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巻頭言

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本論文は、2016 年秋に逝去された数理情報研究所の吉村和美氏によるものである。本論文の投稿手続き中に亡くなられたため、その後ご家族の同意のもと下妻隆（NIFS）及び学部・大学の同窓生（鈴木孝利、上原和也、三尾幸治）で、論文原稿の主張点を保持して、発表できる形に校正を加えたものである。

概要は下記のとおりである。

流体の流れなどの観測にレーザーシートを照射する場合がある。この論文では 2 成分混合分子気体にレーザーシートなどを照射して発生する擾乱によって生ずる非線形波動を表す式を、流体の基礎方程式から出発して逓減摂動法により求めている。　その結果気体分子密度の非線形変調波動が、超電導他、種々の物理事象を記述することで知られる Ginzburg-Landau(GL)方程式で表されることを示している。また、GL 方程式の一つの解である、非線形孤立波の発生条件を求め、数値計算によりその挙動を調べている。本論文は、この孤立波だけでなく、2 成分混合気体の GL 方程式であらわされる他の様々な事象を検討する道を開いたことで意味あるものと考えられる。今後、本論文が関連分野の研究の参考になれば幸いである。

Nonlinear Wave Propagations in Binary-Gas Mixture

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Nonlinear wave propagations in binary-gas mixture in one-dimensional half-space $[0,\infty)$ are discussed, based on two Boltzmann equations. The nonlinear integro-partial differential equations are derived applying the reductive perturbation method to the three conservation equations with relaxation terms and the equation of state. Further, the Ginzburg-Landau equation for the gas densities is derived using Fourier perturbed expansion from the nonlinear integro-partial differential equations. Based on the Ginzburg-Landau equation derived, the necessary conditions and the numerical calculation results for the stationary propagating nonlinear waves in binary-gas mixture are discussed.

Key word: Binary-Gas Mixture, Stationary Wave Propagation, Boltzmann Equation, Ginzburg-Landau Equation, Reductive Perturbation Method, Fourier Perturbed expansion method, Solitary wave

1. Introduction

In this paper, the propagation of disturbance in a binary gas-mixture is discussed based on the fluid mechanics. The fluid mechanical equations for a binary gas-mixture where the temperatures of both gases are equal are known [1]. The fluid mechanical equations for the binary gas mixture in which the temperatures as well as the velocities, the densities, the pressures and the stresses are not equal between the gas components have been obtained [2], [3].

Here the nonlinear wave propagation phenomena in binary-gas mixture are further studied with theoretical and numerical standpoints. In section 2, the basic equations for binary-gas mixture are presented. In section 3, the nonlinear integro-partial differential equation for a gas density is derived applying the reductive perturbation method to the basic equations. In section 4, in order to theoretically analyze the equations derived, the Fourier expansion method is applied to them. Then the equation derived finally is the Ginzburg-Landau (GL) equation for the gas density. In section 5, based on the known solution [4] of the GL equations the necessary conditions for stationary propagating waves in binary gas-mixture are discussed. In section 6, the stationary wave propagation by the local disturbance is numerically investigated based on the GL equations under the condition discussed in section 5.

2. Basic Equations

One dimensional fluid motion of binary-gas mixture may be described by the following basic equations based on the Boltzmann equations [2]:

$$\frac{\partial n^{(i)}}{\partial t}+\frac{\partial n^{(i)}v^{(i)}}{\partial x}=0. \tag{2.1}$$

$$m^{(i)}n^{(i)}\left(\frac{\partial v^{(i)}}{\partial t}+v^{(i)}\frac{\partial v^{(i)}}{\partial x}\right)+\frac{\partial p_{11}^{(i)}}{\partial x}-\eta^{(ij)}\left(v^{(j)}-v^{(i)}\right)=0. \tag{2.2}$$

$$\frac{3k_B}{2}n^{(i)}\left(\frac{\partial T^{(i)}}{\partial t}+v^{(i)}\frac{\partial T^{(i)}}{\partial x}\right)-\frac{\partial q^{(i)}}{\partial x}+p_{11}^{(i)}\frac{\partial v^{(i)}}{\partial x}=\kappa^{(ij)}k_B\left(T^{(j)}-T^{(i)}\right)+M^{(j)}\eta^{(ij)}\left(v^{(j)}-v^{(i)}\right)^2. \tag{2.3}$$

Equations (2.1), (2.2) and (2.3) are the continuity equation for gas molecule i (i = 1 or 2, i $\neq$ j), the motion equation for the gas molecules and the energy conservation equation, respectively; $n^{(i)}$ denotes the density of gas molecule i, $v^{(i)}$ the average velocity of gas molecule i, $m^{(i)}$ the mass of gas molecule i, $p_{11}^{(i)}$ the momentum and $q^{(i)}$ heat flux are described as follows [1],

$$p_{11}^{(i)}=p^{(i)}-2\mu_V^{(i)}\frac{\partial v^{(i)}}{\partial x},\quad q^{(i)}=\lambda^{(i)}\frac{\partial T^{(i)}}{\partial x}$$

where the coefficients $\mu_V^{(i)}$ and $\lambda^{(i)}$ are the proportional constants of the velocity gradient and the heat gradient respectively. k_B Boltzmann constant, $T^{(i)}$ the temperature of gas molecule i, $M^{(j)}=m^{(j)}/m_0$, $m_0=m^{(i)}+m^{(j)}$.

The velocity dissipation $\eta^{(ij)}$ and the heat dissipation $\kappa^{(ij)}$ between the different gas components are expressed as follows,

$$\eta^{(ij)}(x,t)=\frac{2m^{(i)}m^{(j)}}{m_0}n^{(i)}(x,t)n^{(j)}(x,t)S_\sigma^{(ij)}(x,t), \tag{2.4}$$

$$\kappa^{(ij)}(x,t)=\frac{2N^{(i)}m^{(i)}m^{(j)}}{m_0^{\,2}}n^{(i)}(x,t)n^{(j)}(x,t)S_\sigma^{(ij)}(x,t), \tag{2.5}$$

$$n^{(i)}(x,t)n^{(j)}(x,t)S_\sigma^{(ij)}(x,t)=\int f^{(i)}\left(\tilde{\mathbf{c}}^{(i)'},x,t\right)f^{(j)}\left(\tilde{\mathbf{c}}^{(j)'},x,t\right)k_C^{(ij)}d^2\boldsymbol{k}d^3\tilde{\mathbf{c}}^{(j)'}d^3\tilde{\mathbf{c}}^{(i)'}. \tag{2.6}$$

where $k_C^{(ij)}$ is written as

$$k_C^{(ij)}\left(b,\left|\mathbf{c}^{(j)}-\mathbf{c}^{(i)}\right|\right)=\left|\mathbf{c}^{(j)}-\mathbf{c}^{(i)}\right|\frac{b}{\sin\chi^{(ij)}}\left|\frac{\partial b}{\partial\chi^{(ij)}}\right|, \tag{2.7}$$

$b, \chi^{(ij)}$ and $g=\left|\mathbf{c}^{(j)}-\mathbf{c}^{(i)}\right|$ are respectively the impact parameter, the deflection angle and the relative velocity between $i-$ and $j-$component molecules. $\mathbf{c}^{(i)}$ denotes the velocity of gas molecule i. $S_\sigma^{(ij)}$ denotes the total cross-section of the collision for different component gases.

The total collision cross-section $S_\sigma^{(ij)}$ may be estimated by the interactive potential energy $U^{(ij)}=\kappa_a/r^4-\kappa_b/r^2$ where κ_a, κ_b are respectively the potential coefficients of the repulsive and attractive forces [5]. The interactive potential energies $U^{(ii)}, U^{(jj)}$ between same components gases also exist.

2.1 *Total Collision Cross-section*

The deflection angle $\chi^{(ij)}$ is obtained by the orbit integration, that is,

$$\chi^{(ij)}=\pi-2\int_{r_m}^{\infty}\frac{dr}{\sqrt{\left(r^2/b\right)^2\left(1-U^{(ij)}/E^{(i)}\right)-r^2}} \tag{2.8}$$

Where the total energy $E^{(i)}$ of a molecule is $\mu g^2/2$ and r_m is the most neighboring distance between two molecules. $\mu=m^{(i)}m^{(j)}/m_0$ is the effective mass of binary gas and g is the relative velocity expressed by the average of thermal velocity and the relative velocity as follows[6],

$$g=\sqrt{\frac{1}{n^{(i)}n^{(j)}}\int\left(c^{(j)}-c^{(i)}\right)^2 f^{(i)}f^{(i)}dc^{(j)}dc^{(i)}}=\sqrt{3k_B\left(T^{(i)}/m^{(i)}+T^{(j)}/m^{(j)}\right)+\left(v^{(j)}-v^{(i)}\right)^2}\ . \tag{2.9}$$

Putting $y=b/r$, then the range $\left[r_m,\infty\right]$ of r integration are transformed into $\left[y_0,0\right]$ of y integration. Furthermore, we introduce a variable u for simplifying of integrations as follows, $u=\sqrt{1-2\alpha_0\left(y^2/\nu^2\right)}$, using parameter as follows $\alpha_0=2\kappa_a/\mu g^2 b^4$, $\beta_0=2\kappa_b/\mu g^2 b^2$,

$$\lambda^2=\sqrt{\left(1-\beta_0\right)^2+4\alpha_0}+1-\beta_0,\ \nu^2=\sqrt{\left(1-\beta_0\right)^2+4\alpha_0}-\left(1-\beta_0\right), \tag{2.10}$$

and $k_0{}^2=\nu^2/\left(\lambda^2+\nu^2\right)$, where the ending point y_0 of the y integration is transformed to $u_0=\sqrt{1-2\alpha_0\left(y_0^2/\nu^2\right)}$ of the u integration. The distance r between two molecule centers become most neighbor r_m when the differential distance to the angle $\varphi=\left(\pi-\chi^{(ij)}\right)/2$ between the direction of the relative velocity and apse line becomes zero, that is, $\partial r/\partial\varphi = 0$. Because, in that case, $u_0 = 0$, the first kind of Legendre-Jacobi's elliptic integration is zero

$$\gamma^{(ij)} \equiv \int_0^{u_0} du\Big/\left(\sqrt{1-k_0^2u^2}\sqrt{1-u^2}\right) = 0. \tag{2.11}$$

Therefore, we obtain the integration of the deflection angle as

$$\chi^{(ij)} = \pi - 2\sqrt{2}K(k_0)\Big/\sqrt{\lambda^2+\nu^2}\,, \tag{2.12}$$

where K is the first kind of the complete elliptic integration,

$$K(k_0) = \int_0^1 du\Big/\left(\sqrt{1-k_0^2u^2}\sqrt{1-u^2}\right). \tag{2.13}$$

In the case of $k_0 \ll 1$, then $K(k_0) \approx (\pi^2/4)(1+(1/2)k_0^2+\cdots)$.

