§19. A Common Lagrangian Mechanical Structure of Dissipationless, Incompressible Fluid and Plasmas in Three-dimensional Space

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The dynamics of dissipationless, incompressible hydrodynamic (HD), magnetohydrodynamic (MHD), and Hall magnetohydrodynamic (HMHD) media, which are formulated as dynamical systems on some appropriate Lie groups equipped with Riemannian metrics, are considered from the Lagrangian mechanical viewpoint.

The key of the Lagrangian formalism on Lie groups is the appropriate choice of the inner product and the Lie bracket, which are respectively denoted by $\langle * \,|\, * \rangle$ and $[*, *]$ hereafter. Once these two mathematical structures are defined, the variational formulation is formally derived as follows.

First, the Lagrangian is defined using the inner product by $L = \frac{1}{2}\langle \dot\gamma(t) | \dot\gamma(t)\rangle_{\gamma(t)} = \frac{1}{2}\langle \boldsymbol{V}(t) | \boldsymbol{V}(t)\rangle_e$, where $\boldsymbol{V}(t)$ is the tangent vector of integral path, $\gamma(t)$.

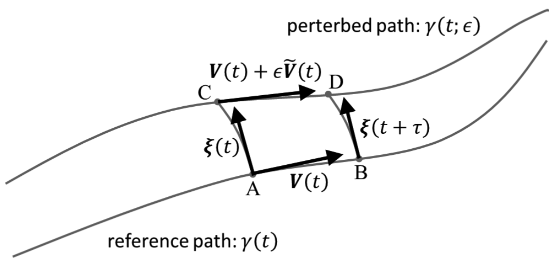

Fig. 1: Derivation of Lin constraints, i.e., the relation among the velocity, the displacement, and the perturbation part of the velocity associated with the variation of a path.

Next, the variation of the path, $\gamma(t;\epsilon)$, induces the perturbation on the velocity, say $\widetilde{\boldsymbol{V}}$: i.e, approximating the small segments by exponential map, the point D is evaluated as

$$\begin{aligned}\gamma(t+\tau;\epsilon) &\approx \exp[\tau(\boldsymbol{V}(t) + \epsilon\widetilde{\boldsymbol{V}}(t))] \circ \gamma(t;\epsilon) \\ &\approx e^{\epsilon\boldsymbol{\xi}(t+\tau)} \circ e^{\tau\boldsymbol{V}(t)} \circ e^{-\epsilon\boldsymbol{\xi}(t)} \circ \gamma(t;\epsilon)\end{aligned}$$

(see Fig. 1). Applying Hausdorff's formula and comparing the $O(\epsilon\tau)$ terms, we obtain the following relation among $\boldsymbol{V}$, $\boldsymbol{\xi}$, and $\widetilde{\boldsymbol{V}}$:

$$\widetilde{\boldsymbol{V}} = \dot{\boldsymbol{\xi}} + [\boldsymbol{\xi}, \boldsymbol{V}], \tag{1}$$

which is known as *Lin constraints*.

Thus, the first variation of the action becomes

$$\begin{aligned}\left.\frac{\partial S_\epsilon}{\partial \epsilon}\right|_{\epsilon=0} &= \int_0^1 \mathrm{d}t \langle \boldsymbol{V} | \widetilde{\boldsymbol{V}}\rangle = \int_0^1 \mathrm{d}t \langle \boldsymbol{V} | \dot{\boldsymbol{\xi}} + [\boldsymbol{\xi}, \boldsymbol{V}]\rangle \\ &= \langle \boldsymbol{V} | \boldsymbol{\xi}\rangle\Big|_{t=0}^{t=1} - \int_0^1 \mathrm{d}t \langle \dot{\boldsymbol{V}} | \boldsymbol{\xi}\rangle + \int_0^1 \mathrm{d}t \langle \mathrm{ad}^\dagger_{\boldsymbol{V}} \boldsymbol{V} | \boldsymbol{\xi}\rangle,\end{aligned}$$

where $\mathrm{ad}^\dagger$ is the dual operator of the Lie bracket with respect to the inner product: $\langle \mathrm{ad}^\dagger_{\boldsymbol{c}} \boldsymbol{a} | \boldsymbol{b}\rangle := \langle \boldsymbol{a} | [\boldsymbol{b}, \boldsymbol{c}]\rangle$. Using Hamilton's principle $(\partial S_\epsilon/\partial\epsilon)_{\epsilon=0} = 0$, and $\boldsymbol{\xi} = 0$ at $t = 0$, 1, we obtain the Euler-Poincare equation,

$$\dot{\boldsymbol{V}} = \mathrm{ad}^\dagger_{\boldsymbol{V}} \boldsymbol{V}, \tag{2}$$

as the Euler-Lagrange equation.

