turning points. Therefore these spikes are expected to be artificial, which is not avoidable by taking finer meshes.

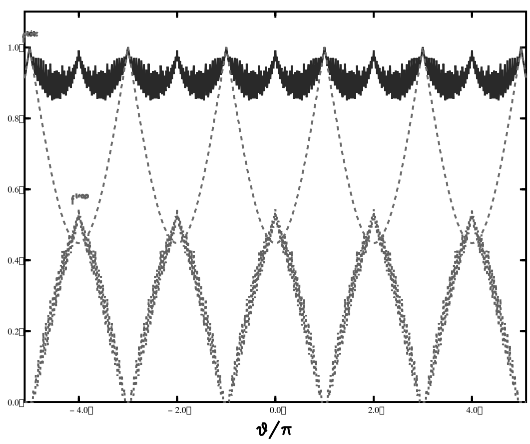

Fig. 1: An integral of the normalized Maxwellian of the passing (dashed line) and trapped (dotted line) parts and thier sum (solid line).

To estimate the improper integral accurately, following method is used. First, we consider a function of θ to be decomposed by the orthnormal polynomials. In particular, we consider

$$h_l(\theta) = (2^l l! \sqrt{\pi}/c)^{-1/2} H_l(c\theta) e^{-(c\theta)^2/2}, \qquad (2)$$

where H_l is a l-th Hermite polynomial and c is a scaling factor. Applying an integral operator $\int d\theta h_l$ for $l = 0, \cdots, l_{\max}$, to a velocity integral as,

$$I_l = \int d\theta h_l I(\theta), \qquad (3)$$

where $I(\theta) = \int d^3v f$, interchanging the order of the integral (first the θ integral and next the velocity integral), and finally the θ dependence is recovered by $I(\theta) = \sum_l h_l I_l$.

In these operation, our concern is treatment of the integral for the trapped particle. We consider an integral of the form,

$$\int_a^b d\theta d^3v W(\theta) f(\theta, \varepsilon, \mu), \qquad (4)$$

where $W = 1/\sqrt{(\theta - a)(b - \theta)}$, and a and b are turning points defined for fixed ε and μ. The abscissae and weights for the weight function can be found in the handbook [1] (In the actual application, one-sided improper Gaussian rule may be used for a and b seperately, and the intermediate domain is treated normally such as by the Simpson's rule). The point is that $\sqrt{(\theta - a)(b - \theta)/(\varepsilon - \mu B)}$ is usually regular, and the singularity is absorbed in the known weight function. This Gaussian quadrature technique along a field line is used in the FULL code [2], and is followed in the goblin code [3], so that these codes will be accurate in (ε, μ) formulation.

The resulting Maxwellian integral is shown in Fig.2. Except for the edge of the integral domain where the trapped particles are not treated completely, the integral is almost 1, as is desired.

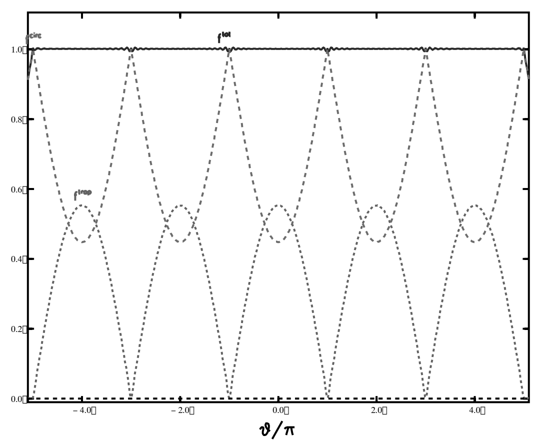

Fig. 2: An integral of the normalized Maxwellian by using Gaussian quadrature method.

1) M. Abramowiz and I.A. Stegun, Handbook of mathematical functions, Dover

2) G. Rewoldt, private communication

3) O. Yamagishi, et al., Physics of Plasmas, (2007)