Consequently, when $\chi^{(ij)} = 0$, then

$$b(0)^2 \approx \frac{\kappa_a}{2\kappa_b}\left(1+\frac{1}{s_d}+s_d\right)+\frac{8}{5}\left(1-\frac{4}{s_d}\right)\frac{\kappa_b}{\mu g^2}+\frac{1}{s_d}\frac{2^3\kappa_b^{\,3}}{25\kappa_a\mu^2g^4}. \tag{2.14}$$

In the case of $\chi^{(ij)} = \pi$, $b(\pi)^2 = 0$. Then the total collision cross-section may be estimated as

$$\begin{aligned} S_\sigma^{(ij)} &= g\Omega = g\int_0^{2\pi}\int_0^{\pi}\left|b\frac{db}{d\chi^{(ij)}}\right|d\chi^{(ij)}d\varepsilon^{(ij)} = 2\pi g\left|\frac{1}{2}\left(b(\pi)^2-b(0)^2\right)\right| \\ &\approx \pi\left\{\frac{\kappa_a}{2\kappa_b}\left(1+\frac{1}{s_d}+s_d\right)g+\left(1-\frac{4}{s_d}\right)\frac{8\kappa_b}{5\mu g}+\frac{8\kappa_b^{\,3}}{25\kappa_a s_d\mu^2g^3}\right\}. \end{aligned} \tag{2.15}$$

where $\varepsilon^{(ij)}$ is deflection angle viewed from i – component molecule and,

$$s_d = \left(1+\frac{32\kappa_b^{\,2}}{\kappa_a\mu g^2}-\frac{154\times4\kappa_b^{\,4}}{5^2\kappa_a^{\,2}\mu^2g^4}-\frac{64\kappa_b^{\,6}}{5^3\kappa_a^{\,3}\mu^3g^6}+\frac{8}{5}\sqrt{\frac{\kappa_b^{\,2}}{\kappa_a\mu g^2}\left(2+\frac{\kappa_b^{\,2}}{\kappa_a\mu g^2}\right)^3\left(5+\frac{16\kappa_b^{\,2}}{\kappa_a\mu g^2}\right)}\right)^{\frac{1}{3}}. \tag{2.16}$$

3. Derivation of Nonlinear Integro-Partial Differential Equation by Reductive Perturbation Method

The propagation of disturbance in a binary gas-mixture could be expressed as slowly varying phenomena when it would be observed on coordinate moving with near velocity of the disturbance propagation. Then, in the two component gases, the coordinates (ζ,τ) moving on the velocity V is introduced using an parameter ε (<1) as follows,

$$\zeta = \varepsilon(x-Vt),\quad \tau = \varepsilon^2 t \tag{3.1}$$

Then the spatial and time derivative of phenomena are expressed as

$$\frac{\partial}{\partial x} = \varepsilon \frac{\partial}{\partial \zeta}, \quad \frac{\partial}{\partial t} = \varepsilon^2 \frac{\partial}{\partial \tau} - \varepsilon V \frac{\partial}{\partial \zeta} \tag{3.2}$$

The physical variable $n^{(i)}, v^{(i)}, T^{(i)}$ are expanded with a perturbation parameter ε as follows,

$$n^{(i)} = n_0^{(i)} + \varepsilon n_1^{(i)} + \varepsilon^2 n_2^{(i)} + \varepsilon^3 n_3^{(i)} + \varepsilon^4 n_4^{(i)} + \cdots \tag{3.3}$$

$$v^{(i)} = v_0^{(i)} + \varepsilon v_1^{(i)} + \varepsilon^2 v_2^{(i)} + \varepsilon^3 v_3^{(i)} + \varepsilon^4 v_4^{(i)} + \cdots \tag{3.4}$$

$$T^{(i)} = T_0^{(i)} + \varepsilon T_1^{(i)} + \varepsilon^2 T_2^{(i)} + \varepsilon^3 T_3^{(i)} + \varepsilon^4 T_4^{(i)} + \cdots \tag{3.5}$$

Where $n_0^{(i)} and T_0^{(i)}$ are the spatial uniform constant values. As a binary-gas is initially stationary state, $v_0^{(i)} = 0$. The relative velocity (2.9) is expanded as,

$$g^2 = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \varepsilon^3 a_3 + \varepsilon^4 a_4 \cdots, \tag{3.6}$$

where the coefficients $a_0, a_1, a_2, a_3, \cdots$ are expressed by $v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \cdots$ and $T_0^{(i)}, T_1^{(i)}, T_2^{(i)}, T_3^{(i)}, \cdots$. These values are symmetric for the gas component i and j which can be replaced each other in the equations.. The total collision cross-section are also expanded as

$$S_\sigma^{(ij)} = S_{\sigma 0}^{(ij)} + \varepsilon S_{\sigma 1}^{(ij)} + \varepsilon^2 S_{\sigma 2}^{(ij)} + \varepsilon^3 S_{\sigma 3}^{(ij)} + \varepsilon^4 S_{\sigma 4}^{(ij)} + \cdots. \tag{3.7}$$

where $S_{\sigma 0}^{(ij)}, S_{\sigma 1}^{(ij)}, S_{\sigma 2}^{(ij)}, S_{\sigma 3}^{(ij)}, \cdots$, are expressed by $a_0, a_1, a_2, a_3, \cdots$ and $n^{(i)}$.

Introducing the above equations (3.1) ~ (3.7) into the basic equations (2.1) ~ (2.3), the following sets of equations for each power of ε are derived.

3.1. Each ε power equation for equation (2.1)

$$\varepsilon^1 : -V \frac{\partial n_0^{(i)}}{\partial \zeta} + \frac{\partial}{\partial \zeta}\left(n_0^{(i)} v_0^{(i)}\right) = 0, \tag{3.8}$$

$$\varepsilon^2 : \frac{\partial n_0^{(i)}}{\partial \tau} - V \frac{\partial n_1^{(i)}}{\partial \zeta} + \frac{\partial}{\partial \zeta}\left(n_1^{(i)} v_0^{(i)} + n_0^{(i)} v_1^{(i)}\right) = 0, \tag{3.9}$$

$$\varepsilon^3 : \frac{\partial n_1^{(i)}}{\partial \tau} - V \frac{\partial n_2^{(i)}}{\partial \zeta} + \frac{\partial}{\partial \zeta}\left(n_2^{(i)} v_0^{(i)} + n_1^{(i)} v_1^{(i)} + n_0^{(i)} v_2^{(i)}\right) = 0, \tag{3.10}$$

$$\varepsilon^4 : \frac{\partial n_2^{(i)}}{\partial \tau} - V \frac{\partial n_3^{(i)}}{\partial \zeta} + \frac{\partial}{\partial \zeta}\left(n_3^{(i)} v_0^{(i)} + n_2^{(i)} v_1^{(i)} + n_1^{(i)} v_2^{(i)} + n_0^{(i)} v_3^{(i)}\right) = 0. \tag{3.11}$$

3.2. Each ε power equation for equation (2.2)

Where due to simplify the expression, we use the notation $\delta v_m^{(ji)} \equiv v_m^{(j)} - v_m^{(i)}$, and $\delta T_m^{(ji)} \equiv T_m^{(j)} - T_m^{(i)}$.

$$\varepsilon^0 : -2\mu S_{\sigma 0}^{(ij)} n_0^{(i)} n_0^{(j)} \delta v_0^{(ji)} = 0 \quad , \quad (3.12)$$

$$\varepsilon^1 : \frac{1}{2} m^{(i)} n_0^{(i)} \frac{\partial v_0^{(i)2}}{\partial \zeta} - V m^{(i)} n_0^{(i)} \frac{\partial v_0^{(i)}}{\partial \zeta} + k_B \frac{\partial}{\partial \zeta}\left(n_0^{(i)} T_0^{(i)}\right) - 2\mu S_{\sigma 0}^{(ij)} n_0^{(i)} n_0^{(j)} \delta v_1^{(ji)} = 0, \quad (3.13)$$

$$\varepsilon^2 : m^{(i)} n_0^{(i)} \left(v_0^{(i)} - V\right) m^{(i)} n_0^{(i)} \frac{\partial v_1^{(i)}}{\partial \zeta} + k_B \frac{\partial}{\partial \zeta}\left(n_0^{(i)} T_1^{(i)} + n_1^{(i)} T_0^{(i)}\right) - 2\mu S_{\sigma 0}^{(ij)} n_0^{(i)} n_0^{(j)} \delta v_2^{(ji)} = 0, \quad (3.14)$$

$$\varepsilon^3 :$$

$$m^{(i)} n_0^{(i)} \frac{\partial v_1^{(i)}}{\partial \tau} - V m^{(i)} \left(n_0^{(i)} \frac{\partial v_2^{(i)}}{\partial \zeta} + n_1^{(i)} \frac{\partial v_1^{(i)}}{\partial \zeta} \right) + \frac{1}{2} m^{(i)} \left(n_0^{(i)} \frac{\partial}{\partial \zeta}\left(v_1^{(i)2} + 2 v_0^{(i)} v_2^{(i)}\right) + 2 n_1^{(i)} \frac{\partial}{\partial \zeta}\left(v_0^{(i)} v_1^{(i)}\right)\right)$$

$$+ k_B \frac{\partial}{\partial \zeta}\left(n_0^{(i)} T_2^{(i)} + n_1^{(i)} T_1^{(i)} + n_2^{(i)} T_0^{(i)}\right) - 2\mu_V^{(i)} \frac{\partial^2 v_1^{(i)}}{\partial \zeta^2}$$

$$-2\mu \left\{ S_{\sigma 0}^{(ij)} \left(n_0^{(i)} n_0^{(j)} \delta v_3^{(ji)} + \left(n_0^{(i)} n_1^{(j)} + n_1^{(i)} n_0^{(j)} \right) \delta v_2^{(ji)} \right) + S_{\sigma 1}^{(ij)} n_0^{(i)} n_0^{(j)} \delta v_2^{(ji)} \right\} = 0, \quad (3.15)$$