Note that, once we find such a variable $\boldsymbol{\xi}$ that satisfies $\dot{\boldsymbol{\xi}} + [\boldsymbol{\xi}, \boldsymbol{V}] = 0$ (i.e. $\widetilde{\boldsymbol{V}} = 0$) along the solution path to the Euler-Lagrange equation (2), we obtain the conservation law $\langle \boldsymbol{V} | \boldsymbol{\xi}\rangle_{t=0} = \langle \boldsymbol{V} | \boldsymbol{\xi}\rangle_{t=1}$ as Noether's first theorem

For the HMHD case, the generalized velocity is the pair of the ion velocity and current fields, $\vec{\boldsymbol{V}} = (\boldsymbol{V}, -\alpha\boldsymbol{J})$. The Riemannian metric and the Lie bracket for the HMHD system are given by

$$\begin{aligned}\langle \vec{\boldsymbol{V}}_1 | \vec{\boldsymbol{V}}_2\rangle &= \int_{\vec{x}\in M} \mathrm{d}^3\vec{x} \Big(\boldsymbol{V}_1 \cdot \boldsymbol{V}_2 + \boldsymbol{B}_1 \cdot \boldsymbol{B}_2\Big), & (3) \\ [\vec{\boldsymbol{V}}_1, \vec{\boldsymbol{V}}_2] &= \Big(\nabla \times (\boldsymbol{V}_1 \times \boldsymbol{V}_2), -\alpha\nabla \times (\boldsymbol{V}_1 \times \boldsymbol{J}_2 \\ &\qquad + \boldsymbol{J}_1 \times \boldsymbol{V}_2 - \alpha\boldsymbol{J}_1 \times \boldsymbol{J}_2)\Big), & (4)\end{aligned}$$

where $\boldsymbol{B}$ is the magnetic field induced by the current field $\boldsymbol{J}$ ($\boldsymbol{J} = \nabla \times \boldsymbol{B}$, $\nabla \cdot \boldsymbol{B} = 0$). According to the procedure described above, we obtain the Euler-Lagrange equations for these structures as follows:

$$\begin{aligned}\partial_t \boldsymbol{V} + \boldsymbol{\Omega} \times \boldsymbol{V} + \boldsymbol{B} \times \boldsymbol{J} &= -\nabla P, & (5) \\ \partial_t \boldsymbol{A} + \boldsymbol{B} \times \boldsymbol{V} - \alpha\boldsymbol{B} \times \boldsymbol{J} &= -\nabla\phi, & (6)\end{aligned}$$

where $\boldsymbol{\Omega}$, P, ϕ are the vorticity ($\boldsymbol{\Omega} = \nabla\times\boldsymbol{V}$), generalized pressure, and the scalar potential of electro-magnetic field, respectively. Taking the curl of (5) and (6) and doing some manipulations, we find that the $\vec{\boldsymbol{V}}$-variable

$$\vec{\boldsymbol{\Omega}} = C_C \begin{pmatrix} \alpha\boldsymbol{\Omega} + \boldsymbol{B} \\ -\alpha\boldsymbol{\Omega} \end{pmatrix} - C_M \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{B} \end{pmatrix} \tag{7}$$

(where C_C, and C_M are arbitrary constants) satisfies $\partial_t\vec{\boldsymbol{\Omega}} + [\vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{V}}] = \vec{\boldsymbol{0}}$, i.e., the variation in the direction of $\vec{\boldsymbol{\Omega}}$ yields the conservation law of Noether's first theorem. The associated constants of motion is

$$H = C_C \int [\alpha\boldsymbol{V} \cdot \boldsymbol{\Omega} + 2\boldsymbol{V} \cdot \boldsymbol{B}]\mathrm{d}^3\vec{x} + \frac{C_M}{\alpha} \int \boldsymbol{A} \cdot \boldsymbol{B}\mathrm{d}^3\vec{x},$$

which is the linear combination of the hybrid helicity, H_H, and the magnetic heliccity, H_M: $H = \alpha^{-1}(C_C H_H + (C_M - C_C)H_M)$.

Similar mathematical structures are found for the HD and MHD cases.[1)]

1) K. Araki, "Particle relabeling symmetry, generalized vorticity, and normal-mode expansion of ideal, incompressible fluids and plasmas in three-dimensional space", (2016) arXiv:1601.05477