$$\varepsilon^4 : m^{(i)} \left(n_0^{(i)} \frac{\partial v_2^{(i)}}{\partial \tau} + n_1^{(i)} \frac{\partial v_1^{(i)}}{\partial \tau} \right) - V m^{(i)} \left(n_0^{(i)} \frac{\partial v_3^{(i)}}{\partial \zeta} + n_1^{(i)} \frac{\partial v_2^{(i)}}{\partial \zeta} + n_2^{(i)} \frac{\partial v_1^{(i)}}{\partial \zeta} \right)$$

$$+ \frac{1}{2} m^{(i)} \left(n_0^{(i)} \frac{\partial}{\partial \zeta}\left(2 v_1^{(i)} v_2^{(i)} + 2 v_0^{(i)} v_3^{(i)}\right) + n_1^{(i)} \frac{\partial}{\partial \zeta}\left(v_1^{(i)2} + 2 v_0^{(i)} v_2^{(i)}\right) + 2 n_2^{(i)} \frac{\partial}{\partial \zeta}\left(v_0^{(i)} v_1^{(i)}\right) \right)$$

$$+ k_B \frac{\partial}{\partial \zeta}\left(n_0^{(i)} T_3^{(i)} + n_1^{(i)} T_2^{(i)} + n_2^{(i)} T_1^{(i)} + n_3^{(i)} T_0^{(i)}\right) - 2\mu_V^{(i)} \frac{\partial^2 v_2^{(i)}}{\partial \zeta^2}$$

$$-2\mu \left\{ S_{\sigma 0}^{(ij)} \left(n_0^{(i)} n_0^{(j)} \delta v_4^{(ji)} + \left(n_0^{(i)} n_1^{(j)} + n_1^{(i)} n_0^{(j)} \right) \delta v_3^{(ji)} + \left(n_0^{(i)} n_2^{(j)} + n_1^{(i)} n_1^{(j)} + n_2^{(i)} n_0^{(j)} \right) \delta v_2^{(ji)} \right) \right.$$

$$\left. + S_{\sigma 1}^{(ij)} \left(n_0^{(i)} n_0^{(j)} \delta v_3^{(ji)} + \left(n_0^{(i)} n_1^{(j)} + n_1^{(i)} n_0^{(j)} \right) \delta v_2^{(ji)} \right) + S_{\sigma 2}^{(ij)} n_0^{(i)} n_0^{(j)} \delta v_2^{(ji)} \right\} = 0. \quad (3.16)$$

Where, $\delta v_m^{(ji)} \equiv v_m^{(j)} - v_m^{(i)}$ and $\delta T_m^{(ji)} \equiv T_m^{(j)} - T_m^{(i)}$ (m = 0, 1. 2, 3. ..).

3.3. Each ε power equation for equation (2.3)

$$\varepsilon^0 : (4\mu/m_0) S_{\sigma 0}^{(ij)} n_0^{(i)} n_0^{(j)} \delta T_0^{(ji)} = 0, \quad (3.17)$$

$$\varepsilon^1 : -\frac{4\mu}{m_0} S_{\sigma 0}^{(ij)} n_0^{(i)} n_0^{(j)} \delta T_1^{(ji)} = 0, \quad (3.18)$$

ε^2 :

$$n_0^{(i)}\left(v_0^{(i)}-V\right)\frac{\partial T_1^{(i)}}{\partial\zeta}+\frac{2}{3}n_0^{(i)}T_0^{(i)}\frac{\partial v_1^{(i)}}{\partial\zeta}-\left(4\mu/m_0\right)S_{\sigma 0}^{(ij)}\left(n_0^{(i)}n_0^{(j)}\delta T_2^{(ji)}+\left(n_0^{(i)}n_1^{(j)}+n_1^{(i)}n_0^{(j)}\right)\delta T_1^{(ji)}\right)=0, \tag{3.19}$$

$$\varepsilon^3: n_0^{(i)}\frac{\partial T_1^{(i)}}{\partial\tau}-V\left(n_0^{(i)}\frac{\partial T_2^{(i)}}{\partial\zeta}+n_1^{(i)}\frac{\partial T_1^{(i)}}{\partial\zeta}\right)+\left(n_0^{(i)}v_1^{(i)}+n_1^{(i)}v_0^{(i)}\right)\frac{\partial T_1^{(i)}}{\partial\zeta}+n_0^{(i)}v_0^{(i)}\frac{\partial T_2^{(i)}}{\partial\zeta}$$
$$-\frac{2\lambda^{(i)}}{3k_B}\frac{\partial^2 T_1^{(i)}}{\partial\zeta^2}+\frac{2}{3}\left(\left(n_0^{(i)}T_1^{(i)}+n_1^{(i)}T_0^{(i)}\right)\frac{\partial v_1^{(i)}}{\partial\zeta}+n_0^{(i)}T_0^{(i)}\frac{\partial v_2^{(i)}}{\partial\zeta}\right)$$
$$-\frac{4\mu}{m_0}\left\{S_{\sigma 0}^{(ij)}\left(n_0^{(i)}n_0^{(j)}\delta T_3^{(ji)}+\left(n_0^{(i)}n_1^{(j)}+n_1^{(i)}n_0^{(j)}\right)\delta T_2^{(ji)}\right)+S_{\sigma 1}^{(ij)}n_0^{(i)}n_0^{(j)}\delta T_2^{(ji)}\right\}=0, \tag{3.20}$$

$$\varepsilon^4: n_0^{(i)}\frac{\partial T_2^{(i)}}{\partial\tau}+n_1^{(i)}\frac{\partial T_1^{(i)}}{\partial\tau}-V\left(n_0^{(i)}\frac{\partial T_3^{(i)}}{\partial\zeta}+n_1^{(i)}\frac{\partial T_2^{(i)}}{\partial\zeta}+n_2^{(i)}\frac{\partial T_1^{(i)}}{\partial\zeta}\right)$$
$$+\left(n_0^{(i)}v_2^{(i)}+n_1^{(i)}v_1^{(i)}+n_2^{(i)}v_0^{(i)}\right)\frac{\partial T_1^{(i)}}{\partial\zeta}+\left(n_0^{(i)}v_1^{(i)}+n_1^{(i)}v_0^{(i)}\right)\frac{\partial T_2^{(i)}}{\partial\zeta}+n_0^{(i)}v_0^{(i)}\frac{\partial T_3^{(i)}}{\partial\zeta}-\frac{2\lambda^{(i)}}{3k_B}\frac{\partial^2 T_2^{(i)}}{\partial\zeta^2}$$
$$+\frac{2}{3}\left(\left(n_0^{(i)}T_2^{(i)}+n_1^{(i)}T_1^{(i)}+n_2^{(i)}T_0^{(i)}\right)\frac{\partial v_1^{(i)}}{\partial\zeta}+\left(n_0^{(i)}T_1^{(i)}+n_1^{(i)}T_0^{(i)}\right)\frac{\partial v_2^{(i)}}{\partial\zeta}+n_0^{(i)}T_0^{(i)}\frac{\partial v_3^{(i)}}{\partial\zeta}\right)-\frac{4\mu_V^{(i)}}{3k_B}\left(\frac{\partial v_1^{(i)}}{\partial\zeta}\right)^2$$
$$-\left(4\mu/m_0\right)\left\{S_{\sigma 0}^{(ij)}\left(n_0^{(i)}n_0^{(j)}\delta T_4^{(ji)}+\left(n_0^{(i)}n_1^{(j)}+n_1^{(i)}n_0^{(j)}\right)\delta T_3^{(ji)}\right.\right.$$
$$\left.+\left(n_0^{(i)}n_2^{(j)}+n_1^{(i)}n_1^{(j)}+n_2^{(i)}n_0^{(j)}\right)\delta T_2^{(ji)}+\frac{m^{(j)}}{3k_B}n_0^{(i)}n_0^{(j)}\delta v_2^{(ji)2}\right)$$
$$\left.+S_{\sigma 1}^{(ij)}\left(n_0^{(i)}n_0^{(j)}\delta T_3^{(ji)}+\left(n_0^{(i)}n_1^{(j)}+n_1^{(i)}n_0^{(j)}\right)\delta T_2^{(ji)}\right)+S_{\sigma 2}^{(ij)}n_0^{(i)}n_0^{(j)}\delta T_2^{(ji)}\right\}=0. \tag{3.21}$$

3.4. Physical Variable of ε^0 and ε^1 order Approximation

The notation of $V^{(i)}\equiv v_0^{(i)}-V$ and $V^{(j)}\equiv v_0^{(j)}-V$ are introduced to simplify the expression. Integrating the first ε^1 order term of equation (3.8), $n_0^{(i)}V^{(i)}=const$. As $n_0^{(i)}$ and V are assumed the constant parameters, then $v_0^{(i)}=const$.

From ε^0 order term equation (3.12), the average velocities of i, j component, $v_0^{(j)}=v_0^{(i)}$. Similar, $T_0^{(j)}=T_0^{(i)}$ from ε^0 order-term equation (3.17) because $\left(4m^{(i)}m^{(j)}/m_0\right)S_{\sigma 0}^{(ij)}n_0^{(i)}n_0^{(j)}\neq 0$.

3.5. Physical Variable of ε^1 Order Approximation

From ε^1 order term equations (3.13) and (3.18), $v_1^{(j)} = v_1^{(i)}$ and $T_1^{(j)} = T_1^{(i)}$ because $\partial v_0^{(i)}/\partial\zeta = 0$, $\partial n_0^{(i)}/\partial\zeta = 0$ and $\partial T_0^{(i)}/\partial\zeta = 0$.

3.6. Physical Variable of ε^2 Order Approximation

Integrating the ε^2 order term equation (3.9), the variable $v_1^{(i)}$ can be expressed by using $n_1^{(i)}$, that is $v_1^{(i)} = -\left(V^{(i)}/n_0^{(i)}\right)n_1^{(i)}$. As $v_1^{(j)} = v_1^{(i)}$, $n_1^{(j)}$ can be expressed by using $n_1^{(i)}$ as,

$$n_1^{(j)} = \left(n_0^{(j)}V^{(i)}/n_0^{(i)}V^{(j)}\right)n_1^{(i)}. \tag{3.22}$$

Furthermore, integrating the sum of the exchanged i and j of ε^2 order equation (3.14) for ζ and using the relations between $v_1^{(j)}$ and $n_1^{(j)}$, following relation can be derived.

$$k_B T_1^{(i)} = a_{T10}^{(ij)} n_1^{(i)}, \tag{3.23}$$

Where, $a_{T10}^{(ij)} = \dfrac{n_0^{(i)}V^{(j)}\left(m^{(i)}V^{(i)2} - k_B T_0^{(i)}\right) + n_0^{(j)}V^{(i)}\left(m^{(j)}V^{(j)2} - k_B T_0^{(i)}\right)}{n_0^{(i)}V^{(j)}\left(n_0^{(i)} + n_0^{(j)}\right)}$.

Similarly, $k_B T_1^{(j)}$ is also expressed by $n_1^{(i)}$, using the relation $k_B T_1^{(j)} = k_B T_1^{(i)}$. $S_{\sigma 1}^{(ij)}$ is proportional to $n_1^{(i)}$ as

$$S_{\sigma 1}^{(ij)} = \pi\left(c_0/\left(2\sqrt{a_0}\right) + c_1\sqrt{a_0}\right)\left(3a_{T10}^{(ij)}/\mu\right)n_1^{(i)} \equiv a_{S10}^{(ij)} n_1^{(i)} \tag{3.24}$$

3.7 Physical Variables of ε^2 and ε^3 order Approximation

From the ε^3 order equation (3.10), $\partial n_2^{(i)}/\partial\zeta$ and $\partial n_2^{(j)}/\partial\zeta$ are expressed by $\partial v_2^{(i)}/\partial\zeta$, $\partial v_2^{(j)}/\partial\zeta$ and $n_1^{(i)}$ as follows,

$$\frac{\partial n_2^{(i)}}{\partial\zeta} = \frac{1}{n_0^{(i)}}\frac{\partial n_1^{(i)2}}{\partial\zeta} - \frac{1}{V^{(i)}}\frac{\partial n_1^{(i)}}{\partial\tau} - \frac{n_0^{(i)}}{V^{(i)}}\frac{\partial v_2^{(i)}}{\partial\zeta}, \tag{3.25}$$

$$\frac{\partial n_2^{(j)}}{\partial\zeta} = \frac{n_0^{(j)}V^{(i)2}}{n_0^{(i)}V^{(j)2}}\left(\frac{1}{n_0^{(i)}}\frac{\partial n_1^{(i)2}}{\partial\zeta} - \frac{1}{V^{(i)}}\frac{\partial n_1^{(i)}}{\partial\tau}\right) - \frac{n_0^{(j)}}{V^{(j)}}\frac{\partial v_2^{(j)}}{\partial\zeta}. \tag{3.26}$$

Using above equations, the sum of i and j component exchanged equations for the ε^3 order term equation (3.15) are expressed with $v_2^{(i)}, v_2^{(j)}, T_2^{(i)}, T_2^{(j)}, n_1^{(i)}$ as follows,

$$a_{vi}\frac{\partial v_2^{(i)}}{\partial\zeta} + a_{vj}\frac{\partial v_2^{(j)}}{\partial\zeta} + a_{Ti}\frac{\partial k_B T_2^{(i)}}{\partial\zeta} + a_{Tj}\frac{\partial k_B T_2^{(j)}}{\partial\zeta} + a_{n1\tau}\frac{\partial n_1^{(i)}}{\partial\tau} + a_{n1s\zeta}\frac{\partial n_1^{(i)2}}{\partial\zeta} + a_{n1\zeta 2}\frac{\partial^2 n_1^{(i)}}{\partial\zeta^2} = 0. \tag{3.27}$$

where, the coefficients, $a_{vi}, a_{vj}, a_{Ti}, a_{Tj}, a_{n1\tau}, a_{n1s\zeta}$, and $a_{n1\zeta 2}$ are respectively constants depend on only $n_0^{(i)}, n_0^{(j)}, T_0^{(i)}, V^{(i)}, V^{(j)}, m^{(i)}, m^{(j)}, \mu_V^{(i)}$, and $\mu_V^{(j)}$. Next, the derivative equations for ζ about the ε^2 order equation (3.14) is expressed by $v_2^{(i)}, v_2^{(j)}, n_1^{(i)}$ as follows,

$$\frac{\partial}{\partial \zeta}\left(v_2^{(j)} - v_2^{(i)}\right) + b_{n1\zeta 2}\frac{\partial^2 n_1^{(i)}}{\partial \zeta^2} = 0. \tag{3.28}$$

where,

$$b_{n1\zeta 2}{}^{(i)} = \frac{m_0\left(\left(m^{(i)}V^{(i)} - m^{(j)}V^{(j)}\right)V^{(i)}V^{(j)} + \left(V^{(i)} - V^{(j)}\right)k_B T_0{}^{(i)}\right)}{2m^{(i)}m^{(j)}S_{\sigma 0}^{(ij)}n_0{}^{(i)}}. \tag{3.29}$$

Furthermore, the derivative equation for ζ about the ε^2 order equation (3.19) is expressed by $T_2^{(i)}, T_2^{(j)}, n_1^{(i)}$ as follows,

$$k_B\frac{\partial}{\partial \zeta}\left(T_2^{(j)} - T_2^{(i)}\right) + c_{n1\zeta 2}\frac{\partial^2 n_1^{(i)}}{\partial \zeta^2} = 0. \tag{3.30}$$

Where,

$$c_{n1\zeta 2} = -\frac{m_0{}^2V^{(i)}\left(3\left(n_0{}^{(i)}m^{(i)}V^{(i)} + n_0{}^{(j)}m^{(j)}V^{(j)}\right)V^{(i)}V^{(j)} - \left(5n_0{}^{(i)}V^{(j)} + n_0{}^{(j)}\left(3V^{(i)} + 2V^{(j)}\right)\right)k_B T_0{}^{(i)}\right)}{12m^{(i)}m^{(j)}S_{\sigma 0}^{(ij)}n_0{}^{(i)}n_0{}^{(j)}\left(n_0{}^{(i)} + n_0{}^{(j)}\right)V^{(j)}}. \tag{3.31}$$

Last, the sum of i and j component exchanged equations for the ε^3 order equation (3.20) are expressed with $v_2^{(i)}, v_2^{(j)}, T_2^{(i)}, T_2^{(j)}, n_1^{(i)}$ as follows,

$$d_{vi}\frac{\partial v_2{}^{(i)}}{\partial \zeta} + d_{vj}\frac{\partial v_2{}^{(j)}}{\partial \zeta} + d_{Ti}\frac{\partial k_B T_2{}^{(i)}}{\partial \zeta} + d_{Tj}\frac{\partial k_B T_2{}^{(j)}}{\partial \zeta} + d_{n1\tau}\frac{\partial n_1{}^{(i)}}{\partial \tau} + d_{n1s\zeta}\frac{\partial n_1{}^{(i)2}}{\partial \zeta} + d_{n1\zeta 2}\frac{\partial^2 n_1{}^{(i)}}{\partial \zeta^2} = 0. \tag{3.32}$$

where the coefficients, $d_{vi}, d_{vj}, d_{Ti}, d_{Tj}, d_{n1\tau}, d_{n1s\zeta}$, and $d_{n1\zeta 2}$ are respectively constants expressed by $n_0^{(i)}, n_0^{(j)}, T_0^{(i)}, V^{(i)}, V^{(j)}, m^{(i)}, m^{(j)}, \lambda^{(i)}$, and $\lambda^{(j)}$. Solving above four equations (3.27), (3.28), (3.30) and (3.32) about $\partial v_2{}^{(i)}/\partial \zeta, \partial v_2{}^{(j)}/\partial \zeta, \partial T_2^{(i)}/\partial \zeta, \partial T_2^{(j)}/\partial \zeta$, these variables are expressed by $n_1{}^{(i)}$ as,

$$\left.\begin{aligned}
\frac{\partial v_2^{(i)}}{\partial\zeta} &= p_{v2i\tau}\frac{\partial n_1^{(i)}}{\partial\tau} + p_{v2i\zeta2}\frac{\partial^2 n_1^{(i)}}{\partial\zeta^2} + p_{v2is\zeta}\frac{\partial n_1^{(i)2}}{\partial\zeta} \\
\frac{\partial v_2^{(j)}}{\partial\zeta} &= p_{v2j\tau}\frac{\partial n_1^{(i)}}{\partial\tau} + p_{v2j\zeta2}\frac{\partial^2 n_1^{(i)}}{\partial\zeta^2} + p_{v2js\zeta}\frac{\partial n_1^{(i)2}}{\partial\zeta} \\
\frac{\partial T_2^{(i)}}{\partial\zeta} &= p_{T2i\tau}\frac{\partial n_1^{(i)}}{\partial\tau} + p_{T2i\zeta2}\frac{\partial^2 n_1^{(i)}}{\partial\zeta^2} + p_{T2is\zeta}\frac{\partial n_1^{(i)2}}{\partial\zeta} \\
\frac{\partial T_2^{(j)}}{\partial\zeta} &= p_{T2j\tau}\frac{\partial n_1^{(i)}}{\partial\tau} + p_{T2j\zeta2}\frac{\partial^2 n_1^{(i)}}{\partial\zeta^2} + p_{T2js\zeta}\frac{\partial n_1^{(i)2}}{\partial\zeta}
\end{aligned}\right\}. \tag{3.33}$$

Where, the coefficients $p_{n2i\tau},\sim,p_{T2js\zeta}$ are expressed by ε^0 order physical parameters, the potential parameters κ_a,κ_b and the physical parameters $\mu_V^{(i)},\mu_V^{(j)},\lambda^{(i)},\lambda^{(j)}$. Integrating these differential equations for ζ, these variable $v_2^{(i)},v_2^{(j)},T_2^{(i)},T_2^{(j)}$ are described with the first order approximation of the number density $n_1^{(i)}$ and it's integration for ζ, which is described by $N_1=\int^{\zeta} n_1^{(i)}d\zeta'$. Thus, we get the solutions of the second order approximation $v_2^{(i)},v_2^{(j)},T_2^{(i)}$, and $T_2^{(j)}$ as,

$$\left.\begin{aligned}
v_2^{(i)} &= p_{v2i\tau}\frac{\partial N_1}{\partial\tau} + p_{v2i\zeta2}\frac{\partial n_1^{(i)}}{\partial\zeta} + p_{v2is\zeta}n_1^{(i)2} \\
v_2^{(j)} &= p_{v2j\tau}\frac{\partial N_1}{\partial\tau} + p_{v2j\zeta2}\frac{\partial n_1^{(i)}}{\partial\zeta} + p_{v2js\zeta}n_1^{(i)2} \\
T_2^{(i)} &= p_{T2i\tau}\frac{\partial N_1}{\partial\tau} + p_{T2i\zeta2}\frac{\partial n_1^{(i)}}{\partial\zeta} + p_{T2is\zeta}n_1^{(i)2} \\
T_2^{(j)} &= p_{T2j\tau}\frac{\partial N_1}{\partial\tau} + p_{T2j\zeta2}\frac{\partial n_1^{(i)}}{\partial\zeta} + p_{T2js\zeta}n_1^{(i)2}
\end{aligned}\right\}. \tag{3.34}$$

Where, the integral constants are zero as the disturbances are regarded as zero at the infinity.

3.8Physical Variables of ε^4 order Approximation

Using the ε^4 order equation (3.11), $n_3^{(i)}$ and $n_3^{(j)}$ are expressed by using variables $v_3^{(i)},v_3^{(j)}$ as follows,

$$\left.\begin{aligned}
\frac{\partial n_3^{(i)}}{\partial\zeta} &= -\frac{1}{V^{(i)}}\left(\frac{\partial n_2^{(i)}}{\partial\tau} + \frac{\partial}{\partial\zeta}\left(n_2^{(i)}v_1^{(i)} + n_1^{(i)}v_2^{(i)}\right) + n_0^{(i)}\frac{\partial v_3^{(i)}}{\partial\zeta}\right) \\
\frac{\partial n_3^{(j)}}{\partial\zeta} &= -\frac{1}{V^{(j)}}\left(\frac{\partial n_2^{(j)}}{\partial\tau} + \frac{\partial}{\partial\zeta}\left(n_2^{(j)}v_1^{(j)} + n_1^{(j)}v_2^{(j)}\right) + n_0^{(j)}\frac{\partial v_3^{(j)}}{\partial\zeta}\right)
\end{aligned}\right\}. \tag{3.35}$$

Now, $\delta v_4^{(ji)} = v_4^{(j)} - v_4^{(i)}$ in eq.(3.16) and $\delta T_4^{(ji)} = T_4^{(j)} - T_4^{(i)}$ in eq.(3.21) are neglected to close the high expansion order calculation. Then substituting $\delta v_3^{(ji)} = v_3^{(j)} - v_3^{(i)}$ get from eq.(3.15) and $\delta T_3^{(ji)} = T_3^{(j)} - T_3^{(i)}$ get from eq.(3.20) into eqs.(3.16) and (3.21), following set of equations are obtained,

$$\left.\begin{aligned} &\alpha_{vi}\frac{\partial v_3^{(i)}}{\partial\zeta} + \alpha_{Ti}k_B\frac{\partial T_3^{(i)}}{\partial\zeta} + \Phi_{v3}^{(i)} = 0, \quad \alpha_{vj}\frac{\partial v_3^{(j)}}{\partial\zeta} + \alpha_{Tj}k_B\frac{\partial T_3^{(j)}}{\partial\zeta} + \Phi_{v3}^{(j)} = 0 \\ &\beta_{vi}\frac{\partial v_3^{(i)}}{\partial\zeta} + \beta_{Ti}k_B\frac{\partial T_3^{(i)}}{\partial\zeta} + \Phi_{E3}^{(i)} = 0, \quad \beta_{vj}\frac{\partial v_3^{(j)}}{\partial\zeta} + \beta_{Tj}k_B\frac{\partial T_3^{(j)}}{\partial\zeta} + \Phi_{E3}^{(j)} = 0 \end{aligned}\right\}. \tag{3.36}$$

where, $\Phi_{v3}^{(i)}, \Phi_{v3}^{(j)}, \Phi_{E3}^{(i)}, \Phi_{E3}^{(j)}$ are the function of $n_1^{(i)}$ and $\alpha_{vi}, \alpha_{vj}, \beta_{vi}, \beta_{vj}, \beta_{Ti}, \beta_{Tj}, \gamma_v, \delta_T$ are expressed by the ε^0 order physical variables. From above four equations, following equations are obtained.

$$\left.\begin{aligned} &\frac{\partial v_3^{(i)}}{\partial\zeta} = \frac{\alpha_{Ti}\Phi_{E3}^{(i)} - \beta_{Ti}\Phi_{v3}^{(i)}}{\alpha_{vi}\beta_{Ti} - \alpha_{Ti}\beta_{vi}}, \qquad \frac{\partial v_3^{(j)}}{\partial\zeta} = \frac{\alpha_{Tj}\Phi_{E3}^{(j)} - \beta_{Tj}\Phi_{v3}^{(j)}}{\alpha_{vj}\beta_{Tj} - \alpha_{Tj}\beta_{vj}} \\ &\frac{\partial k_B T_3^{(i)}}{\partial\zeta} = \frac{\beta_{vi}\Phi_{v3}^{(i)} - \alpha_{vi}\Phi_{E3}^{(i)}}{\alpha_{vi}\beta_{Ti} - \alpha_{Ti}\beta_{vi}}, \qquad \frac{\partial k_B T_3^{(j)}}{\partial\zeta} = \frac{\beta_{vj}\Phi_{v3}^{(j)} - \alpha_{vj}\Phi_{E3}^{(j)}}{\alpha_{vj}\beta_{Tj} - \alpha_{Tj}\beta_{vj}} \end{aligned}\right\}. \tag{3.37}$$

Substituting eq.(3.37) into eq.(3.16), following nonlinear integro-partial differential equations for $n_1^{(i)}$ is derived.

$$\frac{\partial}{\partial\zeta}\left(\frac{\partial n_1^{(i)}}{\partial\tau} + P\frac{\partial^2 n_1^{(i)}}{\partial\zeta^2} + Qn_1^{(i)3} + R\frac{\partial n_1^{(i)2}}{\partial\zeta}\right) + A_1\frac{\partial n_1^{(i)2}}{\partial\tau}$$

$$+ A_2\left(\frac{\partial n_1^{(i)}}{\partial\zeta}\right)^2 + A_3\frac{\partial^2}{\partial\tau^2}\int^{\zeta} n_1^{(i)}d\zeta' + A_4\frac{\partial n_1^{(i)}}{\partial\zeta}\frac{\partial}{\partial\tau}\int^{\zeta} n_1^{(i)}d\zeta' = 0, \tag{3.38}$$

Where, the coefficients $P, Q, R, A_1, A_2, A_3, A_4$ are expressed by the ε^0 order physical variables.

4. Derivation of the Ginzburg-Landau Equation by Fourier Perturbed Expansion Method

In order to study the characteristic of the nonlinear wave propagations based on the equation (3.38), we expand furthermore the gas density $n_1^{(i)}$ with Fourier “perturbed” expansion method [7] as,

$$n_1^{(i)}(\zeta,\tau) = \sum_{m=1}^{\infty}\varepsilon^m\sum_{\ell=-\infty}^{\infty}\hat{n}_\ell^{(m)(i)}(\xi,\sigma)e^{i\ell(k\zeta-\omega\tau)}. \tag{4.1}$$

Where, k and ω are respectively the wave number and the frequency, which are complex parameters. A higher Fourier expanded term of $n_1^{(i)}$ may become to be small rapidly. The coordinate (ξ,σ) describes an envelope part and the coordinate (ζ,τ) describes a carrier part of the wave. The variables of (ξ,σ) is the coordinate of slowly moving than (ζ,τ). Then (ξ,σ) may be described, using a small parameter ε (< 1) as

$$\xi = \varepsilon(\zeta - \upsilon_g \tau), \quad \sigma = \varepsilon^2 \tau . \tag{4.2}$$

Therefore the time and the spatial derivative of $n_1^{(i)}$ are described as,

$$\frac{\partial n_1^{(i)}}{\partial \tau} \Rightarrow \left(\frac{\partial}{\partial \tau} - \varepsilon \upsilon_g \frac{\partial}{\partial \xi} + \varepsilon^2 \frac{\partial}{\partial \sigma}\right) n_1^{(i)}, \quad \frac{\partial n_1^{(i)}}{\partial \zeta} \Rightarrow \left(\frac{\partial}{\partial \zeta} + \varepsilon \frac{\partial}{\partial \xi}\right) n_1^{(i)} . \tag{4.3}$$

From eqs.(4.1), (4.2) and (4.3), the indefinite integration of $n_1^{(i)}$ about ζ is,

$$\int^{\zeta} n_1^{(i)} d\zeta' = \sum_{m=1}^{\infty} \varepsilon^m \sum_{\ell=-\infty}^{\infty} \frac{1}{i\lambda k}\left(\hat{n}_\ell^{(m)(i)}(\xi,\sigma) e^{i\ell(k\zeta' - \omega\tau)} - \varepsilon \int^{\zeta} \left(\frac{\partial \hat{n}_\ell^{(m)(i)}(\xi,\sigma)}{\partial \xi} e^{i\ell(k\zeta' - \omega\tau)}\right) d\zeta'\right). \tag{4.4}$$

Furthermore performing integration of right side of eq.(4.4) and neglecting higher order of ε, following approximation is derived,

$$\int^{\zeta} n^{(i)} d\zeta' \approx \frac{1}{i\lambda k}\left(n^{(i)} - \varepsilon \frac{1}{i\lambda k} \frac{\partial n^{(i)}}{\partial \xi}\right). \tag{4.5}$$

Above relation means that the integration can be described approximately using the differential term. The derivative of the indefinite integration of $n_1^{(i)}$ for ζ is returned nearly to $n_1^{(i)}$ from eq. (4.2), as follows,

$$\frac{\partial}{\partial \zeta} \int^{\zeta} n^{(i)} d\zeta' \Rightarrow n^{(i)} + \varepsilon^2 \frac{1}{\lambda^2 k^2} \frac{\partial^2 n^{(i)}}{\partial \xi^2} \approx n^{(i)} . \tag{4.6}$$

Next, as a higher perturbed expansion term of $\hat{n}_\ell^{(m)(i)}$ become to be small rapidly, the higher perturbed terms of $m \geq 2$ are neglected. And 0 order perturbed gas density is zero. Further, neglecting the terms of $|\lambda| \geq 2$, the nonlinear term in the equation (3.38) is described as

$$\hat{n}_{\ell'}^{(m')(i)} \hat{n}_{\ell''}^{(m'')(i)} \hat{n}_{\ell'''}^{(m''')(i)} \rightarrow \hat{n}_1^{(1)(i)} \hat{n}_{-1}^{(1)(i)} \hat{n}_1^{(1)(i)} = \left|\hat{n}_1^{(1)(i)}\right|^2 \hat{n}_1^{(1)(i)} . \tag{4.7}$$

We could describe $\hat{n}_1^{(1)(i)}$ to be $n_1^{(i)}$ if above conditions of λ, m are satisfied.

Furthermore, the terms of Fourier expansion of $\lambda = 0$ are the constants. Substituting the Fourier perturbed expansion of $n_1^{(i)}$ into the nonlinear integro-partial differential equations (3.38), following equations are obtained for each ε order,

$$\varepsilon^1 : k\omega + i\left(-k^3 P + A_3^{(i)} \omega^2 / k\right) = 0, \tag{4.8}$$

$$\varepsilon^2 : -3P^{(i)} k^2 + 2A_3^{(i)} \omega \upsilon_g / k - i(k\upsilon_g + \omega) = 0, \tag{4.9}$$

$$\varepsilon^3: \left(ik - \frac{A_3^{(i)}}{k} 2\omega\right)\frac{\partial n_1^{(i)}}{\partial \sigma} + \left(-\upsilon_g + i\left(3P^{(i)}k + \frac{A_3^{(i)}}{k}\left(\frac{\omega \upsilon_g}{k} - \upsilon_g^{\ 2}\right)\right)\right)\frac{\partial^2 n_1^{(i)}}{\partial \xi^2} + Q^{(i)} ik\left|n_1^{(i)}\right|^2 n_1^{(i)} = 0.$$

(4.10)

The wave number k and the frequency ω which are complex numbers can be obtained, solving the equations (4.8) and (4.9). The real and imaginary parts of k, ω, that is, $k_r, k_i, \omega_r, \omega_i$ are expressed by P, A_3 and υ_g. From equation (4.10), the Ginzburg-Landau (GL) equation is derived as,

$$i\frac{\partial n_1^{(i)}}{\partial \sigma} + \left(p_r + ip_i\right)\frac{\partial^2 n_1^{(i)}}{\partial \xi^2} + \left(q_r + iq_i\right)\left|n_1^{(i)}\right|^2 n_1^{(i)} = 0. \quad (4.11)$$

where coefficients p_r, p_i, q_r, q_i are expressed by $k_r, k_i, \omega_r, \omega_i$, Q, A_1, A_2, A_3. A special solution of GL equation are known [4] as

$$n_1^{(i)} = n_a \operatorname{sec}h^{1+i\alpha}\left(\Theta \xi\right)\exp\left(-i\left(\varphi_r + i\varphi_i\right)\sigma\right). \quad (4.12)$$

where the coefficients $\alpha, \Theta, \varphi_r, \varphi_i$ are expressed by p_r, p_i, q_r, q_i. In eq. (4.12), the growing or the damping phenomena are included, depending on the values of $\varphi_r, \varphi_i, \omega_r, \omega_i$.

5. Necessary Conditions for Stationary Wave Propagations

First, the ε^0 order densities of both gas components, $n_0^{(i)}, n_0^{(j)}$ must be real and positive. Then the relations of the densities between gas components derived from the ε^2 order of equation (3.19) must be concluded as follows,

$$n_0^{(j)} = \frac{a_n \pm \sqrt{a_n^{\ 2} - b_i b_j}}{b_j} n_0^{(i)} > 0 \quad (5.1)$$

where,

$$a_n = \left(3V^{(i)2} + 3V^{(j)2} + 4V^{(i)}V^{(j)}\right)k_B T_0^{(i)} - 3\left(m^{(i)} + m^{(j)}\right)V^{(i)2}V^{(j)2}$$

$$b_i = 2V^{(j)}V^{(i)}\left(3m^{(i)}V^{(i)2} - 5k_B T_0\right),\ \ b_j = 2V^{(j)}V^{(i)}\left(3m^{(j)}V^{(j)2} - 5k_B T_0\right)$$

As $n_0^{(j)}$ is real and positive, that is, $a_n^{\ 2} - b_i b_j \geq 0$ and $a_n \pm \sqrt{a_n^{\ 2} - b_i b_j} \big/ b_j \geq 0$,

then the values of $V^{(i)}, V^{(j)}$ must be within the limits as

$$S_a < V^{(i)} < S_i \quad and \quad S_b < V^{(j)} < S_j \quad (5.2)$$

where, S_a and S_b are obtained from the real and positive condition of $n_0^{(j)}$.

$$S_i \equiv \sqrt{\frac{5k_B T_0}{3m^{(i)}}},\quad S_j \equiv \sqrt{\frac{5k_B T_0}{3m^{(j)}}}$$

Second, as the wave number k_r, k_i and the frequency ω_r, ω_i must be real numbers, then

$PA_3 > 0.25$.

Third, when $\omega = \omega_r + \mathrm{i}\,\omega_i$ and $k = k_r + \mathrm{i}\,k_i$ are substituted to the equations (4.8) and (4.9), only trivial solution $v_g = 0$ is obtained. Then instead of ω_r , we introduce the real frequency $\omega_r^* = \dfrac{\omega_r}{{v_g}^2} f_w$, where f_w is weight function.

Then we can get the values of $V^{(i)}, V^{(j)}, T_0^{(i)}$ and $v_g^{(i)}$ by solving the simultaneous four equations

$$\varepsilon_{1r} = 0, \varepsilon_{1i} = 0, \varepsilon_{2r} = 0, \varepsilon_{2i} = 0 \tag{5.3}$$

where $\varepsilon_{1r}, \varepsilon_{1i}, \varepsilon_{2r}\varepsilon_{2i}$ are the real and the imaginary parts of eqs.(4.8) and (4.9),

$$\left.\begin{aligned}
\varepsilon_{1r} &= (4PA_3 - 1)^{\frac{7}{2}}(6PA_3 - 1)(9PA_3 - 1)v_g^2\left(f_w - v_g^2\right)\Big/\left(4P^3(9PA_3 - 2)^4\right)\\
\varepsilon_{1i} &= (4PA_3 - 1)^3(9PA_3 - 1)\left(f_w - v_g^2\right)\left(PA_3 f_w(9PA_3 - 1) + (1 + PA_3(9PA_3 - 7))v_g^2\right)\Big/\left(4P^3(9PA_3 - 2)^4\right)\\
\varepsilon_{2r} &= (4PA_3 - 1)^{\frac{3}{2}}(6PA_3 - 1)^2(9PA_3 - 1)v_g\left(f_w - v_g^2\right)\Big/\left(4P^2(9PA_3 - 2)^3\right)\\
\varepsilon_{2i} &= (4PA_3 - 1)^2(6PA_3 - 1)(9PA_3 - 1)v_g\left(f_w - v_g^2\right)\Big/\left(4P^2(9PA_3 - 2)^3\right)
\end{aligned}\right\}.$$

Fourth, the coordinate transformations (4.2) with use of Fourier perturbed expansion methods have also means of the Galilean transformation. We can get the new GL solution with Galilean invariance by transforming the GL equation (4.11) to new GL equations as follows,

$$i\frac{\partial n^{(i)}}{\partial \sigma} + (p_r + ip_i)\frac{\partial^2 n^{(i)}}{\partial \xi^2} + (q_r + iq_i)\left|n^{(i)}\right|^2 n^{(i)} - iv_G\frac{\partial n^{(i)}}{\partial \xi} = 0 \ . \tag{5.4}$$

Obtaining the new GL solutions with the Galilean invariance, we replace $n^{(i)}$ with a new function as follows,

$$n_1^{(i)} = n_a \sec h^{1+i\alpha}(\Theta\xi)\exp\left(i(K_r + iK_i)\xi + i(\Omega_r + i\Omega K_i)\sigma\right). \tag{5.5}$$

Then we get the necessary conditions for stationary wave propagation without growing or damping in any values ξ and σ as,

$$\left.\begin{aligned}
&k_i + K_i = 0,\ \ k_r + K_r = 0,\ \ \psi_i + \omega_i + \Omega_i = 0,\ \ v_G = 0\\
&f_G = 2p_r\alpha - p_i\left(\alpha^2 - 1\right) = 0,\ \ \Omega_r + \psi_r + \omega_r = -\frac{{n_a}^2\left(\alpha^2 + 1\right)q_r}{\alpha^2 + 2}
\end{aligned}\right\}. \tag{5.6}$$

As $v_G = 0$, the wave propagating velocity of Galilei invariability is the velocity v_g which is used in the coordinate transformation of the Fourier perturbed expansion series. f_G is the Galilei invariant function which describes the condition of the Galilean invariance at $f_G = 0$.

6. Numerical results

In order to investigate the nonlinear wave propagation in binary-gas mixture, we put the physical parameters $m^{(i)}, m^{(j)}, n_0^{(i)}$, and $\nu^{(j)}, \lambda^{(j)}$ respectively as here $m^{(i)} = 130, m^{(j)} = 1, n_0^{(i)} = 100$, and $\nu^{(j)} = 1, \lambda^{(j)} = 1$. Then $V^{(i)}, V^{(j)}$, $T_0^{(i)}\left(= T_0^{(j)}\right)$ and $v_g^{(i)}\left(= v_g^{(j)}\right)$ can be obtained, numerically solving the simultaneous equations (5.3) if f_w is determined. Here, we put f_w as the construction factor to the interpolation function of the experimental data which are presented by D. G. Henshaw and A. D. B. Woodsn [8]. The interpolation function is described as follows,

$$f_w = \left(wv_g\right)\frac{c_a\left(wv_g\right)^3 + c_b\left(wv_g\right)^2 + c_c\left(wv_g\right) + c_d}{c_p\left(wv_g\right)^2 + c_q\left(wv_g\right) + c_r}. \tag{6.1}$$

where, w is the weight parameter of v_g and the coefficients $c_a \sim c_r$ are shown in Table 1. Here we put $w = 1$ and use the values of $\kappa_b \sim \lambda^{(i)}$ as shown in Table 2, then we obtain the values of $V^{(i)}, V^{(j)}$, $T_0^{(i)} = T_0^{(j)}$, and $v_g^{(i)} = v_g^{(j)}$ as Table 2. These values satisfies the conditions $PA_3 > 0.25$ and (5.2). In this case, the condition (5.2) and the values of $\left(V^{(i)}, V^{(j)}\right)$ in Table 2 are shown in Fig.1. We can plot f_G and f_w about v_g with several weight parameter w as Fig2. We put $n^{(i)} = n_r + in_i$ in the new GL equation (5.4), then the real and imaginary parts are,

$$\left.\begin{aligned} &\frac{\partial n_i}{\partial \sigma} - p_r\frac{\partial^2 n_r}{\partial \xi^2} + p_i\frac{\partial^2 n_i}{\partial \xi^2} - \left(n_r{}^2 + n_i{}^2\right)\left(q_r n_r - q_i n_i\right) - v_g\frac{\partial n_i}{\partial \xi} = 0 \\ &\frac{\partial n_r}{\partial \sigma} + p_i\frac{\partial^2 n_r}{\partial \xi^2} + p_r\frac{\partial^2 n_i}{\partial \xi^2} + \left(n_r{}^2 + n_i{}^2\right)\left(q_i n_r + q_r n_i\right) - v_g\frac{\partial n_r}{\partial \xi} = 0 \end{aligned}\right\}. \tag{6.2}$$

In the numerical calculations for the differential equations (6.2), we set the initial and boundary conditions as follows,

$$\left.\begin{aligned}n_r &= n_a \sec h\left(\Theta\left(\xi - v_g(\sigma-\sigma_0)\right)\right)\cos\left(\alpha Log\left[\sec h\left(\Theta\left(\xi - v_g(\sigma-\sigma_0)\right)\right)\right] - \frac{n_a^2(\alpha^2+1)q_r}{\alpha^2+2}(\sigma-\sigma_0)\right)\\ n_i &= n_a \sec h\left(\Theta\left(\xi - v_g(\sigma-\sigma_0)\right)\right)\sin\left(\alpha Log\left[\sec h\left(\Theta\left(\xi - v_g(\sigma-\sigma_0)\right)\right)\right] - \frac{n_a^2(\alpha^2+1)q_r}{\alpha^2+2}(\sigma-\sigma_0)\right)\end{aligned}\right\}. \quad (6.3)$$

where, Θ, α, q_r are the functions of the weight parameter w.

We put $\sigma_0 = 40$, then the numerical calculation is done around σ =40. The direction of wave propagation in the numerical calculation is disposed so that the incident wave propagates from the left hand side to right hand side. From the equation $f_G = 0$ of the Galilei invariant condition (5.6), we get the numerical solution of v_G which must be almost correspond to the value v_g obtained from equation (5.3). It means that v_G which meet the Galilei invariant condition also almost satisfies the equation (5.3) under the proper value for κ_b, $\nu^{(i)}$ and $\lambda^{(i)}$.

By numerical calculations for equation (5.3), we obtain the solitary wave propagation profiles which are different from the soliton of the nonlinear Shrodinger equation [9] because $\alpha \neq 0$. The numerical calculation of the simultaneous differential equations (6.2) is performed with various weight parameter w for the incident wave of $n_a = 4$. The amplitude change with time for the incident waves are shown in Fig.3. In the case of weight parameter $w = 0.187$, the incident wave does not change the amplitude during propagation.

The wave propagation in the case of $w = 0.187$ and $n_a = 8$ is shown in Fig.4. When the amplitudes of the incident wave become high, the amplitudes of the propagating waves temporarily change and then approach to the incident amplitude as shown in Fig.5a. Fig.5b indicates the stationary state amplitudes by numerical calculation for the incident wave amplitudes. The reciprocal Θ of the solitary wave width does not change in the Galilean transformation. In eq.(5.5),

$$\Theta = \mathrm{Ck}\,|n_a| \quad (6.4)$$

where, coefficient Ck is determined by $m^{(i)}, m^{(j)}, V^{(i)}, V^{(j)}, T_0^{(i)}, v_g^{(i)}, \kappa_a, \kappa_b, \nu^{(i)}, \lambda^{(i)}$.

The numerical results of Θ in propagating waves in case of $w = 0.187$ are plotted for several incident wave amplitudes n_a in Fig.6. This results are compatible with the estimation by eq.(6.4). The solitary wave velocities $(\Delta \xi / \Delta \sigma)$ numerically calculated are plotted in Fig.7 under the same condition as the calculation of Fig.5 and Fig.6.

Fig.8 shows the propagation profiles of $n_1^{(i)}$, $n_1^{(j)}, T_1^{(i)}, v_1^{(i)}$ for the incident wave of amplitude $n_a^{(i)} = 2$. These numerical results are in consistency with the analytical estimates.

On the other hand, under the other values $\kappa_b, \nu^{(i)}, \lambda^{(i)}$ of the matter, we may get also the stationary propagating wave which have the other value of the velocity weight.

The value of the velocity v_g satisfied with the Galilean invariance condition in the profile of f_w on $w = 0.187$ have the position at $df_w/dv_g < 0$. But in the case of the other values of $\kappa_b, \nu^{(i)}, \lambda^{(i)}$ and w, the velocity v_g is at $df_w/dv_g > 0$. There are no relations between the positions of the velocity on the profiles of f_w and the conditions of the stationary propagation of the solitary waves.

7. Conclusions

The nonlinear solitary wave propagation generated by the laser sheet injection in the molecular binary-gas mixture is discussed. First the nonlinear integro-partial differential equations for the first order approximation of the i component gas density are derived by applying the reductive perturbation method to the three macro conservation laws with relaxation terms and the state of the equation based on the two Boltzmann equations. Furthermore, from the nonlinear integro-partial differential equations, the Ginzburg-Landau (GL) equation is derived by expanding the first order approximation gas density using Fourier perturbed expansion method. Then applying Galilean transformation to the GL equation and the particular solution, the conditions of Galilean invariance are obtained. If the velocity of the coordinate in the Galilean transformation correspond to the velocity which satisfy the simultaneous equations constituted of the first and second approximate relations on the Fourier perturbed expansion of the gas density, the velocity of the Galilean invariance also satisfy the first and second approximation relations with the weight function (the construction factor) under the suitable value of the matter which are the parameter of the intermolecular attractive force, the coefficient of the resistance and the heat conductivity under the ratio of the masses of i and j component gas molecules. The propagation profiles of stationary propagating solitary waves based on the new GL equations derived are numerically calculated in the certain values of the matter as shown in Section 6. In other cases of the mass ratio for i and j gas component molecules, the stationary propagating nonlinear wave propagations can be investigated with the GL equation derived here. Moreover, as the solitary wave solution is only a particular solution of the GL equation, other physical phenomena expressed by the GL equation may also exist in the binary-gas mixture.

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Table 1. Coefficients of the interpolation function f_w eq. (6-1).

c_a=-272003
c_b=2117.2
c_c=-5196.0
c_d=4171.6
c_p=42.20
c_q=-171.3
c_r=189.9

Table 2. The solution values of $V^{(i)}, V^{(j)}$, $T_0^{(i)} = T_0^{(j)}$, and $v_g^{(i)} = v_g^{(j)}$ under $\kappa_b \sim \lambda^{(i)}$ given.

$V^{(i)}$=0.1053
$V^{(j)}$=0.8353
$T_0^{(i)}=T_0^{(j)}$=0.9403
$v_g^{(i)}=v_g^{(j)}$=4.5726
κ_b=2.799
$\mu_v^{(i)}$=1.329
$\lambda^{(i)}$=1.200

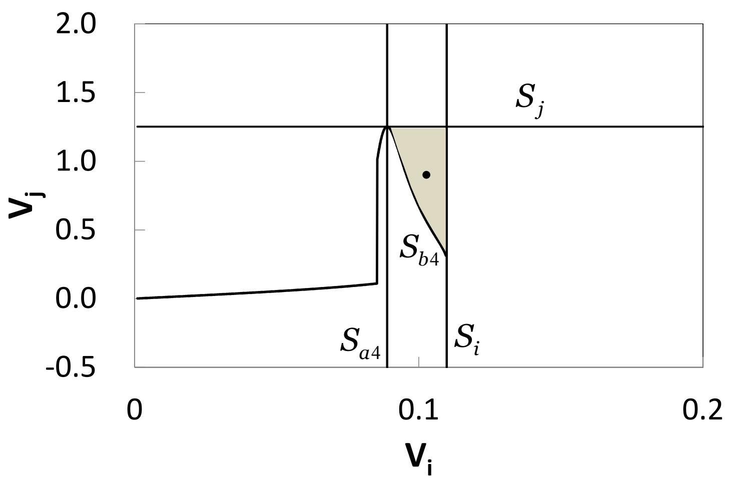

Fig.1. Wave existence condition (gray part) in $\left(V^{(i)}, V^{(j)}\right)$ coordinate by eq.(5.2). Black dot shows a $\left(V^{(i)}, V^{(j)}\right)$ point in case of $m^{(i)} = 130, m^{(j)} = 1$.

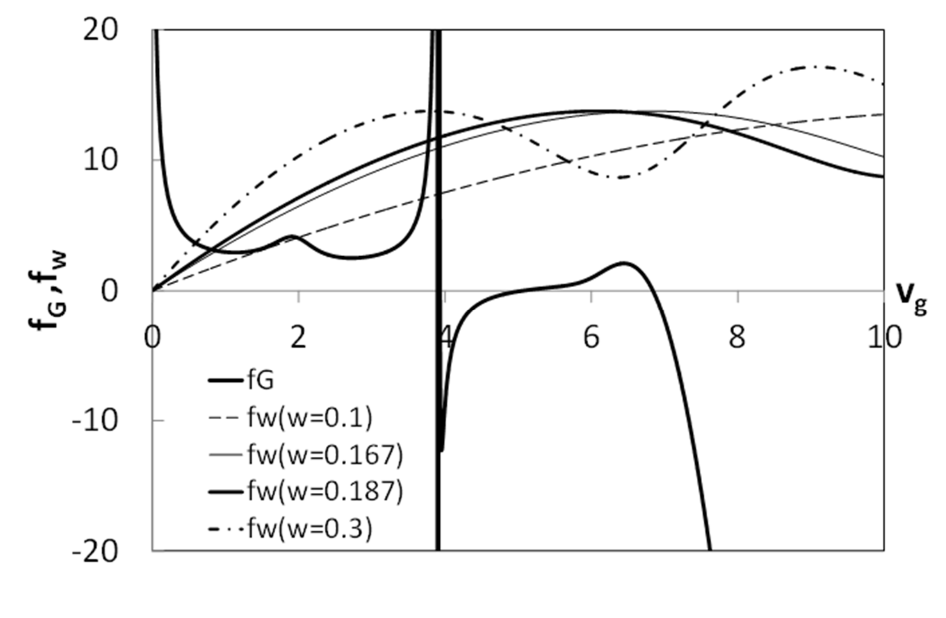

Fig.2. Profiles of the weight function f_w of the real frequency and the Galilei invariant function f_G for velocities v_g .

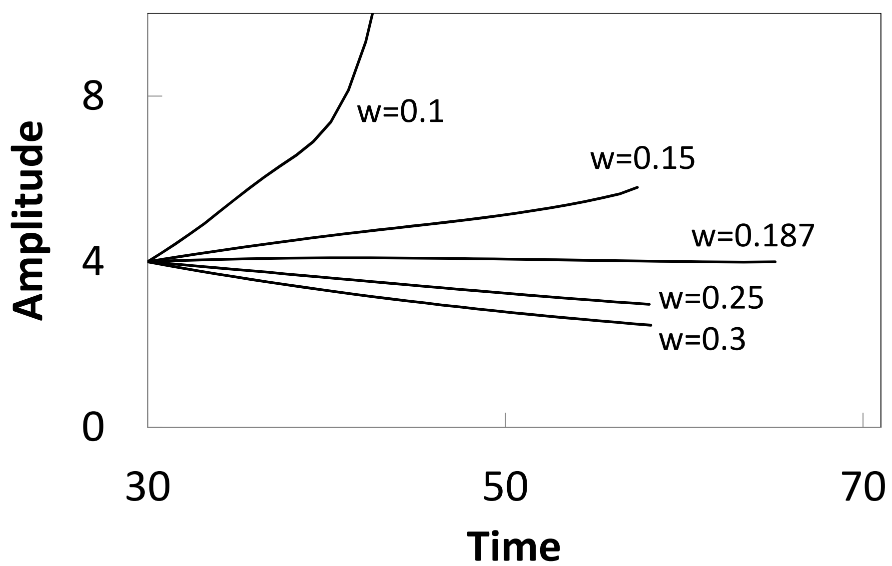

Fig.3. Temporal amplitude development of the incident wave $n_a = 4$ with various weight parameter w .

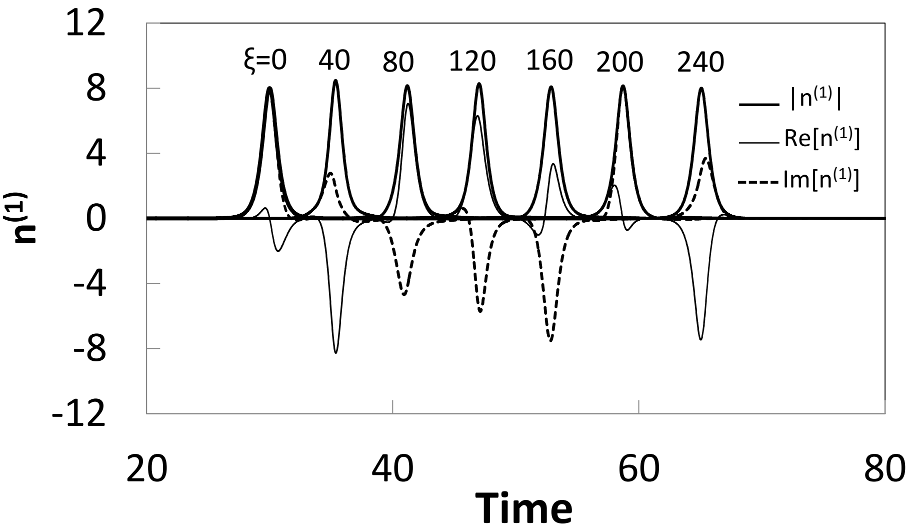

Fig. 4. Plots of the propagating wave at the spatial points of $\xi = 0,40,80,120,160,200,240$ in case of $m^{(i)} = 130$, $m^{(j)} = 1$, the amplitude $n_a = 8$ of the incident wave. The bold, thin and broken lines indicate respectively the absolute, the real and the imaginary part of the gas density.

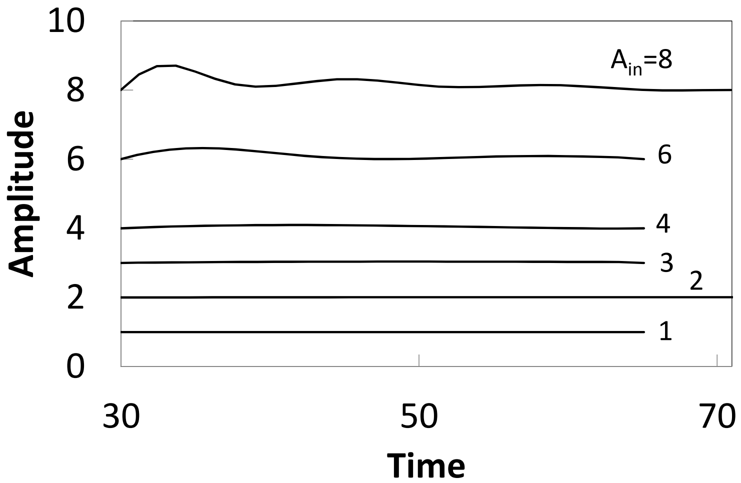

Fig.5a. Temporal amplitude change for incident waves of $n_a = 1,2,3,4,6,8$.

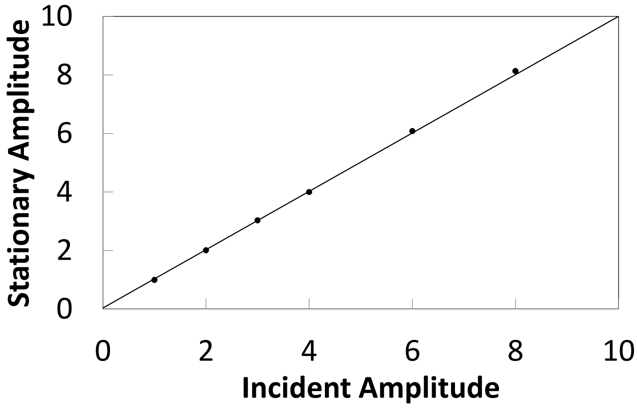

Fig.5b. Stationary state amplitudes (shown by dots) by numerical calculation for the incident wave amplitudes.

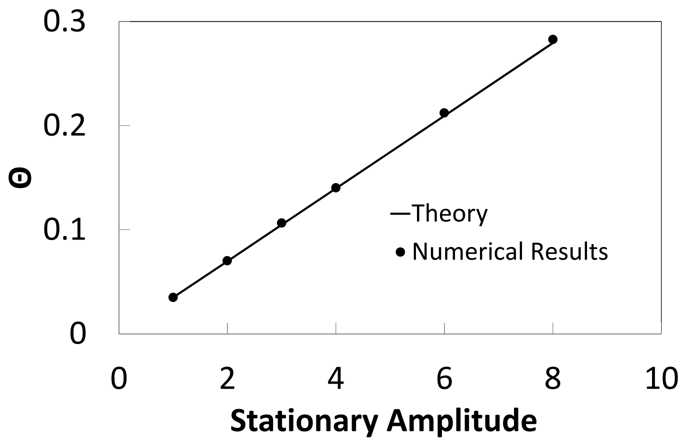

Fig.6. Numerical results of inverse number Θ which is the reciprocal of the solitary wave width are plotted in case of $w = 0.187$. Solid line shows the estimation by eq.(6.4).

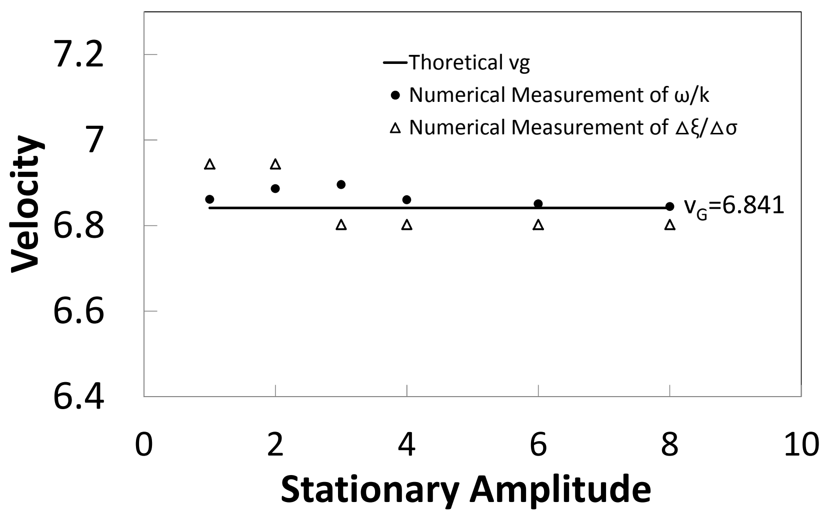

Fig.7. Solitary wave velocities $(\Delta\,\xi/\Delta\,\sigma)$ numerically calculated in case of $w = 0.187$.

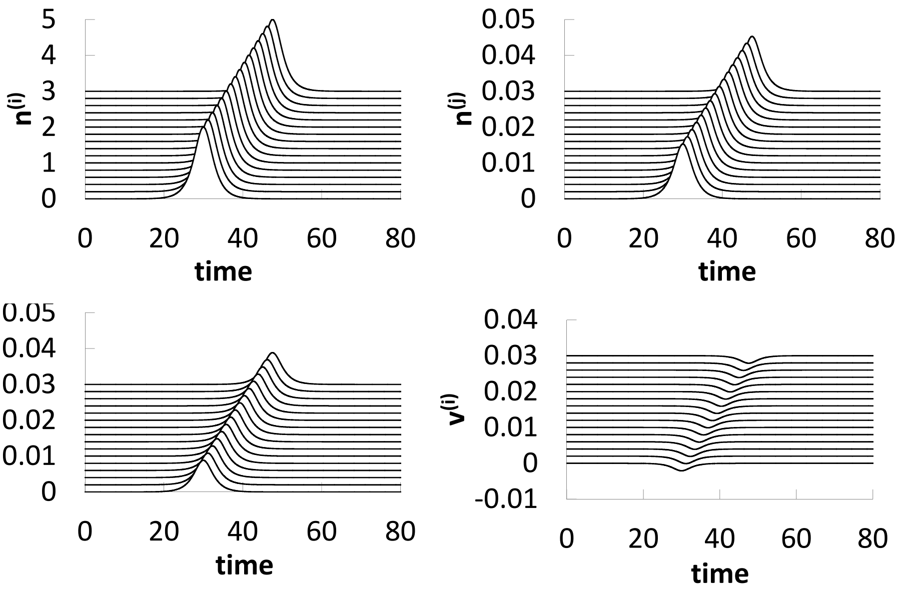

Fig.8. Propagation profiles of $n_1^{(i)}, n_1^{(j)}, T_1^{(i)}, v_1^{(i)}$ for the incident wave of amplitude $n_a^{(i)} = 2$ in case of $w = 0.187$.